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Exact and Serial Rings

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INTRODUCTION

Let R be a left Artinian ring with radical J. Let $R = I_0 \supset I_1 \supset \cdots \supset I_{s-1} \supset I_s = 0$ be a composition series of the two-sided R-module R. R is called an exact ring if, for each i, every endomorphism of the left R-module I_{i-1}/I_i is given by the right-multiplication of an element of R. The notion of the exactness is independent of the choice of the above composition series and is left-right symmetric; in particular, every exact ring is right Artinian too. Let e, f be primitive idempotents of R, and let E, F be the injective envelopes of the simple left R-module Re/Je and the simple right R-module fR/fJ, respectively. Then we show that if R is exact then both E and F have finite composition lengths which are the same as that of the right ideal eR and the left ideal Rf, respectively, and more precisely, the following are equal: (1) the multiplicity of the simple left R-module Rf/Jf in (the composition factor module series of) E, (2) the multiplicity of fR/fJ in eR, (3) the multiplicity of eR/eJ in F, (4) the multiplicity of Re/Je in Rf. Every commutative Artinian ring as well as every semi-simple Artinian ring is obviously an exact ring, while a typical example of noncommutative and nonsemisimple exact rings is given by split algebras, i.e., those finite-dimensional algebras R over a field K for which the factor algebra R/J is the direct sum of full matrix algebras over K. For another interesting example, we show in Section 2 that every serial ring is exact. This is based on a certain property of rings having injective left ideal $\neq 0$, and in this connection, we give a characterization of serial rings that R is serial if and only if every factor ring of R has an injective left ideal $\neq 0$. This characterization may be of interest when compared with the well-known theorem that R is serial if and only if every factor ring of R is of QF-3 type. A conjecture is proposed: If R is an exact

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ring, then R is self-dual, that is, the endomorphism ring of the injective envelope of the left R-module R/J is isomorphic to R itself.

1. Exact Rings

Throughout this paper, we assume that R is a left Artinian ring with identity element 1 and J the radical of R. Let $\bar{R} = R/J$ be the semi-simple factor ring of R and $\bar{R} = \bar{R}_1 \oplus \bar{R}_2 \oplus \cdots \oplus \bar{R}_k$ the direct decomposition of $\bar{R}$ into orthogonal simple components. For any primitive idempotent element e of R its coset $\bar{e}$ modulo J is also a primitive idempotent element of $\bar{R}$ and contained in one of $\bar{R}_\alpha$'s. For each α let e_α be a primitive idempotent element of R such that $\bar{e}_\alpha \in \bar{R}_\alpha$. Then $\bar{R}\bar{e}_\alpha$ is a simple left R-module, and the left R-module $\bar{R}_\alpha$ is the direct sum of, say $n(\alpha)$ copies of $\bar{R}\bar{e}_\alpha$. Every simple left R-module is isomorphic to some $\bar{R}\bar{e}_\alpha$, and ${}_R\bar{R}\bar{e}_\alpha \cong {}_R\bar{R}\bar{e}_\beta$ if and only if $\alpha = \beta$. Moreover, R is a direct sum of indecomposable left ideals each of which is isomorphic to one of Re_α's and the multiplicity of Re_α in the decomposition is $n(\alpha)$, i.e., we have ${}_RR \cong \sum \oplus {}_R(Re_\alpha)^{n(\alpha)}$. The similar facts are also true for simple right R-modules $\bar{e}_\alpha\bar{R}$ and indecomposable right ideals $e_\alpha R$; in particular, we have $(\bar{R}_\alpha)_R \cong (\bar{e}_\alpha\bar{R})_R^{n(\alpha)}$ and $R_R \cong \sum \oplus (e_\alpha R)_R^{n(\alpha)}$.

Let M be a simple two-sided R-module. Since J is nilpotent, M is then annihilated by J on both left- and right-hand sides and so can be regarded as a simple two-sided $\bar{R}$-module. Since furthermore $M = \bar{R}M = \sum \bar{R}_\alpha M$ and each $\bar{R}_\alpha M$ is a two-sided $\bar{R}$-submodule of M, it follows that $\bar{R}_\lambda M \neq 0$ whence $\bar{R}_\lambda M = M$ for some λ; but then $\bar{R}_\alpha M = \bar{R}_\alpha \bar{R}_\lambda M = 0$ for every $\alpha \neq \lambda$. Thus λ is the only index such that $\bar{R}_\lambda M = M$. Similarly, there is a unique index ρ such that $M\bar{R}_\rho = M$, and we have $M\bar{R}_\alpha = 0$ whenever $\alpha \neq \rho$. This means that M can actually be regarded as a simple two-sided $\bar{R}_\lambda$-$\bar{R}_\rho$-module. We shall call $\bar{R}_\lambda$ and $\bar{R}_\rho$ the *left* and the *right simple components* belonging to M, respectively. Since $\bar{R}_\rho$ is a simple ring, $\bar{R}_\rho$ is considered a subring of the endomorphism ring D of the left $\bar{R}_\lambda$-module M. We call M *exact* if $\bar{R}_\rho = D$, i.e., if every endomorphism of the left R-module M is given by the right-multiplication of an element of $\bar{R}$. Now since $\bar{R}_\lambda$ is a direct sum of simple left ideals isomorphic to $\bar{R}\bar{e}_\lambda$, the left $\bar{R}_\lambda$-module M is also a direct sum of copies of $\bar{R}\bar{e}_\lambda$, which implies clearly that M is a (projective) generator. Therefore, it follows from the Morita theorem [4, Lemma 3.3] that if M is exact then the right $\bar{R}_\rho$-module M is finitely generated and projective and $\bar{R}_\lambda$ coincides with its endomorphism ring. Thus we know that the notion of the exactness for M is left–right symmetric and besides that if M is exact, then M is a progenerator with respect to both $\bar{R}_\lambda$ and $\bar{R}_\rho$.

