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Modeling Absolute and Relative Cost Differences in Stochastic User Equilibrium Problem

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Abstract

This paper aims to develop a hybrid closed-form route choice model and the corresponding stochastic user equilibrium (SUE) to alleviate the drawbacks of both Logit and Weibit models by simultaneously considering absolute cost difference and relative cost difference in travelers' route choice decisions. The model development is based on an observation that the issues of absolute and relative cost differences are analogous to the negative exponential and power impedance functions of the trip distribution gravity model. Some theoretical properties of the hybrid model are also examined, such as the probability relationship among the three models, independence from irrelevant alternatives, and direct and indirect elasticities. To consider the congestion effect, we provide a unified modeling framework to formulate the Logit, Weibit and hybrid SUE models with the same entropy maximization objective but with different total cost constraint specifications representing the modelers' knowledge of the system. With this, there are two ways to interpret the dual variable associated with the cost constraint: shadow price representing the marginal change in the entropy level to a marginal change in the total cost, and dispersion/shape parameter representing the travelers' perceptions of travel costs. To further consider the route overlapping effect, a path-size factor is incorporated into the hybrid SUE model. Numerical examples are also provided to illustrate the capability of the hybrid model in handling both absolute and relative cost differences as well as the route overlapping problem in travelers' route choice decisions.

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Keywords: stochastic user equilibrium; logit; weibit; absolute cost difference; relative cost difference; route overlapping

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1. Introduction

The user equilibrium (UE) principle proposed by Wardrop (1952) is perhaps the most widely used route choice model in urban and regional transportation planning practices. It assumes that all travelers are rational in terms of preferring the lower travel time routes, have perfect knowledge of network travel times, and are able to identify the minimum travel time route (Sheffi, 1985). These unrealistic assumptions have been widely recognized and criticized by both researchers and practitioners. Among the various relaxations of these assumptions, Daganzo and Sheffi (1977) suggested the stochastic user equilibrium (SUE) principle to relax the perfect knowledge assumption of the UE model. The SUE incorporates a random perception error term in the route cost function to capture travelers' imperfect knowledge of travel times, such that they do not always end up picking the minimum travel time route. Different distribution assumptions on the random perception error term lead to different specialized probabilistic route choice models. A closed-form probability expression and equivalent mathematical programming (MP) formulation are generally preferred in real-world applications. In the literature, there are two main types of closed-form route choice models: Logit (Dial, 1971) and Weibit (Castillo *et al*., 2008). The Logit and Weibit models were originally derived from the random utility maximization approach by assuming the Gumbel and Weibull random error distributions. Fisk (1980) and Kitthamkesorn and Chen (2013, 2014) provided the equivalent MP formulations for Logit and Weibit models, respectively.

1.1. Motivations and observations

The Logit route choice probability expression is as follows:

$$P_k^w = \frac{\exp\left(-\theta c_k^w\right)}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)} = \frac{1}{\sum_{p\in K^w}\exp\left[-\theta\left(c_p^w - c_k^w\right)\right]},\ \forall k\in K^w,\ w\in W, \tag{1}$$

where W is the set of origin-destination (O-D) pairs, K^w is the set of routes between O-D pair w, c_k^w is the travel cost of route k, θ is the dispersion parameter. This model has two known drawbacks: (1) inability to account for overlapping (or correlation) among routes and (2) inability to account for perception variance with respect to trips of different lengths (Sheffi, 1985). These drawbacks stem from the underlying assumptions that the random error terms are *independently and identically distributed* (IID) Gumbel variates with the same and fixed perception variance.

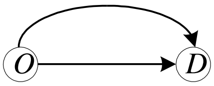
The Weibit route choice probability expression is as follows:

$$P_k^w = \frac{\left(c_k^w\right)^{-\beta}}{\sum_{p\in K^w}\left(c_p^w\right)^{-\beta}} = \frac{1}{\sum_{p\in K^w}\left(c_p^w/c_k^w\right)^{-\beta}},\ \forall k\in K^w,\ w\in W, \tag{2}$$

where β is the shape parameter of Weibull random error distribution. The Weibit model does not require the homogeneous perception variance assumption (i.e., the identically distributed assumption). Therefore, different trip lengths can be identified by the heterogeneous perception variances (Castillo *et al*., 2008). However, it also has its own limitation of not being able to account for any arbitrary multiplier on the route cost (Kitthamkesorn and Chen, 2014).

Below we use a two-route network to demonstrate the drawbacks of the Logit and Weibit models. We assume that the dispersion parameter (θ) of the Logit model is equal to 0.1, and the shape and location parameters (β and ζ) of the Weibit model are equal to 2.1 and 0, respectively. In Case I and Case II, the upper route cost is larger than the lower route cost by 5 units (i.e., absolute cost difference) for both networks. In the short network, the upper route cost is twice larger than the lower route cost, while it is only less than 5% larger in the long network. In Case III and Case IV, the upper route cost is twice larger than the lower route cost (i.e., relative cost difference) for both networks. In the short network, the upper route cost is larger than the lower route cost by 5 units, while it is 50 units larger in the long network. Herein the relative cost difference is based on the route cost ratio of the Weibit model as shown in Eq. (2), and the absolute cost difference is based on the route cost deviation of the Logit model as shown in Eq. (1).

Table 1. Illustration of the drawbacks of Logit and Weibit models.



Case	Route cost		Probability of choosing the lower route: Logit (θ=0.1)	Probability of choosing the lower route: Weibit (β=2.1, ξ=0)
I	Upper: 10 Lower: 5	Identical *absolute* difference (c_{upper}-c_{lower}=5)	$\frac{e^{-0.1(5)}}{e^{-0.1(5)}+e^{-0.1(10)}}=\frac{1}{1+e^{-0.1(5)}}=0.62$	$\frac{5^{-2.1}}{5^{-2.1}+10^{-2.1}}=\frac{1}{1+(10/5)^{-2.1}}=0.81$
II	Upper: 125 Lower: 120		$\frac{e^{-0.1(120)}}{e^{-0.1(120)}+e^{-0.1(125)}}=\frac{1}{1+e^{-0.1(5)}}=0.62$	$\frac{120^{-2.1}}{120^{-2.1}+125^{-2.1}}=\frac{1}{1+\left(\frac{125}{120}\right)^{-2.1}}=0.52$
III	Upper: 10 Lower: 5	Identical *relative* difference (c_{upper}/c_{lower}=2)	$\frac{e^{-0.1(5)}}{e^{-0.1(5)}+e^{-0.1(10)}}=\frac{1}{1+e^{-0.1(5)}}=0.62$	$\frac{5^{-2.1}}{5^{-2.1}+10^{-2.1}}=\frac{1}{1+(10/5)^{-2.1}}=0.81$
IV	Upper: 100 Lower: 50		$\frac{e^{-0.1(50)}}{e^{-0.1(50)}+e^{-0.1(100)}}=\frac{1}{1+e^{-0.1(50)}}=0.99$	$\frac{50^{-2.1}}{50^{-2.1}+100^{-2.1}}=\frac{1}{1+\left(\frac{100}{50}\right)^{-2.1}}=0.81$

In Case I and Case II, the Logit model gives the same results for both short and long networks; while in Case III and Case IV, the Weibit model gives the same results for both short and long networks. On the one hand, the Logit choice probability is solely determined by the *absolute* route cost difference as shown in Eq. (1). Hence, the Logit model is incapable of distinguishing the short and long networks with the same *absolute* route cost difference (i.e., give the same probability for 10-5=125-120=5). On the other hand, the Weibit choice probability (without the location parameter) is solely determined by the *relative* route cost difference as shown in Eq. (2). Hence, the Weibit model is unable to distinguish the short and long networks with the same *relative* route cost difference (i.e., give the same probability for 10/5=100/50=2). In summary, the Logit model is insensitive to a shift, while the Weibit model without the location parameter is insensitive to an arbitrary scale. This paper attempts to alleviate the above drawbacks of both the Logit and Weibit models simultaneously.

1.2. Main contributions of this paper

Motivated by the above issues, this paper aims to develop a hybrid closed-form route choice model that explicitly considers both the *absolute cost difference* of Logit model and the *relative cost difference* of Weibit model. The model development is based on an observation that the issues of absolute cost difference and relative cost difference are analogous to the negative exponential and power impedance functions of the trip distribution gravity model. A combined impedance function (also known as the gamma function) has been suggested to improve the model replication relative to the observed trip length frequency distribution (e.g., Ortuzar and Willumsen, 2011). With this analogy, our hybrid model includes the absolute cost difference in the exponential term and the relative cost difference in the power term. This modeling allows us to capture how travelers evaluate the travel costs: travelers evaluate their travel costs in a linear manner (i.e., linearly increasing), and change to a logarithmic manner for long trips (i.e., less weights to long trips). The linear and logarithmic evaluations correspond to the negative exponential and power functions, respectively (Wong, 1981; Wilson, 2010a,b). For instance, if the power function has a better fitting quality than the exponential function, then the travel disutility/cost may be evaluated in a logarithmic way. This may occur for a model that involves long trips (e.g., intercity O-D pairs). In addition, some theoretical properties are also examined, such as the probability relationship of the three models, independence from irrelevant alternatives (IIA), and direct and indirect elasticities.

