Lifting Lie algebras over the residue field of a discrete valuation ring

William Browder[a,1], Jonathan Pakianathan[b,2,*]

[a] *Department of Mathematics, Princeton University, Princeton, NJ 08544, USA*
[b] *Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA*

Abstract

Let $\mathbf{R}$ be a discrete valuation ring with maximal ideal $\pi\mathbf{R}$ and residue field $\mathbf{k}$. We study obstructions to lifting a Lie algebra $\mathfrak{L}$ over $\mathbf{R}/\pi^k\mathbf{R}$ to one over $\mathbf{R}/\pi^{k+1}\mathbf{R}$. If $\overline{\mathfrak{L}}$ is the Lie algebra over $\mathbf{k}$ obtained from reducing $\mathfrak{L}$ then we show there exists a well-defined class in $H^3(\overline{\mathfrak{L}}, ad)$ which vanishes if and only if $\mathfrak{L}$ lifts. Furthermore, if $\mathfrak{L}$ lifts, the lifts are shown to be in one to one correspondence with $H^2(\overline{\mathfrak{L}}, ad)$. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathbf{R}$ be a discrete valuation ring (DVR) with maximal ideal $\pi\mathbf{R}$. (For basic definitions, see [2].) Let $\mathbf{k}$ be its residue field. For convenience let $\mathbf{R}_k = \mathbf{R}/\pi^k\mathbf{R}$ for $k \geq 1$. An example to keep in mind is $\mathbf{R} = \hat{\mathbb{Z}}_p$, the p-adic integers. In this case, $\mathbf{R}_k = \mathbb{Z}_{p^k}$, the ring of integers modulo p^k and the residue field is $\mathbb{F}_p$, the field with p elements.

A Lie algebra $\mathfrak{L}$ over $\mathbf{R}_k$, $k \geq 1$ is a free $\mathbf{R}_k$-module of finite rank, equipped with a bilinear map

$$[\cdot, \cdot] : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$$

* Corresponding author.
E-mail address: pakianat@math.wisc.edu (J. Pakianathan)

which is alternating, i.e., $[v,v]=0$ for all $v \in \mathfrak{L}$, and which satisfies the Jacobi identity:

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$$

for all $u,v,w \in \mathfrak{L}$.

Given such a Lie algebra we denote $\overline{\mathfrak{L}}$ the **k**-Lie algebra obtained by setting

$$\overline{\mathfrak{L}} = \mathfrak{L} \underset{R}{\otimes} \mathbf{k}.$$

Let $n = \dim(\overline{\mathfrak{L}})$. Recall the existance of a differential Koszul complex (See [1]):

$$0 \to \overset{0}{\bigwedge}(\overline{\mathfrak{L}}, ad) \overset{d}{\to} \cdots \overset{d}{\to} \overset{n}{\bigwedge}(\overline{\mathfrak{L}}, ad) \to 0,$$

where $\bigwedge^i(\overline{\mathfrak{L}}, ad)$ is the vector space of alternating $\overline{\mathfrak{L}}$-valued i-forms on $\overline{\mathfrak{L}}$, and d is given by the formula

$$(d\omega)(x_0,\ldots,x_s) = \sum_{i=0}^{s} (-1)^i [x_i, \omega(x_0,\ldots,\hat{x}_i,\ldots,x_s)]$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([x_i,x_j], x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_s)$$

for a s-form ω. The cohomology of this complex is the Lie algebra cohomology $H^*(\overline{\mathfrak{L}}, ad)$.

In this paper we will study the question of when there exists a $\mathbf{R}_{k+1}$-Lie algebra $\hat{\mathfrak{L}}$ lifting $\mathfrak{L}$, i.e.,

$$\hat{\mathfrak{L}} \overset{\mathrm{mod}}{\to} \mathfrak{L}$$

is a surjective map which preserves the brackets. We will say if this happens that $\mathfrak{L}$ lifts to a Lie algebra over $\mathbf{R}_{k+1}$.