Lemma 1. *Let M be an exact simple two-sided R-module with left simple component $\bar{R}_\lambda$ and right simple component $\bar{R}_\rho$. Then*

(i) ${}_RMe_\rho \cong {}_R\bar{R}\bar{e}_\lambda$, $e_\lambda M_R \cong \bar{e}_\rho \bar{R}_R$,

(ii) ${}_R\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\lambda) \cong {}_R\bar{R}\bar{e}_\rho$, $\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\alpha) = 0$ *if* $\alpha \neq \lambda$, $\mathrm{Hom}(M_R, \bar{e}_\rho \bar{R}_R)_R \cong \bar{e}_\lambda \bar{R}_R$, $\mathrm{Hom}(M_R, \bar{e}_\alpha \bar{R}_R) = 0$ *if* $\alpha \neq \rho$,

(iii) ${}_RM \cong {}_R(\bar{R}\bar{e}_\lambda)^{n(\rho)}$, $M_R \cong (\bar{e}_\rho \bar{R})_R^{n(\lambda)}$.

Proof. (i) Since $\bar{e}_\rho$ is a primitive idempotent element in the endomorphism ring $\bar{R}_\rho$ of ${}_RM$, $Me_\rho = M\bar{e}_\rho$ is an indecomposable submodule of ${}_RM$. But ${}_RM$ and hence ${}_RMe_\rho$ is completely reducible, so it follows that ${}_RMe_\rho$ is a simple submodule of ${}_RM$. Thus ${}_RMe_\rho \cong {}_R\bar{R}\bar{e}_\lambda$. By left–right analogy, we can prove that $e_\lambda M_R \cong \bar{e}_\rho \bar{R}_R$.

(ii) By Morita [4, Theorem 3.4], the functor $\mathrm{Hom}_{\bar{R}_\lambda}(M, \ ) = \mathrm{Hom}({}_RM, \ )$ gives an isomorphism from the category of left $\bar{R}_\lambda$-modules onto the category of left $\bar{R}_\rho$-modules. In particular, since $\bar{R}\bar{e}_\lambda$ is a simple left $\bar{R}_\lambda$-module, the corresponding $\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\lambda)$ must be a simple left $\bar{R}_\rho$-module; but $\bar{R}\bar{e}_\rho$ is (up to isomorphism) the only simple left $\bar{R}_\rho$-module, so that we have ${}_R\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\lambda) \cong {}_R\bar{R}\bar{e}_\rho$. Let now $\alpha \neq \lambda$. Then that $\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\alpha) = 0$ follows from the fact that ${}_R\bar{R}\bar{e}_\alpha \not\cong {}_R\bar{R}\bar{e}_\lambda$ and ${}_RM$ is a direct sum of copies of ${}_R\bar{R}\bar{e}_\lambda$. The other part can be proved in the similar way if we observe $\mathrm{Hom}_{\bar{R}_\rho}(M, \ )$ instead of $\mathrm{Hom}_{\bar{R}_\lambda}(M, \ )$.

(iii) Consider again the isomorphism functor $\mathrm{Hom}_{\bar{R}_\lambda}(M, \ )$. Since ${}_R\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\lambda) \cong {}_R\bar{R}\bar{e}_\rho$ by (ii), we have ${}_R\mathrm{Hom}({}_RM, {}_R(\bar{R}\bar{e}_\lambda)^{n(\rho)})$ ${}_R\mathrm{Hom}({}_RM, {}_R\bar{R}\bar{e}_\lambda)^{n(\rho)} \cong {}_R(\bar{R}\bar{e}_\rho)^{n(\rho)} \cong {}_R\bar{R}_\rho$. On the other hand, that $\bar{R}_\rho$ is the endomorphism ring of ${}_RM$ means that ${}_R\mathrm{Hom}({}_RM, {}_RM) \cong {}_R\bar{R}_\rho$. Thus it follows that ${}_R(\bar{R}\bar{e}_\lambda)^{n(\rho)} \cong {}_RM$. Similarly, we have that $(\bar{e}_\rho \bar{R})_R^{n(\lambda)} \cong M_R$ by considering $\mathrm{Hom}_{\bar{R}_\rho}(M, \ )$.