To consider the congestion effect, we provide a unified modeling framework using the concept of entropy optimization for formulating the Logit, Weibit and hybrid SUE models with different cost constraint specifications. The derivation of the optimality conditions validates the feasibility of interpreting the SUE problem as an entropy optimization problem, besides the original perspective of random utility maximization theory. The cost constraint specification adjusts the entropy maximization of route flow distribution via the consideration of absolute cost difference, relative cost difference, or both in route choice decisions. There are two ways to interpret the dual variable associated with the total cost constraint. From the entropy maximization perspective, it is the shadow price representing the marginal change in the entropy level with respect to a marginal change in the total cost. From the random utility maximization perspective, it is the dispersion/shape parameter representing the travelers' perceptions of travel costs. To be structurally consistent with the existing MP formulations of the Logit SUE (Fisk, 1980) and Weibit SUE (Kitthamkesorn and Chen, 2013), we reformulate the hybrid SUE model as a weighted-sum model of the multi-objective optimization problem by treating the entropy term as the reference objective. There are two integral terms in the objective function. From this viewpoint, the hybrid model can also be considered a bi-criteria route choice problem, i.e., travel time and travel cost. To further alleviate the independence assumption of the hybrid SUE model, a path-size (PS) factor is adopted to handle the route overlapping problem.

In summary, the main contributions of this paper are twofold: (1) the development of a hybrid closed-form route choice model to alleviate the drawbacks of both Logit and Weibit models by simultaneously considering absolute and relative cost differences in the travelers' route choice decisions, and (2) the unified modeling framework to formulate all three models (i.e., Logit, Weibit and hybrid) with the same entropy maximization objective but with different specifications of total cost constraint representing the modelers' knowledge of the system.

The remainder of this paper is organized as follows. Section 2 presents the hybrid closed-form route choice model and explores its properties. Section 3 provides the unified modeling framework of entropy optimization for all three models. Numerical examples are provided in Section 4 to demonstrate the features of the hybrid route choice and SUE models. Finally, concluding remarks are summarized in Section 5.

2. Hybrid closed-form route choice model

In this section, we provide the hybrid closed-form route choice probability expression and explore some fundamental properties.

2.1. Route choice probability expression

In this section, we propose a hybrid closed-form route choice model to alleviate the drawbacks of the Logit and Weibit models simultaneously. The issues of absolute cost difference and relative cost difference are analogous to the negative exponential and power impedance functions of the gravity model used in trip distribution. A combined impedance function (i.e., product of an exponential term and a power term) has been suggested to improve the model replication relative to the observed trip length frequency distribution (e.g., Ortuzar and Willumsen, 2011). The combined impedance function is also known as a gamma function (Wong, 1981). With this analogy, we propose the following hybrid closed-form route choice model:

$$P_k^w = \frac{\exp\left(-\theta c_k^w\right)\cdot\left(c_k^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = \frac{1}{\sum_{p\in K^w}\exp\left[-\theta\left(c_p^w - c_k^w\right)\right]\cdot\left(c_p^w/c_k^w\right)^{-\beta}}. \tag{3}$$

One can see that the above route choice probability expression includes the absolute cost difference in the exponential term and the relative cost difference in the power term. Now let us look at two special cases of the hybrid model. When all routes have equal costs, i.e., the absolute cost difference is zero and the relative cost difference is one, then all three models (i.e., Logit, Weibit, and hybrid) produce the same probability of $1/|K^w|$, where $|K^w|$ is the number of available routes between O-D pair w. On the other hand, when the absolute cost difference and relative cost difference approach infinity, all three models give the same probability of 1. Note that Eq. (3) can be further rewritten as follows:

$$P_k^w = \frac{\exp\left(-\theta c_k^w\right)\cdot\exp\left(-\beta\ln c_k^w\right)}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\exp\left(-\beta\ln c_p^w\right)} = \frac{\exp\left(-\theta c_k^w - \beta\ln c_k^w\right)}{\sum_{p\in K^w}\exp\left(-\theta c_p^w - \beta\ln c_p^w\right)}. \tag{4}$$

Structurally, it is similar to the Logit model as shown in Eq. (1). It is not derived from another random error distribution different from the Gumbel or Weibull distribution. We can derive it by modifying the utility function of the Logit model. However, the deterministic disutility term in the exponential function is not only *linear* with route cost, but also proportional to the *logarithm* of route cost. From this viewpoint, the hybrid model allows us to capture how travelers evaluate travel costs. In the trip distribution model, whether the exponential, power, or gamma function is to be used as the impedance function depends on how travel costs are perceived by the travellers (Wong, 1981). Travelers evaluate/perceive their travel costs in a linear manner (i.e., linearly increasing), and change to a logarithmic manner for long trips (i.e., less weights to long trips). The linear and logarithmic evaluations correspond to the negative exponential and power functions, respectively (Wong, 1981; Wilson, 2010a,b). For instance, if the power function has a better fitting quality than the exponential function, then the travel disutility/cost may be evaluated in a logarithmic way. This may occur, for example, for a model that involves long trips (e.g., intercity O-D pairs). In other words, both the negative exponential and power functions have their own applicable scales. The former is applicable at a smaller spatial scale (e.g., intra-urban area) while the latter is applicable at a larger spatial scale (e.g., inter-urban area). From this viewpoint, the hybrid model has a wider applicable spatial scale, since the deterministic disutility term in the exponential function can be considered as a weighted combination between the linear scale and logarithmic scale of route cost. Despite that c dominates $\ln(c)$ to an increasing degree as c becomes large, the values of 'weighting' parameters θ (corresponding to the linear scale) and β (corresponding to the logarithmic scale of long trips) could implicitly reflect the travel distance.

2.2. Some properties

(1) Probability Relationship of the Three Models

Proposition 1. *The hybrid model gives a non-smaller probability of choosing the shortest route than the Logit and Weibit models, and it assigns a non-larger probability of choosing the longest route than the Logit and Weibit models*.

Proof. For the *shortest* route s between O-D pair w (i.e., $c_s^w \le c_p^w,\ \forall p \in K^w$), we have the following relationship:

$$P_{s,\text{Logit}}^w = \frac{\exp\left(-\theta c_s^w\right)}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)} = \frac{\exp\left(-\theta c_s^w\right)\cdot\left(c_s^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_s^w\right)^{-\beta}} \le \frac{\exp\left(-\theta c_s^w\right)\cdot\left(c_s^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = P_{s,\text{Hybrid}}^w, \tag{5}$$

$$P_{s,\text{Weibit}}^w = \frac{\left(c_s^w\right)^{-\beta}}{\sum_{p\in K^w}\left(c_p^w\right)^{-\beta}} = \frac{\exp\left(-\theta c_s^w\right)\cdot\left(c_s^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_s^w\right)\cdot\left(c_p^w\right)^{-\beta}} \le \frac{\exp\left(-\theta c_s^w\right)\cdot\left(c_s^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = P_{s,\text{Hybrid}}^w, \tag{6}$$

where the two inequalities use the fact that both $\left(c_k^w\right)^{-\beta}$ and $\exp\left(-\theta c_k^w\right)$ are decreasing functions with respect to c_k^w. Similarly, for the *longest* route l between O-D pair w (i.e., $c_l^w \ge c_p^w,\ \forall p \in K^w$), we have

$$P_{l,\text{Logit}}^w = \frac{\exp\left(-\theta c_l^w\right)}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)} \ge \frac{\exp\left(-\theta c_l^w\right)\cdot\left(c_l^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = P_{l,\text{Hybrid}}^w, \tag{7}$$

$$P_{l,\text{Weibit}}^w = \frac{\left(c_l^w\right)^{-\beta}}{\sum_{p\in K^w}\left(c_p^w\right)^{-\beta}} \ge \frac{\exp\left(-\theta c_l^w\right)\cdot\left(c_l^w\right)^{-\beta}}{\sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = P_{l,\text{Hybrid}}^w. \tag{8}$$

Note that the above relationship analysis of the three models is based on the common parameter values for simplicity. Under different parameter values between the hybrid and Logit/Weibit models, we are not able to discuss their general probability relationship. One would not expect that the parameters θ and β in the hybrid model should match the parameter values in the individual Logit and Weibit models.

(2) Independence from Irrelevant Alternatives (IIA)

Like the Logit and Weibit models, the hybrid route choice model still satisfies the independence from irrelevant alternatives (IIA) property. In other words, the ratio of choice probabilities of any two routes is entirely unaffected by the travel costs of any other routes. Mathematically, we have

$$\begin{aligned}\frac{P_k^w}{P_l^w} &= \frac{\exp\left(-\theta c_k^w\right)\cdot\left(c_k^w\right)^{-\beta} \Big/ \sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}}{\exp\left(-\theta c_l^w\right)\cdot\left(c_l^w\right)^{-\beta} \Big/ \sum_{p\in K^w}\exp\left(-\theta c_p^w\right)\cdot\left(c_p^w\right)^{-\beta}} = \frac{\exp\left(-\theta c_k^w\right)\cdot\left(c_k^w\right)^{-\beta}}{\exp\left(-\theta c_l^w\right)\cdot\left(c_l^w\right)^{-\beta}} \\ &= \frac{1}{\exp\left[-\theta\left(c_l^w - c_k^w\right)\right]\cdot\left(c_l^w / c_k^w\right)^{-\beta}},\quad \forall k\neq l\in K^w,\ w\in W\ . \end{aligned} \tag{9}$$

The relative odds of choosing route k and route l keeps intact regardless of the travel costs of any other routes. Also, the relative probabilities do not change regardless of alternatives being added or deleted from the choice set. This well-known property could be both advantageous and disadvantageous (Luce, 1959). As to the advantages, it allows modelers to estimate parameters consistently from a subset of alternatives, which can considerably reduce the computational effort (see, e.g., McFadden *et al*., 1978). As to the disadvantages, all alternatives/routes within the choice set must be distinct.

