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Sum graphs over all the integers

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Abstract

We introduced the sum graph of a set S of positive integers as the graph $G^+(S)$ having S as its node set, with two nodes adjacent whenever their sum is in S. Now we study sum graphs over all the integers so that S may contain positive or negative integers on zero. A graph so obtained is called an integral sum graph. The sum number of a given graph G was defined as the smallest number of isolated nodes which when added to G result in a sum graph. The integral sum number of G is analogous. We see that all paths and all matchings are integral sum graphs. We find the integral sum number of the small graphs and offer several intriguing unsolved problems.

1. Sum graphs

We follow in general the graph-theoretic notation and terminology of [12].

The sum graph $G^+(S)$ of a finite subset $S \subset N = \{1, 2, 3, \ldots\}$ is the graph (V, E) where $V = S$ and $uv \in E$ if and only if $u + v \in S$. Then *a sum graph* G is isomorphic to the sum graph of some $S \subset N$. This concept was discovered in [13], where some basic properties of the family $\mathcal{G}^+$ of all sum graphs were presented. We assume throughout that $|S| \geqslant 3$ and briefly summarize the results on sum graphs:

(1) Obviously the largest member of S is an isolated node of its sum graph $G^+(S)$.

(2) Given any graph G, say with $n \geqslant 3$ nodes v_i and m edges, it is trivial that the union $G \cup mK_1$ of G with m isolated nodes is a sum graph. This follows at once by labeling each v_i by 10^i and the m isolated nodes by $10^i + 10^j$ whenever $v_i v_j \in E$.

(3) It follows that for each G, there is a minimum number $\sigma = \sigma(G)$ such that $G \cup \sigma K_1 \in \mathcal{G}^+$. This number $\sigma(G)$ is the *sum number* of G.

(4) The sum number of cycles is given by

$$\sigma(C_n) = \begin{cases} 3 & \text{when } n = 4, \\ 2 & \text{when } n \neq 4. \end{cases} \tag{1}$$

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(5) We confidently announced as a true conjecture that for all nontrivial trees T,

$$\sigma(T)=1. \tag{2}$$

This was recently proved by Ellingham [6].

Subsequently, the sum numbers of complete graphs were derived by Bergstrand et al. [3], who found that for $n \geqslant 4$,

$$\sigma(K_n)=2n-3. \tag{3}$$

Hartsfield and Smyth [16] obtained the corresponding result for complete bipartite graphs with $r \leqslant s$,

$$\sigma(K_{r,s})=\lceil(3r+s-3)/2\rceil. \tag{4}$$

Bergstrand et al. [3] prove that the family $\mathscr{G}^{\times}$ of all product graphs over $\{2,3,4,\ldots\}$ are just the sum graphs:

$$\mathscr{G}^{\times}=\mathscr{G}^{+}. \tag{5}$$

Harary et al. [14] verified that the family $\mathscr{G}^{+}(R)$ of sum graphs over the positive reals is identical with $\mathscr{G}^{+}$. The proof is analogous to that of Bloom and Burr [4] for difference graphs.

Fricke and Harary [7] studied the *sum cost* of $G \in \mathscr{G}^{+}$, defined as the minimum sum $\sum_{u \in S} u$ taken over all sets $S \subset N$ such that $G \cong G^{+}(S)$.

Other investigations of sum graphs have been made by Grimaldi [9], who studied sum graphs of rings, and Boland et al. [5], who noted that complete graphs are not 'mod sum graphs', i.e. cannot be realized as the sum graph of some $S \subset Z_m=\{0,1,2,\ldots,m-1\}$ with addition modulo m.

Any references to sum graphs not listed above will be gratefully received by the author.

Now our purpose is to introduce and begin to investigate the family $\mathscr{G}^{+}(Z)$ of sum graphs over the set of all integers $Z=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. We shall define the integral sum number $\sigma_Z(G)$, obtain these numbers for the graphs with $n=4$ nodes, demonstrate that the paths P_n and the matchings mK_2 are integral sum graphs, and propose several intriguing open questions.

2. A useful family of sum graphs

In the next section we shall define integral sum graphs and the integral sum number of a graph. In order to specify the structure of an interesting family of integral sum graphs, it is convenient to introduce here a certain family of sum graphs.

Writing $N_n=\{1,2,\ldots,n\}$, this family of sum graphs is defined by

$$G_n=G^{+}(N_n).$$

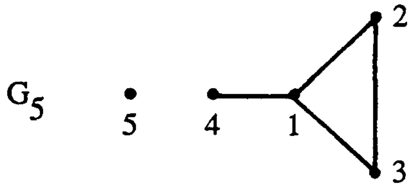

Fig. 1. The fifth graph G_5.

The fifth graph in this family is shown in Fig. 1; the first four are

$$G_1 = K_1, \qquad G_2 = 2K_1, \qquad G_3 = K_1 \cup K_2, \qquad G_4 = K_1 \cup P_3.$$

We have tried without success to find a simple elegant description of the structure of graph G_n.