Now the left Artinian ring R has a two-sided composition series, say

$$R = I_0 \supset I_1 \supset \cdots \supset I_{s-1} \supset I_s = 0;$$

each I_i is a two-sided ideal of R and each factor module I_{i-1}/I_i is a simple two-sided R-module. R is called an *exact ring* if each simple two-sided R-module I_{i-1}/I_i is exact. By the Jordan–Hölder theorem, the composition factor module series $I_0/I_1, I_1/I_2, \ldots, I_{s-1}/I_s$ is, up to isomorphism and order, uniquely determined by R. Therefore, the notion of the exactness for R depends only on R and independent of the choice of the composition series. We now fix the above composition series once for all, and let $\bar{R}_{\lambda(i)}$ and $\bar{R}_{\rho(i)}$ denote the left and the right simple components belonging to I_{i-1}/I_i, respectively. If R is exact, then the right R-module I_{i-1}/I_i is of finite length (indeed, its length is $n(\lambda(i))$ by Lemma 1(iii)) for each i and consequently the right R-module R is of finite length, i.e., R is right Artinian. Thus the concept of the exactness for R is left–right symmetric.

THEOREM 2. *Let R be an exact ring. Then, for any indices i and α, ${}_R(I_{i-1}e_\alpha/I_ie_\alpha) \cong {}_R\bar{R}\bar{e}_{\lambda(i)}$ or $I_{i-1}e_\alpha = I_ie_\alpha$ according to $\alpha = \rho(i)$ or $\alpha \neq \rho(i)$. In particular, the series of left ideals $Re_\alpha = I_0e_\alpha \supset I_1e_\alpha \supset \cdots \supset I_{s-1}e_\alpha \supset I_se_\alpha = 0$ gives a composition series of the left ideal Re_α if those terms I_ie_α for which $\alpha \neq \rho(i)$ are deleted out of the series.*

Proof. If we observe that $I_{i-1}e_\alpha \cap I_i = I_ie_\alpha$, we have the isomorphism ${}_R(I_{i-1}e_\alpha/I_ie_\alpha) \cong {}_R((I_{i-1}e_\alpha + I_i)/I_i) = {}_R(I_{i-1}/I_i)\,e_\alpha$. Since I_{i-1}/I_i is an exact simple two-sided R-module, it follows from Lemma 1(i) that ${}_R(I_{i-1}/I_i)\,e_\alpha \cong {}_R\bar{R}\bar{e}_{\lambda(i)}$ if $\alpha = \rho(i)$. On the other hand, since $\bar{e}_\alpha \in \bar{R}_\alpha$, we have $(I_{i-1}/I_i)\,e_\alpha = (I_{i-1}/I_i)\,\bar{e}_\alpha = 0$ if $\alpha \neq \rho(i)$. This proves our theorem.

From Theorem 2 and its left–right analogy follows

COROLLARY 3. *Let R be an exact ring. Then, for any indices α and β, the following are equal:*

(a) *The number of indices i such that $\lambda(i) = \alpha$ and $\rho(i) = \beta$.*

(b) *The multiplicity of the simple left R-module $\bar{R}\bar{e}_\alpha$ in the composition factor module series of the left ideal Re_β (i.e., the left Cartan invariant of R corresponding to β, α).*

(c) *The multiplicity of the simple right R-module $\bar{e}_\beta\bar{R}$ in the composition factor module series of the right ideal $e_\alpha R$ (i.e., the right Cartan invariant of R corresponding to α, β).*

Let D_i be the endomorphism ring of the left R-module I_{i-1}/I_i. We regard $\bar{R}_{\rho(i)}$ as a subring of D_i. Then Rosenberg and Zelinsky [8, Lemma 3] virtually proved that the injective envelope E_α of the simple left R-module $\bar{R}\bar{e}_\alpha$ is of finite length if and only if D_i is finitely generated as a left $\bar{R}_{\rho(i)}$-module for every i such that $\lambda(i) = \alpha$. Therefore, as a particular case, if R is exact, then E_α is of finite length for all α. However, by using their method, we can get the following more precise result:

THEOREM 4. *Let R be an exact ring. For any indices i and α, let $r_\alpha(I_i)$ denote the right annihilator of I_i in the injective envelope E_α of the simple left R-module $\bar{R}\bar{e}_\alpha$. Then ${}_R(r_\alpha(I_i)/r_\alpha(I_{i-1})) \cong {}_R\bar{R}\bar{e}_{\rho(i)}$ or $r_\alpha(I_i) = r_\alpha(I_{i-1})$ according to $\alpha = \lambda(i)$ or $\alpha \neq \lambda(i)$. In particular, the series $E_\alpha = r_\alpha(I_s) \supset r_\alpha(I_{s-1}) \supset \cdots \supset r_\alpha(I_1) \supset r_\alpha(I_0) = 0$ gives a composition series of ${}_RE_\alpha$ if those terms $r_\alpha(I_i)$ for which $\alpha \neq \lambda(i)$ are deleted out of the series.*

Proof. Consider the exact sequence of two-sided R-modules

$$0 \to {}_R(I_{i-1}/I_i)_R \to {}_R(R/I_i)_R \to {}_R(R/I_{i-1})_R \to 0.$$

Since ${}_RE_\alpha$ is injective, we have then an exact sequence

$$0 \to {}_R\mathrm{Hom}_R(R/I_{i-1}, E_\alpha) \to {}_R\mathrm{Hom}_R(R/I_i, E_\alpha) \\ \to {}_R\mathrm{Hom}_R(I_{i-1}/I_i, E_\alpha) \to 0.$$