(3) Direct Elasticity and Indirect Elasticity

To what extent will the probability of choosing a given route increase if the travel cost is reduced? Derivatives of choice probabilities and elasticities (normalized by the variables' units) can be used as a tool to address this question. Elasticity is the percentage change in one variable in response to a one percent change in another variable. Specifically, the *direct* elasticity represents the change of the probability of choosing a particular route in response to a change in that route cost; whereas the *cross* elasticity determines the extent to which the probability of choosing a particular route changes when the travel cost on another route changes. Below we briefly review the elasticities of the **Logit** choice probability. The same logic will be applied to derive the elasticities of the Weibit and hybrid choice probabilities.

$$E_l^k\left(\text{logit}\right) = \frac{\partial P_k^{\text{logit}}}{\partial c_l}\cdot\frac{c_l}{P_k^{\text{logit}}}, \tag{10}$$

where $E_l^k\left(\text{logit}\right)$ denotes the elasticity of the probability of choosing route k in response to a cost change on route l. The derivative is presented below.

$$\begin{aligned}\frac{\partial P_k^{\text{logit}}}{\partial c_l} &= P_k^{\text{logit}}\cdot\left(-\theta\frac{\partial c_k}{\partial c_l}\right) - \frac{\exp\left(-\theta c_k\right)}{\left(\sum_p \exp\left(-\theta c_p\right)\right)^2}\sum_p \exp\left(-\theta c_p\right)\cdot\left(-\theta\frac{\partial c_p}{\partial c_l}\right) \\ &= \begin{cases} -\theta P_k^{\text{logit}} + \theta\left(P_k^{\text{logit}}\right)^2, & \text{if } k = l \\ \theta P_k^{\text{logit}} P_l^{\text{logit}}, & \text{if } k \neq l \end{cases}. \end{aligned} \tag{11}$$

The direct elasticity (i.e., k=l) and cross elasticity (i.e., k≠l) can be derived as follows.

$$E_k^k\left(\text{logit}\right) = \left(-\theta P_k^{\text{logit}} + \theta\left(P_k^{\text{logit}}\right)^2\right)\cdot\frac{c_k}{P_k^{\text{logit}}} = -\theta c_k\left(1 - P_k^{\text{logit}}\right), \tag{12}$$

$$E_l^k\left(\text{logit}\right) = \left(\theta P_k^{\text{logit}} P_l^{\text{logit}}\right)\cdot\frac{c_l}{P_k^{\text{logit}}} = \theta c_l P_l^{\text{logit}}. \tag{13}$$

Similarly, the derivative and elasticities of the **Weibit** choice probability can be derived as follows.

$$\frac{\partial P_k^{\text{weibit}}}{\partial c_l} = P_k^{\text{weibit}}\cdot\left(-\beta\frac{1}{c_k}\frac{\partial c_k}{\partial c_l}\right) - \frac{\exp\left(-\beta\ln c_k\right)}{\left(\sum_p \exp\left(-\beta\ln c_p\right)\right)^2}\sum_p \exp\left(-\beta\ln c_p\right)\cdot\left(-\beta\frac{1}{c_p}\frac{\partial c_p}{\partial c_l}\right)$$

$$=\begin{cases}-\dfrac{\beta}{c_k}P_k^{\text{weibit}}+\dfrac{\beta}{c_k}\left(P_k^{\text{weibit}}\right)^2, & \text{if } k=l\\ \dfrac{\beta}{c_l}P_k^{\text{weibit}}P_l^{\text{weibit}}, & \text{if } k\neq l\end{cases}, \tag{14}$$

$$E_k^k\left(\text{weibit}\right)=\left(-\frac{\beta}{c_k}P_k^{\text{weibit}}+\frac{\beta}{c_k}\left(P_k^{\text{weibit}}\right)^2\right)\cdot\frac{c_k}{P_k^{\text{weibit}}}=-\beta\left(1-P_k^{\text{weibit}}\right), \tag{15}$$

$$E_l^k\left(\text{weibit}\right)=\left(\frac{\beta}{c_l}P_k^{\text{weibit}}P_l^{\text{weibit}}\right)\cdot\frac{c_l}{P_k^{\text{weibit}}}=\beta P_l^{\text{weibit}}. \tag{16}$$

Now we derive the derivative and elasticities of the **hybrid** choice probability.

$$\begin{aligned}\frac{\partial P_k^{\text{hybrid}}}{\partial c_l}&=P_k^{\text{hybrid}}\cdot\left(-\theta\frac{\partial c_k}{\partial c_l}-\beta\frac{1}{c_k}\frac{\partial c_k}{\partial c_l}\right)\\&-\frac{\exp\left(-\theta c_k-\beta\ln c_k\right)}{\left(\sum_p\exp\left(-\theta c_p-\beta\ln c_p\right)\right)^2}\sum_p\exp\left(-\theta c_p-\beta\ln c_p\right)\cdot\left(-\theta\frac{\partial c_p}{\partial c_l}-\beta\frac{1}{c_p}\frac{\partial c_p}{\partial c_l}\right)\\&=\begin{cases}-\left(\theta+\dfrac{\beta}{c_k}\right)P_k^{\text{hybrid}}+\left(\theta+\dfrac{\beta}{c_k}\right)\left(P_k^{\text{hybrid}}\right)^2, & \text{if } k=l\\ \left(\theta+\dfrac{\beta}{c_l}\right)P_k^{\text{hybrid}}P_l^{\text{hybrid}}, & \text{if } k\neq l\end{cases},\end{aligned} \tag{17}$$

$$E_k^k\left(\text{hybrid}\right)=\left(-\left(\theta+\frac{\beta}{c_k}\right)P_k^{\text{hybrid}}+\left(\theta+\frac{\beta}{c_k}\right)\left(P_k^{\text{hybrid}}\right)^2\right)\cdot\frac{c_k}{P_k^{\text{hybrid}}}=-\left(\theta c_k+\beta\right)\left(1-P_k^{\text{hybrid}}\right), \tag{18}$$

$$E_l^k\left(\text{hybrid}\right)=\left(\left(\theta+\frac{\beta}{c_l}\right)P_k^{\text{hybrid}}P_l^{\text{hybrid}}\right)\cdot\frac{c_l}{P_k^{\text{hybrid}}}=\left(\theta c_l+\beta\right)P_l^{\text{hybrid}}. \tag{19}$$

One can see that the direct elasticity of all three models is negative while the cross elasticity of all three models is positive. Improving one route alternative will increase its probability of being chosen, and the incremental probability comes from other route alternatives. Note that the cross elasticity is uniform for all k other than l. In other words, cost change on route l changes the probabilities of all other routes by the same percentage. This property is a restatement of the IIA property satisfied by the three choice probabilities. Another observation is that the elasticity expressions of the Weibit probability are not explicitly dependent on route costs (c_k or c_l), even though they are implicitly considered in the choice probabilities. This is different from the Logit case. As to the hybrid model, the structure of the elasticity expressions seems to be the summation of the elasticity expressions of Logit and Weibit models, despite with different probability terms.

3. Unified formulation of stochastic user equilibrium

The Logit and Weibit models were originally derived from the random utility maximization approach. Specifically, the Logit model assumes the additive utility function and the independently and identically Gumbel distributed error terms; the Weibit model assumes the multiplicative utility function and the independently Weibull distributed error terms (Fosgerau and Bierlaire, 2009; Kitthamkesorn and Chen, 2013, 2014). Fosgerau and Bierlaire (2009) found that the multiplicative model provides a better fit than the additive model in several empirical data sets. However, it is not universally better, and should not be systematically preferred. Instead, the proposed hybrid model integrates both types of models. In this section, we provide a unified modeling framework for the SUE problem with the Logit, Weibit, and hybrid route choice models– entropy maximization (or information minimization). The three SUE models have the same entropy maximization objective but with different constraint specifications representing the modelers' knowledge of the system. With the unified framework, we have a better understanding on how the absolute/relative cost difference or both affects the entropy maximization of flow distribution via different cost constraint specifications.

3.1. Entropy optimization formulation

Similar to the trip distribution context, the entropy of an aggregate travel demand $\mathbf{q}=\{q^w\}$ is associated with the number of possible combinations resulting from individual route choice decisions:

$$\prod_{w\in W}\frac{q^w!}{\prod_{k\in K^w} f_k^w!}=\frac{\prod_{w\in W} q^w!}{\prod_{w\in W}\prod_{k\in K^w} f_k^w!},\tag{20}$$

where q^w is the travel demand of O-D pair w, and f_k^w is the flow on route k between O-D pair w. By using the monotonic logarithmic transformation and the Stirling's approximation (i.e., $\ln x!\approx x(\ln x-1)$), the most probable state with the maximum entropy is equivalent to

$$\begin{aligned}\arg\max\frac{\prod_{w\in W} q^w!}{\prod_{w\in W}\prod_{k\in K^w} f_k^w!}&=\arg\max\sum_{w\in W}\ln\left(q^w!\right)-\sum_{w\in W}\sum_{k\in K^w}\ln\left(f_k^w!\right)\\&=\arg\max\sum_{w\in W} q^w\left(\ln q^w-1\right)-\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right)=\arg\min\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right).\end{aligned}\tag{21}$$

Below we cast the Logit and Weibit SUE models into the entropy optimization framework. We will show that they have the same objective function (i.e., entropy optimization in Eq. (21)) but with different specifications of total cost constraint.

