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Priestley's Duality from Stone's

Isidore Fleisher

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Priestley's duality for bounded distributive lattices has enjoyed growing attention and has been variously applied in the international literature since its inception in 1970. Whereas its introduction in [10] acknowledged the priority of Stone's 1937 treatment, claiming for itself only a "simpler" "more natural" "reformulation," the following fuller treatment [11] limited Stone to a non-referenced bibliographic item, as does the recently published book [4] which devotes several chapters to a detailed development of Priestley's duality.

In point of fact, the Stone and Priestley duals of a distributive lattice are the *same* space whose topology is described in formally different, but easily seen to be equivalent, ways: Any T_0 space with a distinguished base can be canonically re-equipped with a totally order disconnected topology simply by making the base clopen and retaining the T_0 partial order; in the other direction, the T_0 base can be recovered as the lattice of clopen increasing subsets of the compact dual. This coincidence of the duals is worked out by Cornish [3] (who also derives Priestley's duality from Stone's) in his preliminary lemmas, but who only draws the weaker conclusion that the two dual categories are isomorphic (which suffices for his purpose). This, and the inaccessibility of [3], may be responsible for these facts' lack of recognition.

The text which follows includes principally an exposition of the Stone duality theory and comprises the content of a series of three lectures in the algebra seminar at the University of Tennessee in Knoxville, given during the winter of 1992. I'm very grateful to the participants, especially to the two Davids and Michael S. Gilbert, for being such an alert and attentive audience.

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SOME DEFINITIONS AND THE SET REPRESENTATION

A *lattice* $\mathcal{L}$ is a poset with extrema $p \vee q$, $p \wedge q$ for pairs. It is *distributive* if the identity $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ holds. Any collection of subsets closed under finite union and intersection is an example; it will transpire that this is the most general example.

A subset I of $\mathcal{L}$, containing with p and q the join $p \vee q$ and every $r \leq p$, is an *ideal*. The principal ideal generated by p is thus $(p] = \{x : x \leq p\}$; the ideal generated by a pair of ideals I, J is $\{x \leq p \vee q,\ p \in I,\ q \in J\}$. Ideal P is *prime* if $p \wedge q \in P$ only if p or $q \in P$, equivalently, if $I \cap J \subset P$ for (possibly non-principal) ideals I and J, only if I or $J \subset P$.

Dually, a subset F of $\mathcal{L}$ containing with p and q the meet $p \wedge q$ and every $r \geq p$ is a *filter*; the principal filter generated by p is $[p) = \{x : x \geq p\}$, and filter F is *prime* if $p \vee q \in F$ only for p or $q \in F$. An ideal is prime if and only if its complement is a filter; in a decomposition of $\mathcal{L}$ into a complementary ideal and filter, both components are prime. These decompositions[1] correspond to lattice morphisms of $\mathcal{L}$ to the two-element chain $\{0, 1\}$ and are used in representing $\mathcal{L}$ as a lattice of subsets: indeed the points of a representing space effect such lattice morphisms of $\mathcal{L}$. Conversely, let $\mathcal{P}$ be the set of prime filters of $\mathcal{L}$ (equivalently, prime ideals or $\{0, 1\}$-valued lattice morphisms) and send each element of $\mathcal{L}$ to the subset of prime filters which contain it: this subset's characteristic function may be obtained by assigning every prime filter the image of the element by the lattice morphism having that prime filter as inverse image of one. This assignment is a lattice morphism of $\mathcal{L}$ to a sublattice of subsets of $\mathcal{P}$. That it is an isomorphism will follow from the

KEY LEMMA (Stone). *A filter in a distributive lattice, maximal for being disjoint from an ideal, is prime.*

Proof. If x does not belong to this filter F, there must exist by maximality a $q \geq x \wedge p$ in the ideal I for some $p \in F$; if also $x' \notin F$ there is similarly a $q' \geq x' \wedge p'$. Since I is up-directed and F down-directed, one can arrange to have $q' = q$ in I and $p' = p$ in F. Then $q \geq (x \wedge p) \vee (x' \wedge p) = (x \vee x') \wedge p$; since q does not belong to F, neither does $x \vee x'$.

In particular, if $p \neq q$, say $p \nleq q$, i.e., the principal filter $[p)$ is disjoint from the principal ideal $(q]$, expand $[p)$ to a filter maximal for being disjoint from $(q]$—this will be a prime filter in the subset assigned to p and outside that assigned to q.

[1] Possibly improper, i.e., one component could be void.

TOPOLOGIZING THE REPRESENTATION SPACE $\mathscr{P}$

It is possible to do this so as to characterize the image of the lattice topologically. In essence, one imposes on $\mathscr{P}$ the coarsest topology making continuous the lattice morphisms of $\mathscr{P}$ to $\{0, 1\}$, the latter topologized to have $\{0\}$ as the only proper open subset. However, the desired properties are more readily derived by passing instead through the Galois correspondence between $\mathscr{L}$ and $\mathscr{P}$ based on the relation $p \in P$ between element $p \in \mathscr{L}$ and prime filter $P \in \mathscr{P}$. Recall that this correspondence consists of maps which assign to every $M \subset \mathscr{L}$ the set of P which are related to every $p \in M$—thus the set of prime filters $P \supset M$—and to every $S \subset \mathscr{P}$ the set of p related to every $P \in S$—thus $\cap\{P : P \in S\}$. The latter set is a filter in $\mathscr{L}$ and every filter is obtained in this way: for by the Key Lemma if $p \notin F$ then $p \notin$ some prime filter $\supset F$. Thus the Galois closure on $\mathscr{L}$ assigns every subset the filter it generates; on $\mathscr{P}$ it assigns every subset S the set $\{P : P \supset \cap S\}$. In general one obtains only an algebraic closure from a Galois connection—i.e., increasing, isotone and idempotent—but here $P \in \overline{S \cup S'}$, i.e., $P \supset \cap(S \cup S') = (\cap S) \cap (\cap S')$ entails $P \supset \cap S$ or $P \supset \cap S'$ thus $P \in \overline{S} \cup \overline{S}'$ and one has a topological closure. This is the so-called "hull kernel" topology (the closure being described is "kernel-hull"). Observe that the set images of the singletons $\{p\}$—i.e., the Galois correspondents of the principal filters $[p)$—constitute a base for the closed sets; since these distinguish the points P, the topology is T_0.