The second and the third terms of this sequence are naturally identified with $r_\alpha(I_{i-1})$ and $r_\alpha(I_i)$, respectively, while the fourth term coincides with $\mathrm{Hom}_R(I_{i-1}/I_i, \bar{R}\bar{e}_\alpha)$ since ${}_R(I_{i-1}/I_i)$ is completely reducible and $\bar{R}\bar{e}_\alpha$ is the only simple submodule of ${}_RE_\alpha$. Thus we have ${}_R(r_\alpha(I_i)/r_\alpha(I_{i-1})) \cong {}_R\mathrm{Hom}_R(I_{i-1}/I_i, \bar{R}\bar{e}_\alpha)$. Since, however, I_{i-1}/I_i is an exact simple two-sided R-module, the right side of this isomorphism is isomorphic to ${}_R\bar{R}\bar{e}_{\rho(i)}$ or $=0$ according to $\alpha = \lambda(i)$ or $\alpha \neq \lambda(i)$ by Lemma 1(ii). This completes the proof of our theorem.

The following is an immediate consequence of Theorem 4 and its left–right analogy:

COROLLARY 5. *Let R be an exact ring. Let α and β be any indices, and let E_α and F_β be the injective envelopes of the simple left R-module $\bar{R}\bar{e}_\alpha$ and the simple right R-module $\bar{e}_\beta\bar{R}$, respectively. Then the following are equal*:

(a) *The number of indices i such that $\lambda(i) = \alpha$ and $\rho(i) = \beta$.*

(d) *The multiplicity of the simple left R-module $\bar{R}\bar{e}_\beta$ in the composition factor module series of E_α.*

(e) *The multiplicity of the simple right R-module $\bar{e}_\alpha\bar{R}$ in the composition factor module series of F_β.*

Remark. According to Corollaries 3 and 5, (a)–(e) given there are all equal if R is exact.

EXAMPLE 1. Every commutative Artinian ring is exact. This is because if R is a commutative ring, then every simple R-module is isomorphic to the factor module R/I modulo a maximal ideal I of R and its endomorphism ring is the factor field R/I.

EXAMPLE 2. Every Artinian semi-simple ring R is exact. For, in this case, the simple components $R_1, R_2, \ldots, R_k$ of R form the two-sided composition factor module series of R and the endomorphism ring of each left R-module R_α is the simple ring R_α itself.

EXAMPLE 3. Let R be a finite-dimensional algebra over a field K; R is then a left and right Artinian ring. R is called *split* if each simple component $\bar{R}_\alpha$ of $\bar{R} = R/J$ is a full matrix algebra over K (which is necessarily of degree

$n(\alpha)$). Now let R be split, and consider two simple components $\bar{R}_\alpha$ and $\bar{R}_\beta$. If we denote by $M_{\alpha,\beta}$ the set of all $n(\alpha) \times n(\beta)$ matrices over K, then $M_{\alpha,\beta}$ is regarded as a two-sided $\bar{R}_\alpha$-$\bar{R}_\beta$-module and indeed it is an exact simple two-sided module, as can be seen easily. Moreover, every simple two-sided $\bar{R}_\alpha$-$\bar{R}_\beta$-module (which is element-wise commutative with K) is isomorphic to $M_{\alpha,\beta}$. For, such a two-sided module is, as is well known, converted into a simple left module of the tensor product algebra $\bar{R}_\alpha \times \bar{R}_\beta$ over K (because $\bar{R}_\beta$ is the opposite algebra of itself), but this algebra is also a full matrix algebra over K (of degree $n(\alpha)\,n(\beta)$) and so has, up to isomorphism, only one simple left module. Thus we know that every split algebra is an exact ring. It follows in particular that if R is a split algebra, then (a)–(c) in Corollary 3 are equal. However, the equality of (b) and (c) was already observed by Nakayama [5, Theorem 3], and indeed it was shown that these conditions are also the same as the dimension of $e_\alpha R e_\beta$ over K. On the other hand, in view of the fact that $M_{\alpha,\beta}$ is the only simple two-sided $\bar{R}_\alpha$-$\bar{R}_\beta$-module for split algebra R, we have that (a) is nothing but the multiplicity of $M_{\alpha,\beta}$ in the two-sided composition factor module series of R.