[Logit-based SUE]

The Logit-based SUE problem can be formulated as an entropy optimization problem subject to the total cost constraint specification in Eq. (26).

$$\min Z(\mathbf{f})=\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right),\tag{22}$$

s.t.

$$\sum_{k\in K^w} f_k^w=q^w\ \left(\pi^w\right),\ \forall w\in W,\tag{23}$$

$$f_k^w\geq 0,\ \forall k\in K^w,\ \forall w\in W,\tag{24}$$

$$\sum_{w\in W}\sum_{k\in K^w} f_k^w\delta_{ka}^w=v_a,\ \forall a\in A,\tag{25}$$

$$\sum_{a\in A}\int_0^{v_a} t_a(w)\,dw=C\ \left(\alpha_1\right),\tag{26}$$

where A is the set of directed links in the network; v_a and t_a are flow and travel time of link a; f_k^w is the flow on route k between O-D pair w and $\mathbf{f}$ is its vector form; δ_{ka}^w is the link-route incidence indicator: $\delta_{ka}^w=1$ if link a is on route k between O-D pair w, and 0 otherwise. Eq. (22) is equivalent to the entropy maximization as shown in Eq. (21); Eq. (23) is the conservation constraint, and π^w is the Lagrangian multiplier; Eq. (24) is the non-negativity constraint of route flows; Eq. (25) is the definitional constraint. The left-hand side of Eq. (26) is the well-known Beckmann transformation (Beckmann *et al.*, 1956), the unknown parameter C could be interpreted as an indirect measure of total network congestion (flow-dependent network congestion level, not total travel time); and α_1 is the Lagrangian multiplier. We are not solving the entropy maximization formulation in Eqs. (22)-(26) directly. The role of C is to introduce the hybrid SUE MP in Eqs. (35)-(37) and uses it as a transition to the weighted multi-objective optimization in Section 3.2. The Beckmann transformation being treated as a constraint has also been used in the O-D trip matrix estimation problem (e.g., Fisk and Boyce, 1983). In essence, they formulated a combined distribution and assignment model, in which the link flow data served to furnish an estimate for the sum of the integrals of the link cost functions (i.e., the Beckmann transformation term).

Proposition 2. *The MP formulation in Eqs. (22)-(26) has the solution of Logit model.*
Proof. The Lagrangian function of the above formulation can be constructed as

$$L=\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right)+\pi^w\left(q^w-\sum_{k\in K^w} f_k^w\right)+\alpha_1\left(\sum_{a\in A}\int_0^{v_a} t_a\left(w\right)dw-C\right). \tag{27}$$

The entropy term implicitly determines that all route flows are positive. At the optimum, the following first-order conditions must be satisfied:

$$\frac{\partial L}{\partial f_k^w}=\ln f_k^w-\pi^w+\alpha_1\sum_{a\in A} t_a\delta_{ak}^w=0\Rightarrow f_k^w=\exp\left(\pi^w-\alpha_1 c_k^w\right). \tag{28}$$

In Eq. (28), we make use of the additive route travel time structure: $\sum_{a\in A} t_a\delta_{ka}^w=c_k^w$. Considering the conservation constraint in Eq. (23), the route choice probability can be expressed as follows:

$$P_k^w=\frac{f_k^w}{q^w}=\frac{\exp\left(\pi^w-\alpha_1 c_k^w\right)}{\sum_{p\in K^w}\exp\left(\pi^w-\alpha_1 c_p^w\right)}=\frac{1}{\sum_{p\in K^w}\exp\left[-\alpha_1\left(c_p^w-c_k^w\right)\right]}, \tag{29}$$

which is the Logit route choice model. □

Remark 1. The above derivation shows the feasibility of interpreting the Logit-based SUE problem as an *entropy optimization* problem, besides the original perspective of *random utility maximization* theory embedded in network equilibrium models (Sheffi, 1985). Note that the relationship of the two perspectives has been discussed in the discrete choice problem (e.g., Anas, 1983; Donoso and de Grange, 2010). Constraint (26) adjusts the entropy maximization of route flow distribution via the consideration of *absolute cost difference* in the travelers' route choice decisions as shown in Eq. (29). If the dual variable α_1 is equal to the dispersion parameter θ of the Logit model in Eq. (1), the above formulation in Eqs. (22)-(26) yields the same optimality conditions as in Fisk' model (1980). Hence, there are two ways to interpret the dual variable α_1.

(a) From the *entropy maximization* perspective, α_1 is the *shadow price* representing the marginal change in the entropy level with respect to a marginal change in the Beckmann transformation term (i.e., a measure of total network congestion).
(b) From the *random utility maximization* perspective, α_1 is the *dispersion parameter* representing the travelers' perceptions of travel costs. Specifically, it is related to the fixed perception variance of $\pi^2/6\alpha_1^2$.

Remark 2. If we only consider the route choice problem (i.e., without congestion effect), route travel times/costs are fixed. In that case, the route choice of an O-D pair is irrelevant of all the other O-D pairs. Accordingly, the total cost constraint in Eq. (26) can be changed to the O-D specific total travel time of $\sum_{k\in K^w} c_k^w f_k^w=C^w$ rather than the Beckmann transformation term. Then, we can obtain the same route choice probability expression as in Eq. (29).

[Weibit-based SUE]

Similar to the Logit case, the Weibit-based SUE problem can also be formulated as an entropy optimization but with the total cost constraint specification as in Eq. (31).

$$\min Z\left(\mathbf{f}\right)=\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right), \tag{30}$$

s.t. Eqs. (23)- (25),

$$\sum_{a\in A}\int_0^{v_a}\ln t_a\left(w\right)dw=C\ \left(\alpha_2\right), \tag{31}$$

where α_2 is the dual variable of the multiplicative Beckmann transformation term.

Proposition 3. *The MP formulation in Eqs. (30)-(31) has the solution of Weibit model.*
Proof. The Lagrangian function can be constructed as

$$L=\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right)+\pi^w\left(q^w-\sum_{k\in K^w} f_k^w\right)+\alpha_2\left(\sum_{a\in A}\int_0^{v_a}\ln t_a\left(w\right)dw-C\right), \tag{32}$$

The first-order derivative gives

$$\frac{\partial L}{\partial f_k^w} = \ln f_k^w - \pi^w + \alpha_2 \sum_{a\in A} (\ln t_a) \delta_{ak}^w = \ln f_k^w - \pi^w + \alpha_2 \ln\left(\prod_{a\in \Upsilon_k^w} t_a \right) = \ln f_k^w - \pi^w + \alpha_2 \ln g_k^w = 0$$
$$\Rightarrow f_k^w = \exp(\pi^w)\exp(-\alpha_2 \ln g_k^w) = \exp(\pi^w)(g_k^w)^{-\alpha_2}, \tag{33}$$

where Υ_k^w is the set of links on route k between O-D pair w. In Eq. (33), we make use of the multiplicative cost structure to compute the route cost (i.e., $\prod_{a\in\Upsilon_k^w} t_a = g_k^w$, multiplication of all link costs on that route). Then, the route choice probability can be expressed as follows:

$$P_k^w = \frac{f_k^w}{q^w} = (g_k^w)^{-\alpha_2} \Big/ \sum_{p\in K^w} (g_p^w)^{-\alpha_2} = 1 \Big/ \sum_{p\in K^w} (g_p^w / g_k^w)^{-\alpha_2}, \tag{34}$$

which is the Weibit route choice model. □

Constraint Eq. (31) adjusts the entropy maximization of route flow distribution via the consideration of *relative cost difference* in the route choice decision as shown in Eq. (34). Note that the common parameter β implies that all routes of all O-D pairs have the same coefficient of variance (CoV) $\sqrt{\Gamma(1+2/\beta)\big/\left(\Gamma(1+1/\beta)\right)^2 - 1}$. However, the perception variance is still route-specific since it is dependent on the route-specific mean travel cost (Kitthamkesorn and Chen, 2014).

