JOURNAL OF COMBINATORIAL THEORY, Series B **46**, 240–245 (1989)

On the Δ-Subgraph of Graphs Which Are Critical with Respect to the Chromatic Index

A. G. CHETWYND

Department of Mathematics, University of Lancaster, Cartmel College, Bailrigg, Lancaster LA1 4YL, United Kingdom

A. J. W. HILTON

Department of Mathematic, University of Reading, P.O. Box 220, Whiteknights, Reading RG6 2AX, United Kingdom

AND

D. G. HOFFMAN

Department of Algebra, Combinatories and Analysis, Auburn University, Auburn, Alabama 36849

Communicated by Adrian Bondy

Received March 4, 1986

The Δ-subgraph of a simple graph G is the subgraph of G induced by the vertices of maximum degree $\Delta = \Delta(G)$. We show that a non-empty simple graph H can be a Δ-subgraph of a graph G, which is critical with respect to the chromatic index, if and only if H has no vertices of degree 0 or 1.

1. INTRODUCTION

In this paper we consider simple graphs (that is, graphs which have no loops or multiple edges). An *edge-colouring* of a graph G is a map $\phi: E(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours and $E(G)$ is the set of edges of G, such that no two incident edges receive the same colour. The *chromatic index*, or edge-chromatic number, $\chi'(G)$ of G is the least value of $|\mathscr{C}|$ for which an edge-colouring of G exists. A well-known theorem of Vizing [3] states that, for a simple graph G,

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum degree of G. Graphs G for which

0095-8956/89 $3.00

$\Delta(G)=\chi'(G)$ are said to be *Class* 1, and otherwise they are *Class* 2. The degree of a vertex v of G is denoted by $d_G(v)$ (or $d(v)$).

A graph G is *critical* if it is Class 2, connected, and, for each edge e of G, $\chi'(G\backslash e)<\chi'(G)$. In his "adjacency lemma," Vizing showed that, in a critical graph, each vertex v is adjacent to at least two vertices of maximum degree. Let $\Delta=\Delta(G)$ be the maximum degree of G, and let G_Δ, the *Δ-subgraph of* G, be the subgraph of G induced by the vertices of maximum degree. Thus, from Vizing's adjacency lemma, it follows that, if G is critical, then G_Δ has no vertices of degree 0 or 1. Critical graphs are not well understood, and so it is natural to ask whether G_Δ has to have any other property, for example, must G_Δ be connected? We show here that the answer to such questions is "no."

THEOREM 1. *Let H be a non-empty graph. There is a critical graph G for which $G_\Delta=H$ if and only if H has no vertices of degree* 0 *or* 1.

In fact we argue not so much in terms of critical graphs, but rather in terms of overfull graphs. A graph G is said to be *overfull* if it satisfies

$$|E(G)|\geqslant\Delta(G)\cdot\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1.$$

If there is equality here then G is called *just overfull.* An overfull graph necessarily has odd order. Since no colour class can consist of more than $\lfloor|V(G)|/2\rfloor$ edges, it follows that, if G is overfull, then G is Class 2. Thus if a graph is overfull, then it cannot be critical unless it is *just overfull.* However, not all graphs which are just overfull are critical: an example is a graph formed from Petersen's graph by deleting a vertex and then joining two vertices of degree 2. We propose the following conjecture, linking these two ideas.

CONJECTURE 1. *Let $\Delta(G)\geqslant\frac{1}{2}|V(G)|$. Then G is critical if and only if G is just overfull.*

The connection we use between the two ideas is the following result of Chetwynd and Hilton [1, from Theorems 6 and 7], which establishes the conjecture under quite a wide circumstance.

LEMMA 1. *Let G be a graph with r vertices of maximum degree. Let $n=\lfloor|V(G)|/2\rfloor$. Let*

$$\Delta(G)\geqslant n+\tfrac{7}{2}r-2.$$

Then G is critical if and only if G is just overfull.

2. Proof of Theorem 1

Necessity. If G is a critical graph for which $G_\Delta = H$, then, by Vizing's adjacency lemma [4], H has no vertices of degree 0 or 1.