The promised topological characterization of this base is that it consists of those closed sets whose complements are compact. An element c in a lattice is said to be compact if, whenever some infinite sup dominates it, $\vee d_\alpha \geq c$, then already some finite subsup does, $d_{\alpha_1} \vee \ldots \vee d_{\alpha_n} \geq c$. In the lattice of open subsets the compact elements are just the compact open subsets—"copens" in the sequel! Since both complementation and the Galois connection between the sublattices of Galois closed subsets are anti-isomorphisms, it suffices (for the characterization) to see that the representation by sets preserves no proper infinite sup (for in a lattice of subsets, precisely the proper infinite sups are not compact: hence if this isomorphism preserves no such sup, the images will all be compact). This follows from the Key Lemma, since the elements dominated by the finite subsups constitute an ideal and if the sup p is not among these there will be a prime filter P at which p takes the value one and each of these is zero. An open set not among the complements of the base, being a proper infinite updirected union of basic opens, cannot be compact.

One further topological property of $\mathscr{P}$ should be noted: When a down-directed family of (basic) copens traces on a Galois closed set C, then their intersection meets it. To see that this holds, observe that down-directed copens have updirected complements, which represent updirected p's in $\mathscr{L}$; and that a copen traces on C, just when its complement p fails to contain C, say the set of prime filters containing filter F, i.e., just when there is a prime filter $P \supset F$ with $P \not\supset [p)$; thus $p \notin P$ hence $p \notin F$. By the Key Lemma there is a prime filter P containing F and excluding all the updirected p's: this P is a point in the closed set and in none of the subsets representing the p's, hence in all their copen complements.

This topological condition, as well as compactness of complements of basic closed sets, is implied by: Every filter base of basic closed sets and their complements (which includes some of each) has non-void intersection. The latter is however equivalent to their conjunction since there is either a smallest copen in the filter base or the copens trace on the intersection of the basic closed sets in the filter base.

TOPOLOGICAL CHARACTERIZATION OF STONE SPACES

Let X be a set equipped with a lattice of subsets $\mathscr{L}$. There is a coarsest topology in which $\mathscr{L}$ is a base of closed sets: its closed sets are arbitrary intersections from $\mathscr{L}$ augmented with X and ϕ. Complements of $\mathscr{L}$ will be compact just when none is a proper infinite union of others. The topology need not be T_0 but by identifying pairs split by no set in $\mathscr{L}$ one gets the largest T_0 quotient; $\mathscr{L}$ is naturally a lattice of subsets on this. The quotient's points then correspond one-one to (some of the) prime filters of $\mathscr{L}$; to get them all, impose the topological condition at the end of the last section. Then every subset in $\mathscr{L}$ either belongs to a given one of its prime filters P, or if not, contains no subset in P hence has a (copen) complement which meets every subset in P. The complements meeting P are intersection-closed (since their complements are union-closed by primeness of P) so there is a point common to the intersection of the subsets in P and the complements of those not in P; i.e., P consists exactly of the subsets in $\mathscr{L}$ which contain this point.

STRENGTHENING T_0 TO TOTAL (ORDER-)DISCONNECTEDNESS

This may be done by augmenting any base (henceforth including X) with its complements, e.g., by making the copens clopen (this is Nerode's "strong topology" and Hochster's "patch topology" also known as "constructible"). There results a (separated) totally disconnected space—i.e., with a Boolean base of clopens. The above topological condition in the T_0-space, i.e., that every filter of basic or complementary basic sets has non-void intersection, translates as compactness of the totally disconnected strengthening. However, the original T_0 base is not recoverable from this clopen base.

Every T_0-space is partially ordered by $P \leq Q$ just when $Q \in \{\overline{P}\}$ (T_0 = anti-symmetry; for the hull-kernel topology this gives the inclusion order among the prime filters). Then every closed is an "upset" (i.e., closed under passage to larger elements) and dually every open a "down-set." This order is no longer recoverable from the totally disconnected strengthening either; however, retaining it will enable one to recover the T_0 closed base—i.e., the original lattice—as the upsets in the clopen base. Thus one is led to a totally disconnected compact space with (Nachbin's notion of) a "closed order," as a separated topological dual for a distributive lattice: A "totally order-disconnected space" is defined to be a topological space equipped with an order such that for every $P \not\geq Q$ there is a clopen upset excluding P and including Q. Every strengthened T_0 space is such with the separating clopen from the T_0 closed base.

PRIESTLEY'S DUALITY

Priestley uses the same space as Stone, with the T_0 topology of $\mathscr{P}$ strengthened to total disconnectedness by making the lattice clopen and retaining the partial order as a supplementary datum. As noted above, Stone's topological condition is just that this strengthened topology be compact. Then every open downset is T_0 open, since every P in it is separable from the Q in its compact complement (thus $\not\leq P$) by T_0 copens; in particular, every clopen downset is T_0 copen. Conversely, the clopen upsets of any totally order disconnected space constitute a closed base for a coarser T_0 topology (on the same set); if the former is compact then the latter is a Stone dual topology; and the latter's order disconnected strengthening, being separated, must coincide with the initial finer compact topology.

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