[Hybrid SUE]

When we have the two cost constraints in Eqs. (26) and (31) simultaneously, the entropy optimization explicitly accounts for both *absolute cost difference* and *relative cost difference* in the travelers' route choice decisions. For completeness, the formulation is presented below. We add subscripts 1 and 2 to differentiate the two constraints.

$$\min Z(\mathbf{f}) = \sum_{w\in W}\sum_{k\in K^w} f_k^w (\ln f_k^w - 1), \tag{35}$$

s.t. Eqs. (23)- (25),

$$\sum_{a\in A}\int_0^{v_a} t_a(w)dw = C_1 \quad (\alpha_1), \tag{36}$$

$$\sum_{a\in A}\int_0^{v_a} \ln t_a(w)dw = C_2 \quad (\alpha_2). \tag{37}$$

Proposition 4. *The MP formulation in Eqs. (35)-(37) has the solution of hybrid route choice model.*
Proof. Similar to the previous two models, the first-order derivative of the above formulation gives

$$\frac{\partial L}{\partial f_k^w} = \ln f_k^w - \pi^w + \alpha_1 c_k^w + \alpha_2 \ln g_k^w = 0$$
$$\Rightarrow f_k^w = \exp(\pi^w)\exp(-\alpha_1 c_k^w)\exp(-\alpha_2 \ln g_k^w) = \exp(\pi^w)\exp(-\alpha_1 c_k^w)(g_k^w)^{-\alpha_2}. \tag{38}$$

Then, we have the following probability expression:

$$P_k^w = \frac{f_k^w}{q^w} = \frac{\exp(-\alpha_1 c_k^w)(g_k^w)^{-\alpha_2}}{\sum_{p\in K^w}\exp(-\alpha_1 c_p^w)(g_p^w)^{-\alpha_2}} = \frac{1}{\sum_{p\in K^w}\exp\left[-\alpha_1(c_p^w - c_k^w)\right]\cdot(g_p^w/g_k^w)^{-\alpha_2}}, \tag{39}$$

which is the hybrid route choice model given in Section 2. □

Remark 3. Here we cannot use the same route cost term c_k^w or g_k^w in the choice probability expression. The reason is that the Logit model with the absolute difference consideration has an additive route travel time structure $\sum_{a\in A} t_a \delta_{ka}^w = c_k^w$, while the Weibit model with the relative difference consideration has a multiplicative route travel cost structure $\prod_{a\in\Upsilon_k^w} t_a = g_k^w$ (or $\sum_{a\in A}(\ln t_a)\delta_{ka}^w = \ln g_k^w$). Otherwise, $c_k^w = \sum_{a\in A} t_a \delta_{ka}^w$ conflicts with $c_k^w = \prod_{a\in\Upsilon_k^w} t_a$. The exception is that: if each route only consists of a single link (i.e., each link is a route), we do not need to differentiate c_k^w and g_k^w since they are equal. However, in the general SUE problem where each route consists of multiple links, we need to propagate the link travel time/cost (t_a) to the route travel time/cost (c_k^w and g_k^w) via either summation or multiplication.

3.2. Multi-objective optimization reformulation

To be structurally consistent with the existing MP formulations of the Logit SUE (Fisk, 1980) and Weibit SUE (Kitthamkesorn and Chen, 2013), this section reformulates the entropy optimization in Eqs. (35)-(37) as a weighted-sum model of the multi-objective optimization problem.

$$\min Z(\mathbf{f}) = \alpha_1 \sum_{a\in A} \int_0^{v_a} t_a(w)dw + \alpha_2 \sum_{a\in A} \int_0^{v_a} \ln t_a(w)dw + \sum_{w\in W} \sum_{k\in K^w} f_k^w \left(\ln f_k^w - 1\right), \tag{40}$$

s.t. Eqs. (23)- (25).

Herein we can also use a more general form of $\ln \tau_a$ to replace $\ln t_a$, where τ_a is a travel cost and it is a strictly increasing function of t_a (e.g., $\tau_a = \exp(\gamma t_a)$) like in Kitthamkesorn and Chen (2013, 2014). Note that the multi-objective entropy optimization has also been used in a set of trip distribution models (De Grange *et al*., 2010) and combined models with hierarchical demand choices (De Cea *et al*., 2008).

Remark 4. In the above reformulation, we consider three objective functions simultaneously: minimization of the additive Beckmann transformation term, minimization of the multiplicative Beckmann transformation term, and maximization of the route flow entropy term. The first two objectives are respectively associated with the consideration of absolute cost difference and relative cost difference in travelers' route choice decisions, while adjusting the entropy maximization of route flow distribution. Here we treat the entropy term as the reference objective and assign relative *weights* α_1 and α_2 to the two total cost objectives. There are two integral terms in the objective function. From this viewpoint, the hybrid route choice model could be considered a bi-criteria route choice problem, i.e., travel time t_a and travel cost $\ln t_a$ weighted by the coefficients of α_1 and α_2, respectively. They constitute two types of attributes in the route choice disutility. In the literature, nonlinear utility functions have been suggested to model travel choice behaviors (Tapley, 2008; Stathopoulos and Hess, 2012). For example, Koppelman (1981) demonstrated that nonlinear transformations of travel time and cost can significantly improve the model estimation, and also provide a theoretically appealing interpretation and managerially important differences in policy assessment. Rotaris *et al*. (2012) compared a set of nonlinearities and marginally changing attribute sensitivity in freight transportation service evaluation. As to the collective level, the additive Beckmann transformation term considers the travel time, and the multiplicative Beckmann transformation term considers the nonlinear travel cost. As such, it is reasonable to weight them differently (α_1 and α_2) from the multi-objective optimization perspective.

Proposition 5. *The MP formulation in Eq. (40) has unique route flow solution under the usual assumption of monotone link travel time functions*.

Proof. The feasible region is convex. To prove that the objective function is strictly convex, we look at the Hessian matrix. Similar to the additive Beckmann transformation term, the multiplicative Beckmann transformation term has a positive semi-definite (PSD) Hessian matrix. However, the Hessian matrix of the entropy term is positive definite (PD). The summation of two PSD matrices and a PD matrix results in a PD matrix. Thus, the objective function in Eq. (40) is strictly convex. The route flow solution to the MP in Eq. (40) is unique. □

3.3. An extension to route overlapping consideration

Even though the hybrid SUE model can alleviate the drawbacks of both the Logit and Weibit models by simultaneously considering absolute and relative cost differences, it still inherits another common limitation of both the Logit and Weibit models: route independence assumption. Similar to Kitthamkesorn and Chen (2013), we use a path-size (PS) factor defined below to further handle the route overlapping problem. Conceptually, it accounts for different route sizes determined by the length of links within a route and the relative lengths of routes that share a link (Ben-Akiva and Bierlaire, 1999).

$$\rho_k^w = \sum_{a\in\Upsilon_k^w} \frac{l_a}{L_k^w} \frac{1}{\sum_{k\in K^w} \delta_{ak}^w} = \frac{1}{L_k^w} \sum_{a\in\Upsilon_k^w} \frac{l_a}{\sum_{k\in K^w} \delta_{ak}^w}, \ \forall k\in K^w, \ w\in W, \tag{41}$$

where l_a is the length of link a; L_k^w is the length of route k between O-D pair w. The lengths in the common part and the route ratio (i.e., l_a/L_k^w) is a plausible approximation of the route correlation, and $\sum_{k\in K^w} \delta_{ak}^w$ measures the contribution of link a in the route correlation (Frejinger and Bierlaire, 2007). This PS factor is between 0 and 1. Routes with a heavy overlapping with other routes (i.e., more links with $\sum_{k\in K^w} \delta_{ak}^w$ of greater than one) will have a smaller value of ρ_k^w. For other functional forms of the PS factor, see Bovy *et al.* (2008).

Consider the following MP for the hybrid SUE with route overlapping consideration:

$$\min Z(\mathbf{f}) = \alpha_1 \sum_{a\in A} \int_0^{v_a} t_a(w)dw + \alpha_2 \sum_{a\in A} \int_0^{v_a} \ln t_a(w)dw + \sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w - 1\right) - \sum_{w\in W}\sum_{k\in K^w} f_k^w \ln \rho_k^w, \tag{42}$$

s.t. Eqs. (23)- (25).

The first-order optimality condition of the above formulation gives

$$\begin{aligned} &\alpha_1 c_k^w + \alpha_2 \ln g_k^w + \ln f_k^w - \ln \rho_k^w - \pi^w = 0 \\ &\Rightarrow f_k^w = \exp\left(-\alpha_1 c_k^w - \alpha_2 \ln g_k^w + \ln \rho_k^w\right)\exp\left(\pi^w\right) = \exp\left(\pi^w\right)\exp\left(-\alpha_1 c_k^w\right)\left(g_k^w\right)^{-\alpha_2} \rho_k^w. \end{aligned} \tag{43}$$

Then, we have the following probability expression:

$$P_k^w = \frac{f_k^w}{q^w} = \frac{\rho_k^w \exp\left(-\alpha_1 c_k^w\right)\left(g_k^w\right)^{-\alpha_2}}{\sum_{p\in K^w} \rho_p^w \exp\left(-\alpha_1 c_p^w\right)\left(g_p^w\right)^{-\alpha_2}}, \ \forall k\in K^w, \ w\in W. \tag{44}$$

This is the hybrid route choice probability with route overlapping consideration. It adjusts the probability of routes coupling with other routes via the PS factor. The last term in Eq. (42) is introduced to capture the size/length of the routes in order to correct the hybrid route choice probability. Note that when there is no route overlapping, i.e., ρ_k^w =1, the above MP model collapses to the one in Eq. (40).

Remark 5. Mathematically, the hybrid SUE MP formulation is a combination of the Logit SUE and Weibit SUE MP formulations. Hence, existing algorithms that work well for large-scale Logit (e.g., Xu *et al*., 2012; Zhou *et al*., 2012; Chen *et al*., 2013, 2014) and Weibit SUE (e.g., Kitthamkesorn and Chen, 2013) models could be readily modified for solving the hybrid SUE model.

4. Numerical results

In this section, we demonstrate the capability and features of the proposed hybrid route choice model and its corresponding SUE models in handling both absolute and relative cost differences. Example 1 provides numerical illustrations of the hybrid route choice model and its properties. Example 2 investigates the effect of absolute cost difference and relative cost difference under the congestion effect. Example 3 further considers the route overlapping problem in the hybrid SUE model.