Sufficiency. Suppose that H has no vertices of degree 0 or 1. Let $r = |V(H)|$. Then $r \geqslant 3$. Let $n \geqslant \frac{9}{2}r - 5$. Choose Δ so that

$$2n - r + \varepsilon(H) \geqslant \Delta \geqslant 2n - r + 3,$$

where

$$\varepsilon(H) = \min\left(1 + \delta(H), 1 + \left\lceil r - \frac{r(r-1)}{2n+1-r}\right\rceil\right);$$

here $\delta(H)$ denotes the minimum degree of H. This is possible since $\delta(H) \geqslant 2$, so that $1 + \delta(H) \geqslant 3$, and since

$$\begin{aligned} r - \frac{r(r-1)}{2n+1-r} &> r - \frac{r^2}{2n+1-r} \\ &\geqslant r - \frac{r^2}{8r-7}, && \text{since} \quad n \geqslant \tfrac{9}{2}r - 5, \\ &\geqslant r - \frac{r^2}{4r}, && \text{since} \quad r \geqslant 3, \\ &= \frac{3r}{4} > 2. \end{aligned}$$

We construct a just overfull graph G of order $2n+1$ with $\Delta(G) = \Delta$ and $G_\Delta = H$. We start with the graph H on the vertex set $V(H)$, and with the further set $V(G) \backslash V(H)$ of $2n+1-r$ vertices. Place edges between $V(H)$ and $V(G) \backslash V(H)$ in such a way that there are altogether Δ edges on each vertex of $V(H)$ and such that the numbers of edges on the vertices of $V(G) \backslash V(H)$ are within one of each other. This can be done since $|V(G) \backslash V(H)| + \delta(H) = 2n + 1 - r + \delta(H) \geqslant 2n - r + \varepsilon(H) \geqslant \Delta$. The number of edges added in at this stage is

$$r\Delta - 2|E(H)|,$$

and so the maximum degree of vertices of $V(G) \backslash V(H)$ is at this stage

$$\left\lceil \frac{r\Delta - 2|E(H)|}{2n+1-r} \right\rceil.$$

Now to make up the required number, $n\Delta+1$, of edges, add in a further set of $n\Delta - r\Delta + |E(H)| + 1$ edges on the vertex set $V(G)\backslash V(H)$ in such a way that the degrees in G of the vertices of $V(G)\backslash V(H)$ are within one of each other. If possible, do this in such a way that there are no multiple edges. Note that, when this is done, then the degree in G of each vertex of $V(G)\backslash V(H)$ is either $\lfloor\theta\rfloor$ or $\lceil\theta\rceil$, where

$$\theta = \frac{2}{2n+1-r}\{n\Delta - r\Delta + |E(H)| + 1\} + \frac{1}{2n+1-r}\{r\Delta - 2|E(H)|\}$$

$$= \frac{1}{2n+1-r}\{2n\Delta - r\Delta + \Delta - (\Delta - 2)\} = \Delta - \frac{\Delta - 2}{2n+1-r}.$$

Since $\Delta \geqslant 2n - r + 3$, it follows that $\Delta - 2 \geqslant 2n + 1 - r$, so each vertex of $V(G)\backslash V(H)$ has degree at most $\Delta - 1$. It is now easy to see that there need be no multiple edges, since

$$\begin{aligned}\Delta - 1 &\leqslant (2n - r) + (\varepsilon(H) - 1)\\ &\leqslant (|V(G)\backslash V(H)| - 1) + \left\lceil \frac{r(2n+1-r) - r(r-1)}{2n+1-r}\right\rceil\\ &\leqslant (|V(G)\backslash V(H)| - 1) + \left\lceil \frac{r\Delta - 2|E(H)|}{2n+1-r}\right\rceil\end{aligned}$$

which is the sum of the maximum degree of a simple graph on $|V(G)\backslash V(H)|$ and the maximum degree of the vertices of $V(G)\backslash V(H)$ after the first stage. This completes the construction of G. Note that

$$|E(G)| = (n\Delta - r\Delta + |E(H)| + 1) + (r\Delta - 2|E(H)|) + |E(H)| = n\Delta + 1,$$

so G is just overfull. It clearly has all the other properties required of it.

Since $n \geqslant \frac{9}{2}r - 5$ it follows that $2n + 3 - r \geqslant n + \frac{7}{2}r - 2$, and so we have

$$\Delta \geqslant n + \tfrac{7}{2}r - 2.$$

Therefore, by Lemma 1, since G is just overfull, it follows that G is critical, as required.