This question is answered in Theorem 2.1 of Section 2. Suppose $\overline{\mathfrak{L}}$ is the reduction of $\mathfrak{L}$ to a Lie algebra over the residue field **k**. Then we will show that there exists a class in $H^3(\overline{\mathfrak{L}}, ad)$ such that $\mathfrak{L}$ lifts if and only if this class vanishes.

If $\mathfrak{L}$ lifts, one can ask if the lift is unique and if not, how many distinct ones there are. This question is answered in Theorem 3.1 of Section 3. We show that if $\mathfrak{L}$ lifts then the distinct lifts are in one to one correspondence with $H^2(\overline{\mathfrak{L}}, ad)$.

Finally in Section 4 we provide an example of a Lie algebra which does not lift.

2. The obstruction to lifting

Let $\mathfrak{L}$ be a Lie algebra over $\mathbf{R}_k$.

There obviously exists a free $\mathbf{R}_{k+1}$-module B with $B \overset{\mathrm{mod}}{\to} \mathfrak{L}$ surjective. Let $\{\bar{e}_1,\ldots,\bar{e}_n\}$ be a basis for this free $\mathbf{R}_{k+1}$-module B. Then if we set

$$e_i = \mathrm{mod}(\bar{e}_i),$$

then $\{e_1,\ldots,e_n\}$ is a $\mathbf{R}_k$ basis for $\mathfrak{L}$.

So we can define a bilinear map $[\cdot,\cdot] : B \otimes B \to B$ by defining on the basis elements:

$[\bar{e}_i, \bar{e}_j] = $ a lift of $[e_i, e_j]$ for $i < j$,

$[\bar{e}_i, \bar{e}_i] = 0$,

$[\bar{e}_i, \bar{e}_j] = -[\bar{e}_j, \bar{e}_i]$ for $i > j$.

It is now routine to show that

$$[\cdot,\cdot] : B \otimes B \to B$$

is alternating, i.e., $[b, b] = 0$ for all $b \in B$, and also that

$$B \stackrel{\mathrm{mod}}{\to} \mathfrak{L}$$

preserves the brackets. We will call such a pair $(B, [\cdot,\cdot])$ a bracket algebra (Lie algebra minus Jacobi identity) lift of $\mathfrak{L}$.

If we have another bracket $[\cdot,\cdot]'$ on B lifting the bracket of $\mathfrak{L}$, then

$$[\cdot,\cdot]' - [\cdot,\cdot] = \pi^k \langle \cdot, \cdot \rangle,$$

where

$$\pi^k \langle \cdot, \cdot \rangle : B \otimes B \to B$$

is an alternating, bilinear form which takes values in $\pi^k B$.

As $\pi^{k+1} B = 0$, we see that

$$\pi^k \langle \pi x, y \rangle = 0$$

for all $x, y \in B$. Define

$$\chi : \pi^k B \to \overline{\mathfrak{L}}$$

by $\chi(\pi^k x) = \lambda(x)$, where

$$\lambda : B \to \overline{\mathfrak{L}}$$

is the mod reduction from $\mathbf{R}_{k+1}$ to $\mathbf{k}$. It is easy to check that χ is a well-defined isomorphism of abelian groups. Let ψ be its inverse. So we see easily that $\chi(\pi^k \langle \cdot, \cdot \rangle)$ induces an alternating, bilinear form

$$\langle \cdot, \cdot \rangle : \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}} \to \overline{\mathfrak{L}}.$$

Conversely, given such an alternating bilinear form

$$\langle \cdot, \cdot \rangle : \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}} \to \overline{\mathfrak{L}},$$

we can set

$$[\cdot,\cdot]' = [\cdot,\cdot] + \psi(\langle \lambda(\cdot), \lambda(\cdot) \rangle)$$

to get another lift of $\mathfrak{L}$.