4.1. Example 1: Numerical illustration of the hybrid route choice model

We continue to use the example in Table 1 to demonstrate the hybrid route choice model. Table 2 shows the probability of choosing the lower/shorter route under the four cases when using the Logit, Weibit, and Hybrid models. One can readily see that the proposed hybrid route choice model can alleviate the drawbacks of the Logit and Weibit models simultaneously. Specifically, it can distinguish the short and long networks with the identical absolute cost difference (i.e., Case I and Case II) as well as the identical relative cost difference (i.e., Case III and Case IV) in the travelers' route choice decisions.

Table 2. Probability comparison among the Logit, Weibit and hybrid models.

Case	$O \to D$ Route cost		Probability of choosing the lower route		
			Logit	Weibit	Hybrid
I	Upper: 10 Lower: 5	Identical *absolute* difference (5)	**0.62**	0.81	$\dfrac{e^{-0.1(5)}5^{-2.1}}{e^{-0.1(5)}5^{-2.1}+e^{-0.1(10)}10^{-2.1}}=0.88$
II	Upper: 125 Lower: 120		**0.62**	0.52	$\dfrac{e^{-0.1(120)}120^{-2.1}}{e^{-0.1(120)}120^{-2.1}+e^{-0.1(125)}125^{-2.1}}=0.64$
III	Upper: 10 Lower: 5	Identical *relative* difference (2)	0.62	**0.81**	$\dfrac{e^{-0.1(5)}5^{-2.1}}{e^{-0.1(5)}5^{-2.1}+e^{-0.1(10)}10^{-2.1}}=0.88$
IV	Upper: 100 Lower: 50		0.99	**0.81**	$\dfrac{e^{-0.1(50)}50^{-2.1}}{e^{-0.1(50)}50^{-2.1}+e^{-0.1(100)}100^{-2.1}}=0.998$

To be more general, we plot the probability of choosing the shorter route under various combinations of absolute and relative cost differences in Fig. 1. One can see that the probability surface of the hybrid model is above those of Logit and Weibit models. This is consistent with Proposition 1. We should point out that this relationship is due to the hybridization in Eq. (3) and also under the same θ value between the Logit and hybrid models and the same β value between the Weibit and hybrid models. However, this is NOT an assumption of the hybrid model, since it may have different θ (and/or β) values with the Logit (and/or Weibit) model in calibrations. In fact, a route choice model with more parameters would provide more flexibility for accounting for different behavioral responses in travelers' route choice decisions.

Fig. 2 further shows the corresponding contour plots. The Logit and Weibit models have the horizontal and vertical straight line contours due to the *sole* determination of either absolute cost difference or relative cost difference. On the contrary, the curved contour of the hybrid model shows the *joint* determination of both absolute cost difference and relative cost difference in route choice decisions. In addition, as the cost difference (i.e., route dominance) increases, the impact on probability shift becomes less significant for all three models as shown by the gradually sparse gaps of probability contours.

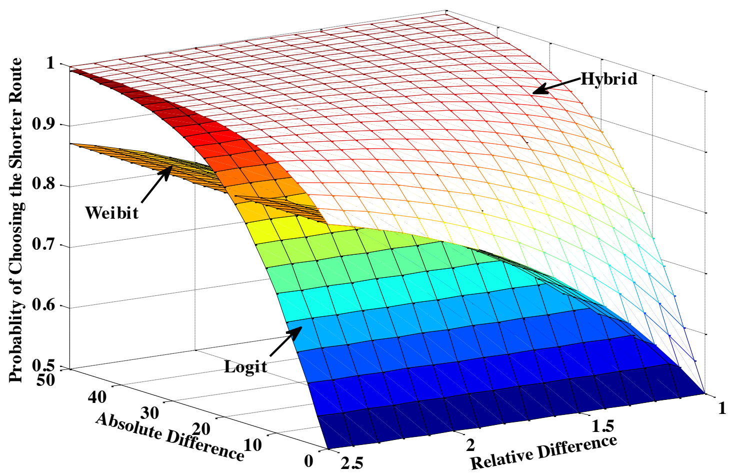


Fig. 1. Probability surfaces of Logit, Weibit and hybrid models.

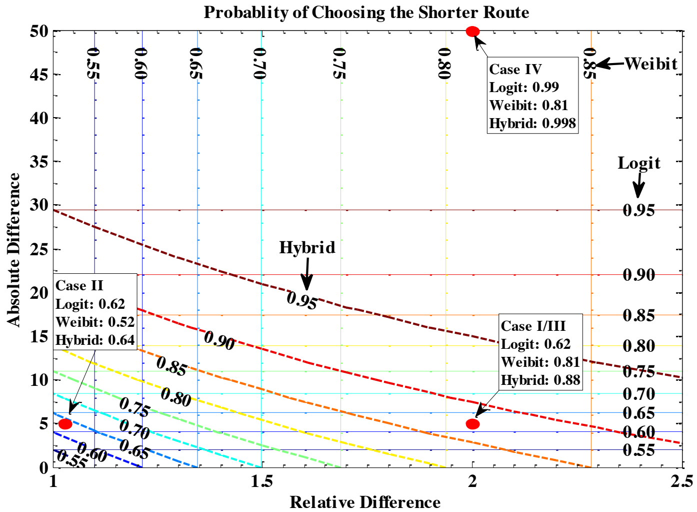


Fig. 2. Probability contours of Logit, Weibit and hybrid models.

To see when the Logit and Weibit probabilities approach the hybrid choice probabilities, Fig. 3 shows the difference in probability of choosing the shorter route between the hybrid and Logit models as well as between the hybrid and Weibit models. Specifically, the probability difference between the hybrid and Logit models (i.e., dashed contour) is decreased with the increase of absolute cost difference, and it is less affected by the relative cost difference. The probability difference between the hybrid and Weibit models (i.e., solid contour) is significantly decreased with the increase of the relative cost difference at large absolute difference, and becomes less affected by the relative cost difference at low absolute cost difference. It means that the three models do not necessarily approach each other at large absolute and relative differences. In general, the absolute and relative cost differences play different roles in the probability differences.

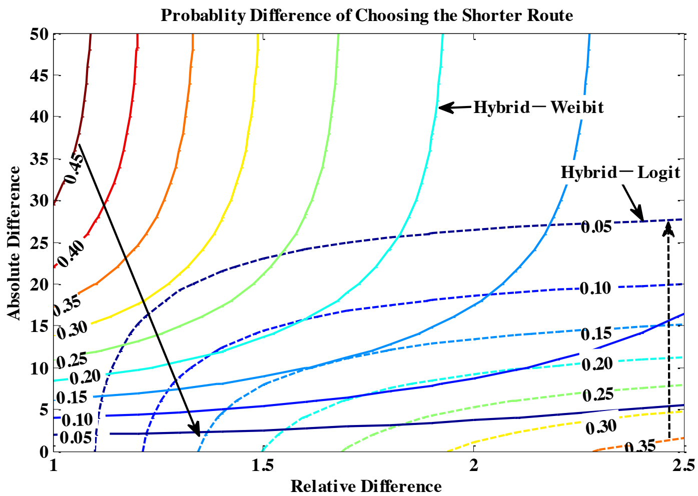


Fig. 3. Probability difference contours of Logit, Weibit and hybrid models.

4.2. Example 2: Two-route network with congestion effect

We modify the above two-route network to examine the effect of different combinations of absolute and relative cost differences under congestion effect. To this end, each route/link incorporates a flow-dependent cost component of $f/10$ as shown in Table 3 to represent congestion effect. The O-D demand is fixed at 100 vehicles per unit of time. The dispersion parameter of the Logit model is set to 0.1 and the shape parameter of the Weibit model is set to 3.7.

Table 3. Flow-dependent route costs for the two-route network under three cases.

Network case	Upper route	Lower route
I/III	$10 + f/10$	$5 + f/10$
II	$125 + f/10$	$120 + f/10$
IV	$100 + f/10$	$50 + f/10$

First of all, we compare the three network cases for each SUE model (i.e., vertical comparison). Table 4 shows the equilibrium route flows assigned by the Logit, Weibit, and hybrid SUE models under the above three network cases. One can see that the Logit-based SUE model produces the same route flow pattern for Case I and Case II, regardless of the overall trip length. The reason is that the flow-dependent route cost difference is identical for the two cases, i.e., $10+f_u/10-(5+f_l/10) = 125+f_u/10-(120+f_l/10) = 5+(f_u-f_l)/10$. Accordingly, they have the same absolute cost difference of 3.34 at the respective SUE state. However, the Logit-based SUE model is incapable of distinguishing their different relative cost differences under congestion effect (i.e., 1.31 versus 1.03). In addition, as shown in Case IV, with the increase of the absolute cost difference, the lower route significantly dominates the upper route in the route choice decisions.

As to the Weibit-based SUE model, it assigns different flow patterns to the three cases to reflect their distinct flow-dependent relative cost differences. Particularly, Case III and Case IV do not have the same route flow pattern as Case I and Case II in the Logit-based SUE model. Case III and Case IV have the same relative difference without congestion consideration, i.e., 10/5=100/50=2. However, their flow-dependent relative cost differences are generally not identical, i.e., $(10+f_u/10)/(5+f_l/10)$ versus $(100+f_u/10)/(50+f_l/10)$, except for f_u=66.67 with the same relative difference of 2. Case II has the smallest relative difference, leading to the largest amount of flows on the upper route.

Table 4. Equilibrium route flows of three models under three cases.