Remark. Let K be a commutative Artinian ring and R a finite algebra over K, that is, R is an Artinian algebra. Let N be the radical of K and F the injective envelope of the K-module K/N (in case K is a field, F obviously coincides with K). Let $Q = \mathrm{Hom}_K(R, F)$. Then Q is converted into a two-sided R-module in the natural manner, and, as a left R-module, Q is a finitely generated injective cogenerator whose socle is isomorphic to $\bar{R} = R/J$ and whose endomorphism ring coincides with (the right operator ring) R (Azumaya [1, Theorem 19]). Therefore Q defines a Morita duality between the category of finitely generated left R-modules and the category of finitely generated right R-modules [4, Theorem 6.4; 1, Theorem 8]. Moreover, the Q-dual $\mathrm{Hom}_R(\bar{R}\bar{e}_\alpha, Q)_R$ ($\cong \mathrm{Hom}_K(\bar{R}\bar{e}_\alpha, F)_R$) of ${}_R\bar{R}\bar{e}_\alpha$ is isomorphic to $\bar{e}_\alpha \bar{R}_R$ and the Q-dual ${}_R\mathrm{Hom}_R(\bar{e}_\alpha \bar{R}, Q)$ of $\bar{e}_\alpha \bar{R}_R$ is isomorphic to ${}_R\bar{R}\bar{e}_\alpha$ for every α [1, Lemma 2, p. 274]. We now consider the projective right ideal $e_\alpha R$. Since it has a unique maximal right subideal $e_\alpha J$ and $(e_\alpha R/e_\alpha J)_R \cong \bar{e}_\alpha \bar{R}_R$, its Q-dual is an injective left R-module having a unique simple submodule, which is isomorphic to $\bar{R}\bar{e}_\alpha$. Thus the Q-dual of $e_\alpha R$ is isomorphic to the injective envelope E_α of the simple left R-module $\bar{R}\bar{e}_\alpha$. Therefore we know that for each β, (c) in Corollary 3 is equal to (d) in Corollary 5 (even if R is not exact). Similarly, for each β, the Q-dual of the left ideal Re_β is isomorphic to the injective envelope F_β of the simple right R-module $\bar{e}_\beta \bar{R}$, and therefore (b) in Corollary 3 is equal to (e) in Corollary 5 for every α. Taking these phenomena into consideration, we are now tempted to conjecture that every exact ring is self-dual, that is, if R is an exact ring and Q the injective envelope of the R-module $\bar{R} = R/J$, or equivalently, $Q = \sum \oplus E_\alpha^{n(\alpha)}$, then the endomorphism ring of Q is isomorphic to R.

2. Serial Rings

We assume again that R is a left Artinian ring and J, $\bar{R}$, $\bar{R}_\alpha$, e_α, $n(\alpha)$, k, etc. have the same meanings as in Section 1. Suppose now that R has a nonzero injective left ideal L. Then L is a direct summand of R, that is, $R = L \oplus L'$ for a suitable left ideal L' of R. We refine this decomposition into an indecomposable direct decomposition of R. Each indecomposable summand is then isomorphic to one of Re_α's, and it is injective provided it is a summand of L. Thus we know that R has a nonzero injective left ideal if and only if Re_α is injective for some α.

Proposition 6. *Let Re_α be injective, and let M be a faithful left R-module. Then M has a direct summand isomorphic to Re_α.*

Proof. Since R is left Artinian, Re_α contains a simple left subideal S_α. Then that Re_α is indecomposable and injective means that Re_α is the injective envelope of S_α and hence S_α is a unique simple left subideal of Re_α. Since M is faithful, there is a $u \in M$ such that $S_\alpha u \neq 0$. Then the right multiplication of u gives a homomorphism ${}_RRe \to {}_RM$ whose kernel does not contain S_α and so is 0. Thus the homomorphism is a monomorphism, which means that M contains the isomorphic image $Re_\alpha u$ of Re_α. Since Re_α and hence $Re_\alpha u$ is injective, this is a direct summand of M.

Corollary 7. *Let R have an injective left ideal $\neq 0$, and M a faithful indecomposable left R-module. Then there is a unique α such that Re_α is injective, and we have that ${}_RRe_\alpha \cong {}_RM$; M is necessarily injective, of finite length, and has a unique maximal submodule as well as a unique simple submodule.*

Consider now a decomposition of R into a direct sum of indecomposable left ideals. Then the multiplicity of Re_α in the decomposition is $n(\alpha)$ for every α, and thus the decomposition can be written in the following form:

$$R = \sum_{i=1}^{n(1)} \oplus L_1^{(i)} \oplus \sum_{i=1}^{n(2)} \oplus L_2^{(i)} \oplus \cdots \oplus \sum_{i=1}^{n(k)} \oplus L_k^{(i)},$$

where ${}_RL_\alpha^{(1)} \cong {}_RL_\alpha^{(2)} \cong \cdots \cong {}_RL_\alpha^{(n(\alpha))} \cong {}_RRe_\alpha$ for every $\alpha = 1, 2, \ldots, k$.

Theorem 8. *Let Re_α be injective. Then the socle of the left ideal $\sum_{i=1}^{n(\alpha)} \oplus L_\alpha^{(i)}$ is an exact simple two-sided ideal of R.*

Proof. For simplicity, denote n and L_i for $n(\alpha)$ and $L_\alpha^{(i)}$, respectively, and let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Then, for each i $(= 1, 2, \ldots, n)$, ${}_RL_i \cong {}_RRe_\alpha$ and so L_i is left injective and has a unique simple left subideal S_i, which is isomorphic to the unique simple left subideal S_α of Re_α. Moreover, $S = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is the socle of the left ideal L.