Now suppose we have a bracket algebra lift $(B, [\cdot, \cdot])$ of $\mathfrak{L}$. Then we can define

$$\bar{J}_{[\cdot,\cdot]}(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$$

for all $x, y, z \in B$.

It is easy to check that $\bar{J}_{[\cdot,\cdot]}$ defines a B-valued alternating 3-form on B. Moreover we note that B lifts $\mathfrak{L}$ which is a Lie algebra hence we see $\bar{J}_{[\cdot,\cdot]}$ has image in $\pi^k B$. Hence using χ as mentioned before we see that $\chi(\bar{J}_{[\cdot,\cdot]})$ induces an $\overline{\mathfrak{L}}$-valued alternating 3-form on $\overline{\mathfrak{L}}$ which we will call $J_{[\cdot,\cdot]}$.

It is easy to see that $J_{[\cdot,\cdot]} = 0$ if and only if $\bar{J}_{[\cdot,\cdot]} = 0$ if and only if $(B, [\cdot, \cdot])$ is a Lie algebra.

Note $J_{[\cdot,\cdot]}$ is an element of $\bigwedge^3(\overline{\mathfrak{L}}, ad)$ of the Koszul complex mentioned above. Let us show it is closed under the differential of the Koszul complex. For simplicity we drop the subscript and denote $J_{[\cdot,\cdot]}$ by J and similarly $\bar{J}_{[\cdot,\cdot]}$ by $\bar{J}$. Then

$$(dJ)(x, y, z, w) = -J([x, y], z, w) + J([x, z], y, w) - J([x, w], y, z)$$

$$-J([y, z], x, w) + J([y, w], x, z) - J([z, w], x, y)$$

$$+[x, J(y, z, w)] - [y, J(x, z, w)]$$

$$+[z, J(x, y, w)] - [w, J(x, y, z)]$$

for all $x, y, z \in \overline{\mathfrak{L}}$.

Now, we apply ψ to both sides and note

$$\psi(J(x, y, z)) = \bar{J}(\bar{x}, \bar{y}, \bar{z}),$$

where $\bar{x}, \bar{y}, \bar{z}$ are lifts of x, y, z to B, so we get

$$\psi(dJ(x, y, z, w)) = -\bar{J}([\bar{x}, \bar{y}], \bar{z}, \bar{w}) + \bar{J}([\bar{x}, \bar{z}], \bar{y}, \bar{w}) - \bar{J}([\bar{x}, \bar{w}], \bar{y}, \bar{z})$$

$$-\bar{J}([\bar{y}, \bar{z}], \bar{x}, \bar{w}) + \bar{J}([\bar{y}, \bar{w}], \bar{x}, \bar{z}) - \bar{J}([\bar{z}, \bar{w}], \bar{x}, \bar{y})$$

$$+[\bar{x}, \bar{J}(\bar{y}, \bar{z}, \bar{w})] - [\bar{y}, \bar{J}(\bar{x}, \bar{z}, \bar{w})]$$

$$+[\bar{z}, \bar{J}(\bar{x}, \bar{y}, \bar{w})] - [\bar{w}, \bar{J}(\bar{x}, \bar{y}, \bar{z})]$$

$$= 0,$$

where to get the last equality we plugged in the definition of $\bar{J}$ and cancelled terms in pairs using that $[\cdot, \cdot]$ is an alternating form.

Thus we conclude that $dJ = 0$, i.e., $J_{[\cdot,\cdot]}$ is closed under the differential of the Koszul complex.