3. Paths and matchings are integral sum graphs

Following customary notation, write the set of all integers as

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The *integral sum graph* $G^+(S)$ is defined just as the sum graph, the difference being that $S \subset Z$ instead of $S \subset N$. We illustrate with the family of integral sum graphs

$$G_{n,n} = G^+\{-n, \ldots, -2, -1, 0, 1, 2, \ldots, n\}. \tag{6}$$

The structure of these graphs is easy to specify in terms of the graphs G_n. We first illustrate in Fig. 2 the graph $G_{n,n}$ for $n = 3$.

In analogy with this example, we can easily demonstrate that

$$G_{n,n} = K_1 + (G_n + G_n). \tag{7}$$

In this equation, the K_1-term is realized by the integer 0, which obviously is adjacent to all other elements of S. The two G_n-terms are $G^+\{1, 2, 3, \ldots, n\}$ and $G^+\{-1, -2, -3, \ldots, -n\}$ where parentheses are omitted for simplicity.

We next show that all paths are integral sum graphs. Figure 3 shows that the paths P_n are integral sum graphs for $n = 1, 2, 3, 4$ and 8.

Clearly this integral sum graph numbering of the path P_8 can be continued indefinitely using the sequence

$$(a_1, a_2, a_3, \ldots) = (-1, 3, -4, 7, -11, 18, -29, 47, \ldots) \tag{8}$$

satisfying

$$a_n = a_{n-2} - a_{n-1}, \tag{9}$$

with $a_1 = -1$ and $a_2 = 3$.

Sequence (8) proves the next result.

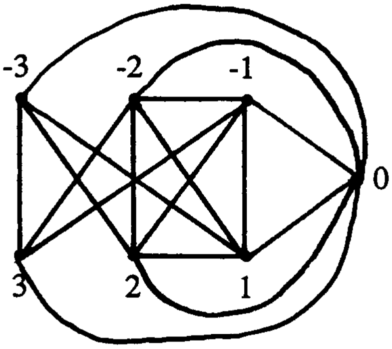

Fig. 2. The integral sum graph $G_{3,3}$.

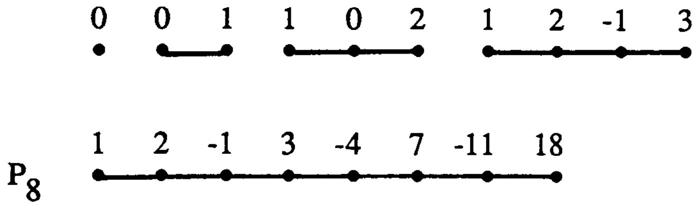

Fig. 3. Five paths.

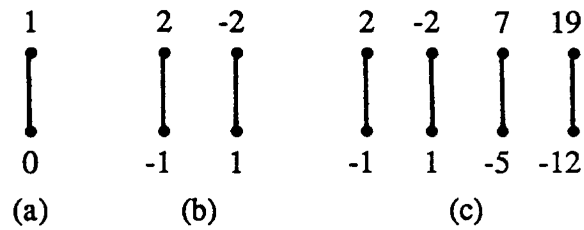

Fig. 4. Three matchings are integral sum graphs.

Theorem 3.1. *For all positive integers n, the path P_n is an integral sum graph.*

We now turn to proving that every matching mK_2 is an integral sum graph. Figure 4 establishes this for $m = 1, 2$ and 4.

Analogous to the construction of sequence (8), it is easy to build a sequence of unordered pairs of nonzero integers such as

$$(\{-1,2\}, \{1,-2\}, \{-5,7\}, \{-12,19\}, \{-40,59\}, \ldots,). \tag{10}$$

Obviously there is so much 'slack' in the choice of pairs of integers that the condition $u+v=w$ for any three integers, u, v, w and u and v in different unordered pairs appearing in (10) can easily be avoided. One can assure that no unwanted edges occur by taking the integers for the next copy of K_2 large enough in absolute value so that it never happens that the sum of one of the two 'new' integers with any one of the 'old' integers is one of the node integers. Thus we have shown how to establish the next result.

Theorem 3.2. *For all positive integers m, the matching mK_2 is an integral sum graph.*

4. The integral sum number and small graphs

The sum number $\sigma(G)$ has been defined as the smallest s such that $G \cup sK_1$ is a sum graph. Analogously, the *integral sum number* $\zeta(G)$ is the smallest nonnegative s such that $G \cup sK_1$ is isomorphic to $G^+(S)$ for some $S \subset Z$, i.e. is an integral sum graph written $\int\sum$-graph. Obviously $\zeta(G) \leqslant \sigma(G)$ for all graphs G.

We have seen some families of $\int\sum$-graphs G, with $\zeta(G)=0$. These include paths, matchings, the graphs G_n and $G_{n,n}$, and also the stars since

$$K_{1,n} \cong G^+\{0, 1, 3, 5, \ldots, 2n-1\}. \tag{11}$$

We already observed that for $n \leqslant 3$ nodes, every graph G satisfies $\zeta(G)=0$. We now display in Fig. 5 the $\int\sum$-graphs with $n=4$.