First we show that S is a two-sided ideal of R. Suppose not, i.e., suppose that there is an $a \in R$ such that $Sa \not\subset S$. Since $Sa = S_1 a + S_2 a + \cdots + S_n a$, this implies that $S_i a \not\subset S$ for some i. We fix this i, and consider the epimorphism $f: {}_R L_i \to {}_R L_i a$ given by the right-multiplication of a. If $\mathrm{Ker}(f) \neq 0$, then it contains S_i, i.e., $S_i a = 0$, which contradicts to that $S_i a \not\subset S$. Thus $\mathrm{Ker}(f) = 0$, and f is an isomorphism. Therefore, the left ideal $L_i a$ is also injective and $S_i a$ is a unique simple left subideal of $L_i a$. Consider now the intersection $L \cap L_i a$. If this is nonzero, then it must contain $S_i a$; therefore we have $S_i a \subset L$ whence $S_i a \subset S$, which contradicts again to our assumption that $S_i a \not\subset S$. Thus $L \cap L_i a = 0$ and so we have $L + L_i a = L \oplus L_i a = L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus L_i a$. This left ideal is injective, because so is each summand. Therefore it is a direct summand of R, i.e., $R = L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus L_i a \oplus L'$ for some left ideal L'. If we replace L' by an indecomposable direct decomposition of L', then we have an indecomposable direct decomposition of R in which the multiplicity of Re_α is at least $n+1$. But this is a contradiction, due to the Krull–Schmidt theorem. Thus, in any case, our assumption that $Sa \not\subset S$ is impossible. This shows that S is a two-sided ideal of R.

Next we show that S is a simple two-sided ideal. Namely, let T be a nonzero two-sided ideal of R such that $T \subset S$. Since S is a direct sum of simple left ideals $S_1, S_2, \ldots, S_n$ isomorphic to S_α, T has the same structure, that is, T is also a direct sum of simple left ideals isomorphic to S_α. Therefore it follows that each S_i is a homomorphic image of T, i.e., there is an epimorphism $g_i: {}_R T \to {}_R S_i$. Since $S_i \subset L_i$ and L_i is left injective, there exists a $b_i \in L_i$ such that $g_i(x) = xb_i$ for all $x \in T$, and this implies that $S_i = Tb_i \subset T$. Since this is the case for every i, we have $S = S_1 + S_2 + \cdots + S_n \subset T$ whence $S = T$, which shows that S is a simple two-sided ideal.

Finally let $h: {}_R S \to {}_R S$ be any endomorphism of ${}_R S$. Since $S \subset L$ and ${}_R L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ is injective, there exists a $c \in L$ such that $h(y) = yc$ for all $y \in S$. This shows that the simple two-sided R-module S is exact.

COROLLARY 9. *Let R be a left Artinian ring such that every factor ring of R has a nonzero injective left ideal. Then R is an exact ring.*

Proof. Since R itself has a nonzero injective left ideal, R has an exact simple two-sided ideal I_1 by Theorem 8. If $I_1 \neq R$, then consider the factor ring R/I_1. Since it is left Artinian and has a nonzero injective left ideal by assumption, it has an exact simple two-sided ideal, say I_2/I_1 again by Theorem 8, where I_2 is a suitable two-sided ideal of R containing I_1. Continuing in this way, we have a properly ascending chain $I_1 \subset I_2 \subset I_3 \subset \cdots$ of two-sided ideals of R such that $I_1, I_2/I_1, I_3/I_2, \ldots,$ are exact simple two-sided R-modules. But since R is left Artinian (whence Noetherian) the chain terminates at, say sth step, which means that $I_s = R$. Thus $I_s \supset I_{s-1} \supset \cdots \supset I_2 \supset I_1 \supset 0$ gives a two-sided composition series of R, and this shows that R is an exact ring.

Now a left Artinian ring R is called a *left QF-3 ring* if for every simple left ideal S_0 of R its injective envelope $E(S_0)$ is isomorphic to Re_α for some α.

PROPOSITION 10. *Let R be a left Artinian ring such that each Re_α has a unique simple left subideal and the injective envelope of every simple left ideal of R has a unique maximal submodule. Then R is a left QF-3 ring.*

Proof. Let S_0 be a simple left ideal of R. Since R is a direct sum of left ideals each of which is isomorphic to one of $Re_1, Re_2, \ldots, Re_k$, S_0 is isomorphic to the unique simple left subideal S_α of Re_α for some α. Consider then the injective envelope $E(Re_\alpha)$ of this left ideal Re_α. Since Re_α is essential over S_α, $E(Re_\alpha)$ is also essential over S_α and therefore is an injective envelope of S_α. It follows then that $E(Re_\alpha)$ is isomorphic to $E(S_0)$ and so has a unique maximal submodule. But this fact implies that $E(Re_\alpha)$ is a homomorphic image of Re_β for some β; indeed, if U is the unique maximal submodule of $E(Re_\alpha)$ and if ${}_R(E(Re_\alpha)/U) \cong {}_R(\bar{R}\bar{e}_\beta)$, then Re_β is the projective cover of $E(Re_\alpha)$. Let $f: {}_RRe_\beta \to {}_RE(Re_\alpha)$ be an epimorphism. The inverse image $f^{-1}(Re_\alpha)$ of the submodule Re_α of $E(Re_\alpha)$ is mapped onto Re_α by f. Since, however, Re_α is projective, it follows that $\mathrm{Ker}(f)$ is a direct summand of $f^{-1}(Re_\alpha)$, i.e., $f^{-1}(Re_\alpha) = \mathrm{Ker}(f) \oplus V$ for a suitable left subideal V of Re_β. V is nonzero and hence contains the unique simple left subideal S_β of Re_β. This and the fact that $\mathrm{Ker}(f) \cap V = 0$ imply that $\mathrm{Ker}(f)$ cannot contain S_β and therefore $\mathrm{Ker}(f) = 0$, i.e., f is an isomorphism. Thus ${}_RRe_\beta \cong {}_RE(Re_\alpha) \cong {}_RE(S_0)$, which proves that R is a left QF-3 ring.