Given another bracket $[\cdot, \cdot]'$ on B which lifts $\mathfrak{L}$ we want to relate $J_{[\cdot,\cdot]}$ and $J_{[\cdot,\cdot]'}$. Recall

$$[\cdot, \cdot]' = [\cdot, \cdot] + \psi(\langle \lambda(\cdot), \lambda(\cdot) \rangle)$$

for some alternating, bilinear two form

$$\langle \cdot, \cdot \rangle : \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}} \to \overline{\mathfrak{L}}$$

which we can view as an element in $\bigwedge^2(\overline{\mathfrak{L}}, ad)$ of the Koszul complex. Then we have

$$\psi(J_{[\cdot,\cdot]'}(x,y,z)) = \bar{J}_{[\cdot,\cdot]'}(\bar{x}, \bar{y}, \bar{z})$$

$$= [[\bar{x}, \bar{y}]', \bar{z}]' + [[\bar{y}, \bar{z}]', \bar{x}]' + [[\bar{z}, \bar{x}]', \bar{y}]'$$

$$= [[\bar{x}, \bar{y}], \bar{z}]' + [\psi(\langle x, y\rangle), \bar{z}]'[[\bar{y}, \bar{z}], \bar{x}]'$$

$$\qquad + [\psi(\langle y, z\rangle), \bar{x}]'[[\bar{z}, \bar{x}], \bar{y}]' + [\psi(\langle z, x\rangle), \bar{y}]'$$

$$= [[\bar{x}, \bar{y}], \bar{z}] + \psi(\langle[x, y], z\rangle) + [\psi(\langle x, y\rangle), \bar{z}]$$

$$\qquad + \psi(\langle\lambda(\psi(\langle x, y\rangle)), z\rangle) + [[\bar{y}, \bar{z}], \bar{x}] + \psi(\langle[y, z], x\rangle)$$

$$\qquad + [\psi(\langle y, z\rangle), \bar{x}] + \psi(\langle\lambda(\psi(\langle y, z\rangle)), x\rangle)$$

$$\qquad + [[\bar{z}, \bar{x}], \bar{y}] + \psi(\langle[z, x], y\rangle) + [\psi(\langle z, x\rangle), \bar{y}]$$

$$\qquad + \psi(\langle\lambda(\psi(\langle z, x\rangle)), y\rangle),$$

where $\lambda : B \to \overline{\mathfrak{L}}$ is the mod map. Since $k \geq 1$, we see easily that $\lambda \circ \psi = 0$ so the equation above simplifies to

$$\psi(J_{[\cdot,\cdot]'}(x,y,z)) = \bar{J}_{[\cdot,\cdot]}(\bar{x}, \bar{y}, \bar{z}) + \psi(\langle[x, y], z\rangle + \langle[y, z], x\rangle + \langle[z, x], y\rangle)$$

$$\qquad + [\psi(\langle x, y\rangle), \bar{z}] + [\psi(\langle y, z\rangle), \bar{x}] + [\psi(\langle z, x\rangle), \bar{y}]$$

$$= \psi(J_{[\cdot,\cdot]}(x,y,z)) + \psi(\langle[x, y], z\rangle + \langle[y, z], x\rangle + \langle[z, x], y\rangle)$$

$$\qquad + \psi([\langle x, y\rangle, z] + [\langle y, z\rangle, x] + [\langle z, x\rangle, y])$$

$$= \psi(J_{[\cdot,\cdot]}(x,y,z) - (d\langle\cdot,\cdot\rangle)(x,y,z)),$$

where in the second to last equality we used the easily verified identity

$$[\psi(\langle x, y\rangle), \bar{z}] = \psi([\langle x, y\rangle, z])$$

for all $x, y, z \in \overline{\mathfrak{L}}$ and in the last equality we thought of $\langle\cdot,\cdot\rangle$ as an element in $\bigwedge^2(\overline{\mathfrak{L}}, ad)$ of the Koszul complex.

We apply χ to both sides of the equation above to get that

$$J_{[\cdot,\cdot]'} = J_{[\cdot,\cdot]} - d(\langle\cdot,\cdot\rangle)$$

so $J_{[\cdot,\cdot]'}$ and $J_{[\cdot,\cdot]}$ differ by a boundary in the Koszul complex and hence represent the same cohomology class $[J]$ in $H^3(\overline{\mathfrak{L}}, ad)$. The following theorem follows immediately:

Theorem 2.1. *Given a Lie algebra $\mathfrak{L}$ over $\mathbf{R}_k$, let $\overline{\mathfrak{L}}$ be its reduction to a **k**-Lie algebra as before. Then there exists a well-defined cohomology class*

$$[J] \in H^3(\overline{\mathfrak{L}}, ad)$$

which vanishes if and only if $\mathfrak{L}$ has a $\mathbf{R}_{k+1}$-Lie algebra lift $\hat{L}$ in the sense mentioned before.