Network case	Route flow and cost		Logit-SUE	Weibit-SUE	Hybrid-SUE
I/III	Flow	Upper	41.72	35.25	33.59
		Lower	58.28	64.75	66.41
	Absolute difference		**3.34**	2.05	**1.72**
	Relative difference		*1.31*	**1.18**	**1.15**
II	Flow	Upper	41.72	46.84	40.27
		Lower	58.28	53.16	59.73
	Absolute difference		**3.34**	4.37	**3.05**
	Relative difference		*1.03*	**1.03**	**1.02**
IV	Flow	Upper	1.74	11.84	0.27
		Lower	98.26	88.16	99.73
	Absolute difference		40.35	42.37	**40.05**
	Relative difference		*1.67*	**1.72**	**1.67**

Secondly, we compare the three SUE models under each network case (i.e. horizontal comparison). From Table 4, the hybrid SUE model has the lowest absolute cost difference and relative cost difference compared to the Logit-based and Weibit-based SUE models. Accordingly, it assigns the smallest amount of flows on the upper route and the largest amount of flows on the lower route. It seems that the hybrid model makes the preferential route more stand out.

Meanwhile, Table 5 provides another way to validate the traffic assignment results. According to the first-order optimality conditions presented in Section 3, each SUE model has a generalized route cost: $\alpha_1 c+\ln(f)$ for the Logit case, $\alpha_2\ln(g)+\ln(f)$ for the Weibit case, and $\alpha_1 c+\alpha_2\ln(g)+\ln(f)$ for the hybrid case. At the respective SUE state, all routes have equal generalized cost, which is the Lagrangian multiplier of the travel demand conservation constraint. For example, in the hybrid SUE model and network case I, the upper route has the actual cost of 10+33.59/10=13.36 and the generalized cost of 0.1×13.36+3.7×ln(13.36)+ln(33.59)=14.44. Similarly, the lower route has the actual cost of 5+66.41/10=11.64 and the generalized cost of 0.1×11.64+3.7×ln(11.64)+ln(66.41)=14.44.

Table 5. Generalized route costs of three models under three cases.

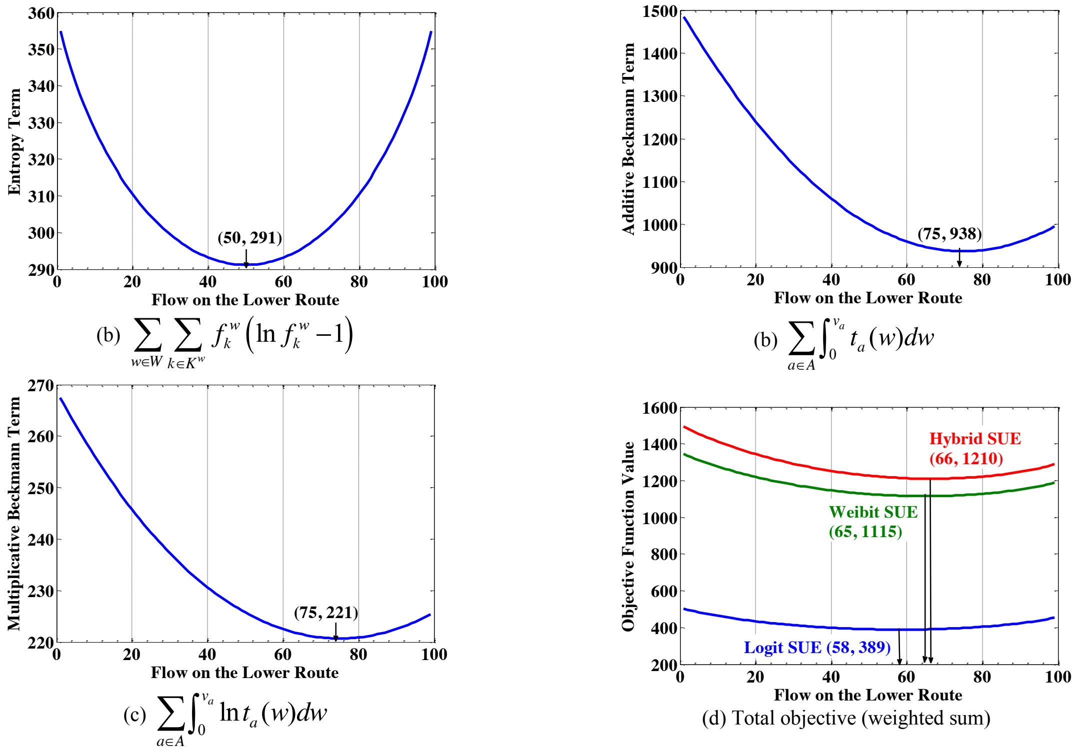
Network case	Generalized route cost	Logit-SUE $\alpha_1 c+\ln(f)$	Weibit-SUE $\alpha_2\ln(g)+\ln(f)$	Hybrid-SUE $\alpha_1 c+\alpha_2\ln(g)+\ln(f)$
I/III	Upper	5.15	13.20	14.44
	Lower	5.15	13.20	14.44
II	Upper	16.65	21.85	34.58
	Lower	16.65	21.85	34.58
IV	Upper	10.57	19.55	25.75
	Lower	10.57	19.55	25.75

As mentioned in Section 3.2, the hybrid SUE formulation simultaneously optimizes three objective functions: minimization of the additive Beckmann transformation term Z_1, minimization of the multiplicative Beckmann transformation term Z_2, and maximization of the route flow entropy term Z_3. Z_1 and Z_2 are respectively associated with the consideration of absolute and relative cost differences in travelers' route choice decisions, while adjusting the route flow entropy Z_3. Table 6 presents the objective values and their components of all three models and all three cases. All three models have a higher total objective value as the overall trip length increases from Case I/III to Case IV and then to Case II, due to the substantial increase of Z_1 and/or Z_2. Also, the hybrid SUE model produces the largest entropy value Z_3 for each network case, due to the two adjustments of Z_1 and Z_2. Accordingly, the hybrid SUE model has a slightly lower value of total cost term (Z_1 and/or Z_2). In fact, this is consistent with the entropy optimization nature. More constraints correspond to a higher objective value with the minimized Z_3. With the above change in Z_1, Z_2, and Z_3, the corresponding ratios between total cost and flow entropy are slightly lower than those of the Logit and Weibit SUE models.

Table 6. Objective function values of three models under three cases.

Network case	Objective	Logit-SUE	Weibit-SUE	Hybrid-SUE
I/III	Z_1	96.54	/	94.49
	Z_2	/	819.71	818.78
	Z_3	**292.58**	**295.62**	**296.69**
	Total	389.13	1115.33	1209.96
	Ratio	**0.33:1**	**2.77:1**	**0.32:2.76:1**
II	Z_1	1246.55	/	1246.08
	Z_2	/	1785.95	1785.24
	Z_3	**292.58**	**291.40**	**293.11**
	Total	1539.13	2077.35	3324.43
	Ratio	**4.26:1**	**6.13:1**	**4.25:6.09:1**
IV	Z_1	556.98	/	551.10
	Z_2	/	1505.28	1482.72
	Z_3	**351.75**	**324.14**	**358.63**
	Total	908.73	1829.42	2392.44
	Ratio	**1.58:1**	**4.64:1**	**1.54:4.13:1**
Note: $Z_1=\alpha_1\sum_{a\in A}\int_0^{v_a} t_a(w)dw$; $Z_2=\alpha_2\sum_{a\in A}\int_0^{v_a}\ln t_a(w)dw$; $Z_3=\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w-1\right)$				

Without loss of generality, we use Case I to further demonstrate the interaction or weighting of different objectives. Fig. 4 shows the entropy term, additive Beckmann term, multiplicative Beckmann term, and the total objective functions of the Logit, Weibit, and Hybrid SUE models. As expected, the single entropy optimization leads to the identical flow allocation on the two routes. The single minimization of the additive Beckmann term assigns 75 units of flows on the lower route (i.e., $10+(100-f_2)/10=5+f_2/10$). For this particular example, the single minimization of the additive Beckmann term has the same solution as that of the multiplicative Beckmann term. The reason is that each route only consists of a single link, and the monotonicity of logarithmic function makes $\ln(10+(100-f_2)/10)=\ln(5+f_2/10)$ equivalent to $10+(100-f_2)/10=5+f_2/10$. However, they have quite different objective values. The objective value of the hybrid model is not a summation of the Logit and Weibit models, since all three models treat the entropy term as the reference objective (i.e., with the weight of 1.0). From Fig. 4(d), we can see that different weighting mechanisms (i.e., additive Beckmann—entropy, multiplicative Beckmann—entropy, and additive Beckmann—multiplicative Beckmann—entropy) lead to different equilibrium route flow allocations between 50 and 75.



(b) $\sum_{w\in W}\sum_{k\in K^w} f_k^w\left(\ln f_k^w - 1\right)$ (b) $\sum_{a\in A}\int_0^{v_a} t_a(w)dw$

(c) $\sum_{a\in A}\int_0^{v_a} \ln t_a(w)dw$ (d) Total objective (weighted sum)

Fig. 4. The optimal flow pattern under various objective functions (Case I).