Let M be a left R-module. M is called *uniserial* if M has a unique composition series. In this case, the finite number of members of the composition series exhaust all submodules of M and so every submodule of M and every homomorphic image of M are uniserial too. Suppose M is of finite length l. Then, as is well known, the following conditions are equivalent:

(a) M is uniserial,

(b) The upper Loewy series $M \supset JM \supset J^2M \supset \cdots \supset J^lM$ $(=0)$ is a composition series of M,

(c) The lower Loewy series $(M=)$ $r_M(J^l) \supset r_M(J^{l-1}) \supset \cdots \supset r_M(J) \supset 0$ is a composition series of M, where $r_M(\;)$ means the right annihilator in M.

Thus if M is uniserial then $JM = r_M(J^{l-1})$, $J^2M = r_M(J^{l-2}), \ldots, J^{l-1}M = r_M(J)$ and l is the least positive integer such that $J^lM = 0$. Now R is called a *serial ring* (generalized uniserial ring in the terminology of Nakayama [6]) if the left ideal Re_α as well as the right ideal $e_\alpha R$ is uniserial for every $\alpha = 1, 2,..., k$. Clearly every serial ring is both left and right Artinian. It was proved by Nakayama [6, Theorem 17] that if R is serial, then every left R-module is a direct sum of submodules each of which is a homomorphic image of one of Re_α's, and in particular every indecomposable left R-module is a homomorphic image of some Re_α and therefore is uniserial. But the injective envelope of every simple left R-module is indecomposable and so uniserial. Therefore it follows from Proposition 10 that *every serial ring is a left QF-3 ring*. This is of course included in the theorem that a ring is serial if and only if every its factor ring is a left QF-3 ring, which was obtained by Morita [4, Theorem 17.8], Wall [10, Theorem 2] (finite-dimensional algebra case), and Fuller [3, Theorem 3.6] (Artinian ring case). Since, however, no proof is given explicitly to the "only if" part of the theorem in any of these papers, the above observation might not be redundant. (Another proof for the proposition is given by using Fuller [2, Theorem (4.1)].) Now, clearly every factor ring of a serial ring is serial and every left QF-3 ring has a nonzero injective left ideal. Thus from Corollary 9 follows

THEOREM 11. *Every serial ring is an exact ring.*

We now want to prove the following characterization of serial rings:

THEOREM 12. *Let R be a left Artinian ring. Then the following conditions are equivalent*:

(1) *R is a serial ring.*

(2) *Every factor ring of R has a nonzero injective left ideal.*

Proof. We need only to prove that (2) implies (1), so that we assume (2). Let M be any indecomposable left R-module. Let I be the annihilator of M in R. Then I is a proper two-sided ideal of R, and M can be regarded as a faithful indecomposable left R/I-module. But since R/I has a nonzero injective left ideal by assumption, it follows from Corollary 7 that M has a unique maximal submodule M_1 and a unique simple submodule U. If $M_1 \neq 0$, then M_1 necessarily contains U and therefore is indecomposable. Then M_1 must contain a unique maximal submodule M_2. If $M_2 \neq 0$, then M_2 contains U and so is indecomposable. Hence M_2 has a unique maximal submodule M_3. In this way we have a sequence $M_1, M_2, M_3,...,$ of submodules of M such that each member is unique maximal in its preceding member. Since, however, M is of finite length by Corollary 7, the sequence must terminate, or equivalently, there is a positive integer r such that $M_r = 0$ and $M, M_1, M_2,..., M_r$ gives a unique composition series of M. Thus we see that condition (2) implies that every indecomposable left R-module is uniserial and in particular

(3) *The left ideal Re_α and the injective envelope E_α of the simple left R-module $\bar{R}\bar{e}_\alpha$ are uniserial for every $\alpha = 1, 2,..., k$.*

It is, however, proved by Fuller [3, Theorem 5.4] that this condition (3) implies condition (1), and thus the proof of our theorem is completed.

Remark. By virtue of Theorem 12, it turns out that Corollary 9 and Theorem 11 represent one and the same fact.

Finally, it might be of some interest to give another proof to the implication (3) ⇒ (1) (Fuller's theorem) by using the following property of exact rings:

PROPOSITION 13. *Let R be an exact ring. Then, for any α, the right ideal $e_\alpha R$ is uniserial if and only if the left R-module E_α is uniserial.*

Proof. Let $R = I_0 \supset I_1 \supset \cdots \supset I_{s-1} \supset I_s = 0$ be a two-sided composition series of the exact ring R, and let $\bar{R}_{\lambda(i)}$ and $\bar{R}_{\rho(i)}$ be the left and the right simple components belonging to I_{i-1}/I_i, respectively. Let t be the number of those integers i between 1 and s for which $\alpha = \lambda(i)$, and let $i(1), i(2),..., i(t)$ be these t integers numbered in increasing order: $i(1) < i(2) < \cdots < i(t)$. Then, according to Theorems 2 and 4, the corresponding series $e_\alpha R \supset e_\alpha I_{i(1)} \supset e_\alpha I_{i(2)} \supset \cdots \supset e_\alpha I_{i(t)} = 0$ and $E_\alpha = r_\alpha(I_{i(t)}) \supset r_\alpha(I_{i(t-1)}) \supset \cdots \supset r_\alpha(I_{i(1)}) \supset 0$ give composition series of $e_\alpha R$ and E_α, respectively, where $r_\alpha(\ )$ means the right annihilator in E_α.