A p-adic Lie-algebra $\mathfrak{L}$ will be a finitely generated, free $\hat{\mathbb{Z}}_p$-module equipped with a bracket as usual which satisfies the Jacobi identity, where $\hat{\mathbb{Z}}_p$ is the p-adic integers. Such a $\mathfrak{L}$ reduces in an obvious way to $\overline{\mathfrak{L}}$, a Lie algebra over $\mathbb{F}_p = \mathbb{Z}_p$. Conversely given $\overline{\mathfrak{L}}$ a Lie algebra over $\mathbb{F}_p$ we say it lifts to the p-adics if such a $\mathfrak{L}$ exists. Now we can state the following corollary as an immediate consequence of the theorem above.

Corollary 2.2. *A Lie algebra $\overline{\mathfrak{L}}$ over $\mathbb{F}_p$ with $H^3(\overline{\mathfrak{L}}, ad) = 0$ lifts to the p-adics.*

The obstruction given in Theorem 2.1 is nontrivial in general as shown by the example in Section 4. However if the reduction map $\mathbf{R}_{k+1} \overset{\text{mod}}{\rightarrow} \mathbf{R}_k$ splits, that is, if there exists a ring map

$$\mathbf{R}_k \overset{\kappa}{\rightarrow} \mathbf{R}_{k+1}$$

with $\text{mod} \circ \kappa = Identity$, then Lie algebras over $\mathbf{R}_k$ always lift to ones over $\mathbf{R}_{k+1}$ using κ so the obstructions obtained must all vanish. This happens for example when $k = 1$ and $\mathbf{R} = \mathbb{F}_p[[x]]$ the power series ring over $\mathbb{F}_p$. Note this ring has residue field $\mathbb{F}_p$ the same residue field as for the p-adic integers where this obstruction theory is nontrivial on the $k = 1$ level and (see Section 4).

3. Classifying lifts of L

Suppose for this section that we are in the situation as in the last section but that $\overline{\mathfrak{L}}$ has a Lie algebra lift $\hat{\mathfrak{L}}_0$ to a $\mathbf{R}_{k+1}$-Lie algebra.

Suppose now that $\hat{\mathfrak{L}}_1$, $\hat{\mathfrak{L}}_2$ are two other lifts. We will view these two lifts as having the same underlying $\mathbf{R}_{k+1}$-module B but with different bracket structures $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$, respectively.

Then as we saw before, there are two alternating 2-forms $\langle \cdot, \cdot \rangle_i : \overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}} \rightarrow \overline{\mathfrak{L}}$ such that

$$[\cdot, \cdot]_i = [\cdot, \cdot]_0 + \psi(\langle \lambda(\cdot), \lambda(\cdot) \rangle_i)$$

for $i = 1, 2$.

Now as

$$J_{[\cdot, \cdot]_i} = J_{[\cdot, \cdot]_0} - d(\langle \cdot, \cdot \rangle_i)$$

and

$$J_{[\cdot, \cdot]_0} = J_{[\cdot, \cdot]_1} = J_{[\cdot, \cdot]_2} = 0,$$

we see that $\langle \cdot, \cdot \rangle_i$ are closed elements of $\bigwedge^2(\overline{\mathfrak{L}}, ad)$ for $i = 1, 2$.