3. Further Remarks

The critical graphs constructed in Theorem 1 are all just overfull. They have the further property that the number of vertices of maximum degree is relatively small. The known graphs which are critical but not overfull all

have a relatively large number of vertices of maximum degree. It seems very unlikely that Theorem 1 would remain true if "critical" were replaced by "critical but not overfull"; however, we do not see how to prove this.

In this connection, it is perhaps of some interest to note the following graph G^* of order $2n+1$, with $\Delta(G^*)=n+2$, where G^*_Δ has order $n+1$ and could be made to be disconnected. Here $\Delta(G^*)$ is relatively low and the number of vertices of maximum degree is high. The graph G^* is just overfull and is critical.

THEOREM 2. *Let G^* be a simple graph constructed from $K_{n,n+1}$ by inserting on the set of $n+1$ independent vertices a regular graph of order $n+1$ and degree 2. Then G^* is critical.*

Proof. It is easy to verify that G^* is just overfull and so is Class 2. Let e be an edge of G^*. We must show that $G^*\backslash e$ is Class 1. Let the independent sets of vertices of $K_{n,n+1}$ be $A=\{a_1, ..., a_n\}$ and $B=\{b_1, ..., b_{n+1}\}$.

Suppose first that e joins two vertices of B. Extend the graph $G^*\backslash e$ by adding a further vertex b^* and join b^* to all vertices of A and to the vertices at each end of e. Then we obtain a $K_{n,n+2}$ with a regular graph of order $n+2$ and degree 2 inserted on the independent set $B^*=B\cup\{b^*\}$ of $n+2$ vertices. We show that this extended graph is Class 1. First colour the $n+2$ edges on B^* with the colours $c_1, ..., c_{n+2}$, using each colour exactly once. Now construct a bipartite graph J with vertex sets $\{b'_1, ..., b'_{n+1}, b^{*\prime}\}$ and $\{c'_1, ..., c'_{n+2}\}$, joining b' to c' if c is not used on b in the extended graph so far. This graph J is bipartite and regular of degree n; it can therefore be edge-coloured with n colours $a'_1, ..., a'_n$. Now if $b'c'$ is coloured a' in J, then in the extended graph colour ab with colour c. This yields an edge-colouring of the extended graph with $n+2$ colours, and so $G^*\backslash e$ is edge-colourable with $n+2$ colours, as required.

Next suppose that e joins a vertex $a_0\in A$ to a vertex $b_0\in B$. Let $e_1=b_0b_1$ be an edge within B adjacent to e. Proceed as before to colour $G\backslash e_1$. Let c_1 be the colour on the edge of the extended graph joining b^* and b_1. When colouring J, ensure that the edge $c'_1b'_0$ is coloured a'_0. Then in the extended graph the edge $e=a_0b_0$ is coloured c_1. In $G\backslash e_1$, colour c_1 is missing at b_1, so we can remove edge e and insert edge e_1 coloured c_1.

This proves Theorem 2.

Given n, let $C(n,p)$ be the set of integers ε for which there exists a critical graph of order n, maximum degree p with ε edges. Clearly if n is odd, $\max C(n,p)\leqslant p\lfloor n/2\rfloor+1$. A number of questions can be asked about $C(n,p)$. Is $C(n,p)$ an interval? Is $\max C(n,p)=p\lfloor n/2\rfloor+1$ for all $p\geqslant 2$ and for all odd $n\geqslant p+1$? What is the value of $\min C(n,p)$? Investigation of the set $C(n,p)$ might help to throw much needed light on the nature of critical graphs.

ACKNOWLEDGMENT

We thank Dr. R. Häggkvist for indicating a proof of Theorem 2 to us.

REFERENCES

1. A. G. CHETWYND AND A. J. W. HILTON, The chromatic index of graphs with large maximum degree, where the number of vertices of maximum degree is relatively small, *J. Combin. Theory*, to appear.
2. S. FIORINI AND R. J. WILSON, "Edge-Colourings of Graphs," Research notes in mathematics, Vol. 16, Pitman, London, 1977.
3. V. G. VIZING, On an estimate of the chromatic class of a p-graph, *Diskret. Analiz.* **3** (1964), 25–30. [Russian]
4. V. G. VIZING, Critical graphs with a given chromatic class, *Diskret. Analiz.* **5** (1965), 9–17.