Assume that $e_\alpha R$ is uniserial. Then we have $e_\alpha J^j = e_\alpha I_{i(j)}$ for every j $(= 1, 2,..., t)$. On the other hand, $r_\alpha(J^j) = r_\alpha(e_\alpha J^j)$ and $r_\alpha(I_{i(j)}) = r_\alpha(e_\alpha I_{i(j)})$ for every j by [3, Lemma 1.1(a)]. Therefore we have $r_\alpha(J^j) = r_\alpha(I_{i(j)})$ for every j. Thus the series $E_\alpha = r_\alpha(J^t) \supset r_\alpha(J^{t-1}) \supset \cdots \supset r_\alpha(J) \supset 0$ is a composition series of E_α, which means that E_α is uniserial.

Conversely, suppose that E_α is uniserial. Then we have $r_\alpha(J^j) = r_\alpha(I_{i(j)})$ for every j. But $r_\alpha(J^j) = r_\alpha(e_\alpha J^j)$ and $r_\alpha(I_{i(j)}) = r_\alpha(e_\alpha I_{i(j)})$, as seen above. Moreover, both $e_\alpha J^j$ and $e_\alpha I_{i(j)}$ are, respectively, the left annihilators of $r_\alpha(e_\alpha J^j)$ and $r_\alpha(e_\alpha I_{i(j)})$ in $e_\alpha R$ by [3, Lemma 2.3(b)]. Thus we have $e_\alpha J^j = e_\alpha I_{i(j)}$ for every j, and therefore $e_\alpha R \supset e_\alpha J \supset e_\alpha J^2 \supset \cdots \supset e_\alpha J^t = 0$ gives a composition series of $e_\alpha R$. This means nothing but that $e_\alpha R$ is uniserial.

Having established Proposition 13, we now assume condition (3) above. Then in particular each Re_α has a unique simple left subideal and each E_α has a unique maximal submodule. Hence, by Proposition 10, R is a left QF-3 ring and so has a nonzero injective left ideal. Let I be any proper two-sided ideal of R. Then the factor ring R/I also satisfies the same condition as (3); this is because of the fact that if $\bar{R}\bar{e}_\alpha$ is right annihilated by I, then the left R/I-modules Re_α/Ie_α and $r_\alpha(I)$ become, respectively, the projective cover and the injective envelope of the simple left R/I-module $\bar{R}\bar{e}_\alpha$ by [3, Lemma 4.5] (or by [1, Theorem 17]). Thus it follows that R/I also has a nonzero injective left ideal, and therefore by Corollary 9 R is an exact ring. This, together with the assumption that each E_α is uniserial, implies that each $e_\alpha R$ is uniserial by Proposition 13. Since Re_α is uniserial also by assumption, we show that R is a serial ring. Thus we complete the second proof to the implication $(3) \Rightarrow (1)$.

Remark. A uniserial (left or right) R-module is called *homogeneously uniserial* if its composition factor modules are all isomorphic, while R is called a *uniserial ring* if the left ideal Re_α as well as the right ideal $e_\alpha R$ is homogeneously uniserial for every α. (Although we use here the conventional terminology of uniserial ring, this is clearly inadequate at the present time. We could change it to homo-serial ring, for instance.) It is the well-known theorem by Nakayama, Ikeda, Osima, and Wall that R is uniserial if and only if every factor ring of R is a quasi-Frobenius ring (Nakayama [7, Lemma 2], Wall [9, Theorem 2]). We now want to point out that we can deduce from Corollary 7 a somewhat simplified proof to the "if" part of the theorem. To do so, let R be a quasi-Frobenius ring. Then the left R-module R is injective, and hence its direct summand Re_α is injective for every $\alpha = 1, 2, \ldots, k$. Therefore, if we assume that there is a faithful indecomposable left R-module, then it follows from Corollary 7 that $k = 1$. Thus we have

PROPOSITION 14. *Let R be a quasi-Frobenius ring and let R have a faithful indecomposable left module. Then R has, up to isomorphism, the only simple left R-module (i.e., R is a primary ring).*

Assume now that every factor ring of R is quasi-Frobenius. Then, of course, condition (2) in Theorem 12 is satisfied. Hence we can deduce from Corollary 7 that every indecomposable left R-module M is uniserial, as was shown in the first half of the proof of Theorem 12. Let I be the annihilator of M in R. Then I is a proper two-sided ideal of R, so R/I is a quasi-Frobenius ring, and M can be regarded as a faithful indecomposable left R/I-module. Therefore, it follows from Proposition 14 that R/I has only one simple left module, so that M is homogeneously uniserial. In particular, every indecomposable left ideal Re_α is homogeneously uniserial. Since the notion of quasi-Frobenius rings is left–right symmetric, we should conclude that every indecomposable right ideal $e_\alpha R$ is homogeneously uniserial too, and thus R is a uniserial ring.

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