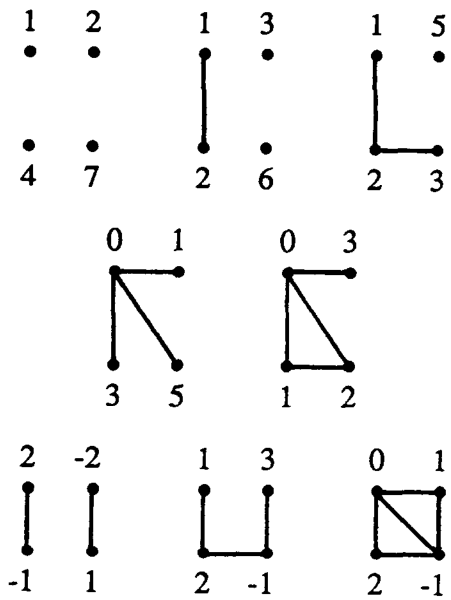

Fig. 5. All eight $\int\sum$-graphs of order 4.

The first row of graphs of Fig. 5 shows all three sum graphs with $n=4$. The second row lists two graphs which can be realized as sum graphs of some $S \subset N \cup \{0\}$. The third and last row presents the three $\int\sum$-graphs which require S to contain both positive and negative integers. Finally there are three graphs with $n=4$ that are not $\int\sum$-graphs. Perhaps surprisingly, it turns out that their ζ-numbers equal their sum numbers:

$$\zeta(K_3 \cup K_1)=1, \qquad \zeta(C_4)=3, \qquad \zeta(K_4)=5. \tag{12}$$

For completeness, we list sets $S_1, S_2, S_3 \subset N$ which realize (12):

$$S_1=\{2,3,5,7,8\}, \qquad S_2=\{1,5,6,9,13,14,22\},$$

$$S_3=\{1,5,6,9,10,13,14,18,22\}. \tag{13}$$

5. Unsolved problems

Some trees T are $\int\sum$-graphs. This holds for all trees with $n \leqslant 5$ nodes, but not when $n \geqslant 6$. A *double star* (introduced in [10]) consists of a single edge K_2 together with a positive number of leaves (end-edges) at each node of K_2. Figure 6 shows the double stars $S(1,3)$ and $S(2,2)$. It is straightforward to verify that neither is an $\int\sum$-graph.

Obviously every tree T has $\zeta(T)=0$ or 1 by Ellingham's result (2) showing $\sigma(T)=1$.

Unsolved Problem I. Characterize the trees T satisfying $\zeta(T)=0$. As a special case, what is the criterion when T is a caterpillar?

Recall that given T, its *pruned tree* T' is obtained by removing all the end-nodes of T. A tree T is a *caterpillar* (introduced by Harary and Schwenk [15]) if T' is a path; this path is called the *spine* of T. Given two distinct n-sequences of nonnegative integers $\alpha=(a_1, \ldots, a_n)$ and $\beta=(b_1, \ldots, b_n)$, we say that α is *smaller* than β if $a_1<b_1$ or for some k, $1 \leqslant k \leqslant n$, $a_i=b_i$ for $i=1$ to $k-1$ and $a_k<b_k$. The *code of a caterpillar* T, or briefly its *cat-code* (introduced in [11]), is the smaller of the two sequences of consecutive end-degrees of the nodes of the spine of T'. We have observed that the two families of caterpillars with codes,

$$(1,1,1,\ldots,1) \quad \text{and} \quad (1,2,1,\ldots,1),$$

are $\int\sum$-graphs. The first of these families are just the coronas $P_n \circ K_1$, a binary operation on graphs studied by Frucht and Harary [8].

Fig. 6. Two double stars.

Conjecture 5.1. Every tree T with $\zeta(T)=0$ is a caterpillar.

Unsolved Problem II. Characterize the graphs G which satisfy

$$\zeta(G)=\sigma(G). \tag{14}$$

We saw that included among these graphs are $K_3 \cup K_1$, the cycle C_4 and the complete graph K_4. We conjecture that this holds for all K_n with $n \geqslant 4$.

However, the pentagon C_5 is an $\int \sum$-graph as it satisfies

$$C_5 \cong G^+\{-2,-1,1,2,3\}. \tag{15}$$

Thus far it appears that C_n satisfies (14) if and only if $n \neq 3, 5$.

It remains to investigate and determine $\zeta(G)$ for other families of graphs including

(a) the graphs $K_n - e$ and, more generally, $K_n - E(K_r)$ for $K_r \subset K_n$,
(b) the so-called 'cocktail party graph', $K_n - E(nK_2)$,
(c) the complete bipartite graphs $K_{r,s}$ and, in particular, $K_{r,r}$,
(d) the dancing version of the cocktail party graph, $K_{r,r} - E(rK_2)$.

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