4.3. Example 3: Route overlapping consideration

In this section, we use the modified loop-hole network shown in Fig. 5 to simultaneously consider route overlapping, absolute cost difference and relative cost difference problems under congestion. This network has three routes. The upper two routes have the same length of y and they have an overlapping section with a length of x. The lower route is truly independent of the other two routes and its length is z. The O-D demand is 100 units and all four links have the same capacity of 100 units. l_a and L_k^w used in the PS factor are set to the link free-flow travel cost and route free-flow travel cost, respectively. For this particular example, the PS factors of the three routes are $\rho_1 = \rho_2 = x/(2y) + (y-x)/y$ and $\rho_3 = 1$. We use the standard Bureau of Public Road (BPR) function to calculate the flow-dependent link travel time. Without loss of generality, we adopt the following exponential function as the link travel cost (or disutility) function (Hensher and Truong, 1985; Mirchandani and Soroush, 1987):

$$\tau_a = \exp(0.075 t_a), \ \forall a \in A. \tag{45}$$

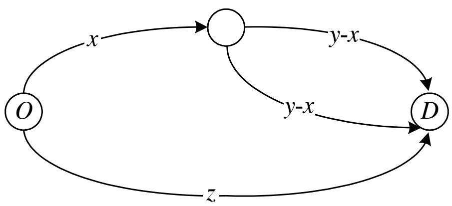


Fig. 5. Modified loop-hole network.

To examine the effect of route overlapping, the length of the overlapped section x is varied from 0 to y with an interval of $0.1y$. Two scenarios shown in Table 7 are considered with different combinations of absolute cost difference and relative cost difference. These two scenarios have the same absolute cost difference of 5 but different relative cost differences: scenario 1 corresponds to a short network with a large relative difference while scenario 2 corresponds to a long network with a small relative difference. We compare the hybrid SUE pattern without route overlapping consideration (denoted as Hybrid SUE) and the hybrid SUE pattern with the PS factor (denoted as PS-Hybrid SUE) under the two scenarios with various values of x.

Table 7. Two scenarios of example 3.

Scenario	y	z	Absolute difference (y-z)	Relative difference (y/z)
1	15	10	5	1.5
2	55	50	5	1.1

Fig. 6 and Fig. 7 show the equilibrium flow on the lower route and the corresponding absolute and relative route cost differences under scenario 1 and scenario 2, respectively. Note that the symmetric characteristics of the upper two routes make it straightforward to infer their flows from the flow on the lower route. One can see that the flow on the lower route has been substantially increased with the increase of the overlapping section x. When x is equal to zero, the three routes are completely independent. In that special situation, the PS-Hybrid SUE collapses to the Hybrid SUE solution. This has been verified by the intersection point of the solid curve and dashed curve in the two figures. When x is equal to y, there only exist two routes and more travelers will use the lower route. The above change shows the capability of the PS-Hybrid SUE model in capturing the route overlapping problem. Specifically, the increase of x decreases the PS factor of the upper two routes (i.e., $\rho_1 = \rho_2 = (0.5x + y - x)/y = (y - 0.5x)/y$). As a result, the PS-Hybrid SUE model assigns a smaller amount of flow to routes that have couplings with other routes, hence a larger amount of flow on the lower route. On the contrary, the Hybrid SUE model does not handle the route overlapping problem while considering each route as an independent alternative. We should point out that the Hybrid SUE model is not totally irrelevant of x. Hence, the flow on the lower route has a slight increase with the increase of x due to the congestion-dependent absolute and relative cost differences. Compared to the PS-Hybrid SUE model, the Hybrid SUE model assigns more flows on the two upper routes as x increases (hence a smaller amount of flows on the lower route).

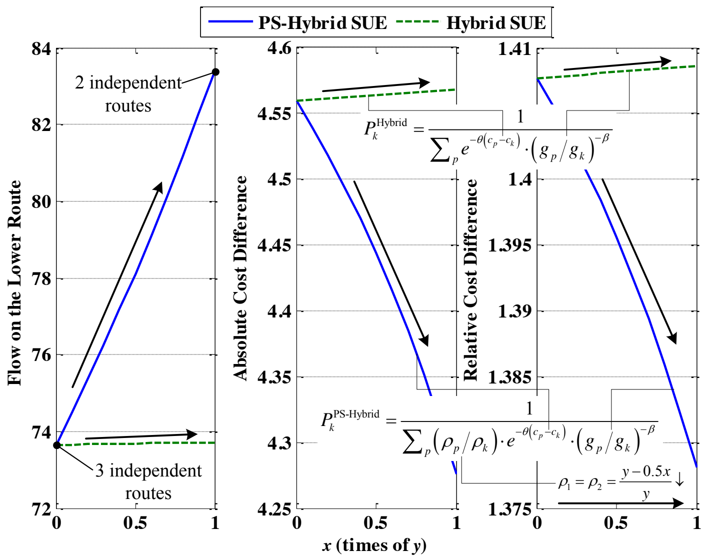


Fig. 6. Effect of route overlap on flow allocation and cost differences (Scenario 1).

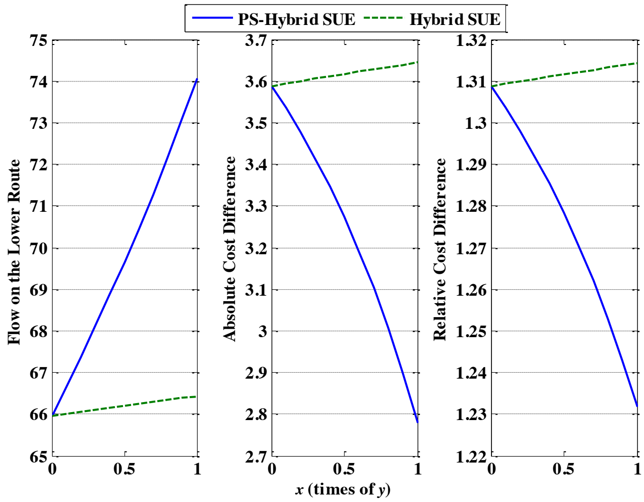


Fig. 7. Effect of route overlap on flow allocation and cost differences (Scenario 2).

On the other hand, we can see that the congestion-dependent absolute and relative cost differences are significantly different from the topological absolute and relative differences shown in Table 7. As x increases, the PS-Hybrid SUE model substantially decreases the absolute and relative cost differences while the Hybrid SUE model slightly increases the absolute and relative differences. Between the two scenarios, the effect of route overlapping on the congestion-dependent absolute and relative cost differences seems to be more significant in the longer network. In the PS-Hybrid SUE model, the decrease of the absolute and relative cost differences (i.e., the increase of the exponential term and power term) seems to counteract the decrease of PS factor. Accordingly, the flow on the lower route is increased as x increases (PS factor decreases) in the PS-Hybrid SUE model. In contrast, the slight increase of the absolute and relative cost differences leads to a slight increase of the lower route flow in the Hybrid SUE model.

5. Concluding remarks

This paper developed a hybrid closed-form route choice model and the corresponding SUE model to alleviate the insensitivity to a shift in the Logit model and the insensitivity to an arbitrary scale in the Weibit model simultaneously. Some theoretical properties of the hybrid model were examined, including the probability relationship among the three models, IIA property, and direct and indirect elasticities. To further consider the congestion effect, a unified modeling framework was provided to formulate the Logit, Weibit and hybrid SUE models with the same entropy maximization objective but with different total cost constraint specifications representing the modelers' knowledge of the system. The derivation of the optimality conditions validated the feasibility of interpreting the SUE problem as an entropy optimization problem, besides the original random utility maximization perspective. To be structurally consistent with the existing Logit and Weibit SUE MP formulations, we reformulated the hybrid SUE model as a weighted multi-objective optimization problem by treating the entropy term as the reference objective. The hybrid SUE formulation was further extended to handle the route overlapping problem by using a path-size factor. Theoretical and numerical analyses demonstrated that the proposed hybrid route choice model was capable of capturing the travelers' concerns on not only absolute difference but also relative difference of travel costs in their route choice decisions. As such, it can alleviate the drawbacks of the Logit and Weibit models simultaneously. In addition, the path-size (PS) factor was incorporated into the hybrid SUE model to alleviate the independence assumption by adjusting the route choice probabilities and flow allocations to routes with overlapping.

Further research should examine the parameter relationship among the Logit, Weibit, and hybrid route choice models through realistic data calibrations. Sensitivity and uncertainty analyses (Yang and Chen, 2009; Yang *et al.*, 2013) could be used to quantify the effect of parameters on the flow allocation. Also, large-scale realistic network applications should be conducted to investigate the distribution of absolute and relative route cost differences among different O-D pairs as well as the flow difference assigned by the Logit, Weibit, and hybrid SUE models. Note that this paper adopted the conventional Stirling's approximation (i.e., $\ln x! \approx x(\ln x-1)$) in the entropy optimization objective. More advanced approximation schemes, e.g., the second-order Stirling approximation and Burnside's formula proposed by De Grange *et al*. (2014) could be considered for improving prediction of small flow values. Moreover, it is of interest to explore different strategies of hybridizing the Logit and Weibit models: our hybrid model, a linear combination of Logit and Weibit probabilities (Yao and Chen, 2014), the Box-Cox model by using the Box-Cox transformation on the utility function of Logit model (Ortuzar and Willumsen, 2011), and the q-generalized Logit model by using a q-exponential function (Nakayama, 2013), where the latter two include Logit and Webit models as special cases. Interested aspects include the hybridization/generalization manner, existence of an equivalent MP formulation for the SUE problem, and their probability and parameter relationships.

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