Suppose that they determine the same cohomology element in $H^2(\overline{\mathfrak{L}}, ad)$, i.e.,

$$\langle \cdot, \cdot \rangle_2 - \langle \cdot, \cdot \rangle_1 = -d\phi,$$

where $\phi \in \bigwedge^1(\overline{\mathfrak{L}}, ad)$ is a linear map $\overline{\mathfrak{L}} \rightarrow \overline{\mathfrak{L}}$. Then

$$\Psi = Id + \psi \circ \phi \circ \lambda : B \rightarrow B$$

is certainly a homomorphism of $\mathbf{R}_{k+1}$-modules. (Here Id stands for the identity map). On the other hand, it is invertible using

$$(1+x)^{-1} = \sum_{i=0}^{\infty} (-x)^i$$

as $\psi \circ \phi \circ \lambda$ takes values in $\pi^k B$ and hence has square zero. So Ψ is an automorphism of B.

Let us show $[\Psi(x), \Psi(y)]_2 = \Psi([x, y]_1)$ so that Ψ gives an isomorphism between the Lie algebras $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$.

$$[\Psi(x), \Psi(y)]_2 = [x + \psi(\phi(\lambda(x))), y + \psi(\phi(\lambda(y)))]_2$$

$$= [x, y]_2 + [x, \psi(\phi(\lambda(y)))]_2 + [\psi(\phi(\lambda(x))), y]_2,$$

where we do not write the 4th term as it is zero because the image of ψ is in $\pi^k B$.

$$[\Psi(x), \Psi(y)]_2 = [x, y]_0 + \psi(\langle \lambda(x), \lambda(y) \rangle_2) + [x, \psi(\phi(\lambda(y)))]_0 + \psi(\langle \lambda(x), 0 \rangle_2)$$

$$+ [\psi(\phi(\lambda(x))), y]_0 + \psi(\langle 0, \lambda(y) \rangle_2)$$

$$= [x, y]_0 + \psi(\langle \lambda(x), \lambda(y) \rangle_2) + [x, \psi(\phi(\lambda(y)))]_0 + [\psi(\phi(\lambda(x))), y]_0$$

$$= [x, y]_0 + \psi(\langle \lambda(x), \lambda(y) \rangle_2)$$

$$+ \psi([\lambda(x), \phi(\lambda(y))]) + \psi([\phi(\lambda(x)), \lambda(y)]).$$

On the other hand, we have

$$\Psi([x, y]_1) = [x, y]_1 + \psi(\phi(\lambda([x, y]_1)))$$

$$= [x, y]_1 + \psi(\phi([\lambda(x), \lambda(y)]))$$

$$= [x, y]_0 + \psi(\langle \lambda(x), \lambda(y) \rangle_1)$$

$$+ \psi(\phi([\lambda(x), \lambda(y)])).$$

So we have

$$[\Psi(x), \Psi(y)]_2 - \Psi([x, y]_1) = \psi(\langle \lambda(x), \lambda(y) \rangle_2 - \langle \lambda(x), \lambda(y) \rangle_1)$$

$$+ \psi([\lambda(x), \phi(\lambda(y))] + [\phi(\lambda(x)), \lambda(y)]$$

$$- \phi([\lambda(x), \lambda(y)]))$$

$$= \psi(\langle \lambda(x), \lambda(y) \rangle_2 - \langle \lambda(x), \lambda(y) \rangle_1)$$

$$+ \psi((d\phi)(\lambda(x), \lambda(y))),$$

where we have used the formula

$$(d\phi)(u, v) = -\phi([u, v]) + [u, \phi(v)] - [v, \phi(u)].$$

Note that up to now to get the formula above we have not used the fact that

$$\langle \cdot, \cdot \rangle_2 - \langle \cdot, \cdot \rangle_1 = -d\phi$$

and have only used the fact that

$$\Psi = Id + \psi \circ \phi \circ \lambda.$$

Now since

$$\langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_1 - d\phi(\cdot, \cdot),$$

we see from the above that $[\Psi(x), \Psi(y)]_2 = \Psi([x, y]_1)$.

So we see indeed that $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ are isomorphic Lie algebras under an isomorphism which induces the identity map on $\mathfrak{L}$.

Conversely suppose $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ are lifts which are isomorphic via an isomorphism Ψ which induces the identity map on $\mathfrak{L}$ then clearly $\Psi = Id + \mu$ where μ is a homomorphism of $\mathbf{R}_{k+1}$-modules which has image in $\pi^k B$. Thus it is easy to see that

$$\Psi = Id + \psi \circ \phi \circ \lambda$$

for some linear map $\phi : \overline{\mathfrak{L}} \to \overline{\mathfrak{L}}$. Reversing the steps above we see that

$$\langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_1 - d\phi$$

so that the elements in $\bigwedge^2(\overline{\mathfrak{L}}, ad)$ corresponding to the two Lie algebras $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ are cohomologous. So from all the preceding facts, the following theorem follows readily:

Theorem 3.1. *Given $\mathfrak{L}$ a Lie algebra over $\mathbf{R}_k$, let $\overline{\mathfrak{L}}$ be the Lie algebra obtained by reducing to the residue field. Suppose there is a Lie algebra lift of $\mathfrak{L}$ to a $\mathbf{R}_{k+1}$ Lie algebra. Then the set of all lifts (up to isomorphism of Lie algebras inducing identity on $\mathfrak{L}$) are in one to one correspondence with $H^2(\overline{\mathfrak{L}}, ad)$.*

4. A Lie algebra which does not lift

Consider the Lie algebra $\mathfrak{M} = \mathfrak{sl}_3(3)$ of trace zero, 3×3 matrices over $\mathbb{F}_3$, the field with three elements. This is an eight dimensional Lie algebra which is easily verified to be perfect, i.e., $[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}$.

Unlike the characteristic zero case, this Lie algebra has a one dimensional center consisting of the multiples of the identity which have trace zero. Form the quotient Lie algebra of $\mathfrak{M}$ by its center and let us call it $\mathfrak{P}$.

This Lie algebra $\mathfrak{P}$ is seven dimensional and is still perfect of course. It is easy to check it is in fact simple. A tedious calculation reveals its Killing form to be identically zero. (This is much different than in the characteristic zero case where the Killing form is nondegenerate if and only if the Lie algebra is semisimple.) On the other hand, there does exist a nondegenerate, invariant, symmetric 3-form and this generates the vector

space of invariant, symmetric 3-forms on $\mathfrak{P}$. One can also show that $\mathfrak{P}$ is unimodular and hence calculate its cohomology relatively easily as

$$H^1(\mathfrak{P}, \mathbb{F}_3) = 0,$$

$$H^2(\mathfrak{P}, \mathbb{F}_3) = \mathbb{F}_3^6,$$

$$H^3(\mathfrak{P}, \mathbb{F}_3) = 0$$

and the rest filled in by Poincaré duality. Also one can show

$$H^0(\mathfrak{P}, ad) = 0,$$

$$H^1(\mathfrak{P}, ad) = \mathbb{F}_3^7.$$

Thus $\mathfrak{P}$ is not complete although one can show directly that it is restricted using the p-power map from $\mathfrak{M}$. Furthermore, this is the unique restricted structure.

As is well known, such cohomology would not be expected for a simple Lie algebra in the characteristic zero case. So we have much reason to believe $\mathfrak{P}$ does not lift to the p-adics. Indeed, computer calculations of the cohomology obstruction in Section 2 have shown that it does not even lift to a $\mathbb{Z}_9$-Lie algebra. Unfortunately, at the current time we have left it at that and not attempted to try to find a "human" proof. Unfortunately although the verification that $[J]$ is not a boundary in this case is merely linear algebra, it is of considerable size!

In general the Lie algebras $\mathfrak{sl}_p(p)$ for p a prime number have a nontrivial center and the quotient Lie algebra obtained by moding out this center should be quite interesting.

For an application of the results of this paper to questions about uniform, p-central, p-groups see [3].

Acknowledgements

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