Some Localizations Which Are Hilbert Rings

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Suppose $K_1, \ldots, K_n$ are fields of transcendence degrees $t_1, \ldots, t_n$, respectively, over a common subfield $F$, where $t_1 \geq t_2 \geq \cdots \geq t_n$, and $t_2 < \infty$. Then the tensor product $R = K_1 \otimes_F K_2 \otimes_F \cdots \otimes_F K_n$ has dimension $t_2 + \cdots + t_n$ [6]. O’Carroll and Qureshi [4] conjectured that $R$ is an equidimensional Hilbert ring, and proved the conjecture in special cases. The conjecture has been proved in general by Trung [7], using results from algebraic geometry.

The object of this paper is to generalize Trung’s result in two directions. Specifically, we prove the following two theorems.

**Theorem 1.** Let $D$ be a commutative domain, and let $\{B_i, i \in I\}$ be a nonempty collection of subrings of $D$, such that:

1. $B = \bigcap_i B_i$ is infinite, of cardinality $\text{Card}(B) > \text{Card}(I)$, or $B$ and $I$ are both finite;
2. $D$ is a finitely generated $B$-algebra.

Let $S$ be the multiplicatively closed set generated by $\bigcup_i (B_i \setminus 0)$. Then $S^{-1}D$ is a Noetherian equidimensional Hilbert ring, of dimension

$$d = \min_i \{t.d.(D \mid B_i)\}.$$
THEOREM 2. Let $D$ be a commutative domain, and let $B_1, \ldots, B_n$ be subrings of $D(n \geq 1)$. Suppose that $B_1$ is chosen so as to satisfy $t.d.(D \mid B_1) = \min_i \{t.d.(D \mid B_i)\}$. Suppose further that:

(i) $B = \bigcap_i B_i$ is infinite;
(ii) $D$ is a finitely generated $B_i$-algebra, $1 \leq i \leq n$;
(iii) $B_2, \ldots, B_n$ are integrally closed.

Let $S$ be the multiplicatively closed set generated by $\bigcup_i (B_i \setminus 0)$. Then $S^{-1}D$ is a Noetherian equidimensional Hilbert ring of dimension

$$d = t.d.(D \mid B_1).$$

(Here, for a given $i$, $t.d.(D \mid B_i)$ denotes the transcendence degree of the quotient field of $D$ over the quotient field of $B_i$.)

The theorem of Trung can readily be recovered from either of these theorems, taking account of the results of [4]. Note also that the Sharp and Vamos formula [6] is a special case of the formula for the dimension of $S^{-1}D$ in either of the theorems.

The similarity of Theorems 1 and 2 might lead one to suspect that they are both special cases of a single, more general, result. However, the following example shows that condition (ii) in Theorem 1 cannot be weakened to agree with condition (ii) in Theorem 2, and also that Theorem 2 cannot be extended to deal with an infinite family of subrings $B_i$. Let $F$ be an arbitrary field and let $I$ be a countably infinite set. Let $\{X_i, i \in I\}$ be a set of indeterminates. Let $D = F[\{X_i, i \in I\}]$, and for each $i \in I$ let $B_i = F[\{X_j, j \in I, j \neq i\}]$. Then $D = \bigcup_i B_i$, so $S = D \setminus 0$ and $S^{-1}D$ is a field, whereas $t.d.(D \mid B_i) = 1$ for all $i \in I$.

However, in the case where $I$ is finite, Theorem 1 follows easily from Theorem 2 or a variant of it (see Section 3).

The proofs of Theorems 1 and 2 follow a similar pattern. In each case the argument uses induction on $d$, and the inductive step boils down to finding a prime ideal of $D$ satisfying certain properties. This is done in two stages. First an element $x \in D$ is found which is transcendental over each $B_i$, and second an element $b \in B$ is found such that consideration of $(x - b)D$ yields the desired ideal. The methods used involve algebraic geometry in the case of Theorem 1, and a "Going down" argument in the case of Theorem 2.

In Section 1 we prove Theorem 1 under the hypothesis that $B$ is infinite with $\text{card}(B) > \text{card}(I)$. The result in the remaining case, where $B$ and $I$ are both finite, can then be deduced by an argument involving standard algebraic technicalities; it seems best therefore to postpone consideration of the latter case until Section 3. This section also contains remarks on the relationship between Theorems 1 and 2 when $I$ is finite and on what can be said in the
situation of Theorem 2 when $B$ is finite. A variant of Theorem 2 is also considered.

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1. Proof of Theorem 1

In this section we suppose that $B$ is infinite, with $\text{card}(B) > \text{card}(I)$.

Let $T = B \setminus 0$ and let $S_0$ denote the multiplicatively closed set in $T^{-1}D$ generated by $\bigcup_i (T^{-1}B_i \setminus 0)$. Then $S_0^{-1}(T^{-1}D) = S^{-1}D$; $T^{-1}D$ is a finitely generated $T^{-1}B$-algebra; $T^{-1}B$ is an infinite subfield of $\bigcap_i T^{-1}B_i$, with $\text{Card}(T^{-1}B) = \text{Card}(B) > \text{Card}(I)$; and $t.d.(T^{-1}D \mid T^{-1}B_i) = t.d.(D \mid B_i)$ for each $i \in I$.

Hence, passing to $T^{-1}D$, etc., there is no loss of generality in assuming that $B$ contains an infinite subfield $F$ such that $\text{Card}(F) > \text{Card}(I)$ and $D$ is a finitely generated $F$-algebra. Clearly, therefore, $S^{-1}D$ is Noetherian. The remaining statements in the theorem are proved simultaneously by induction on $d$.

It is worth remarking at this point that $t.d.(D \mid F) = \dim D$ and $t.d.(B_i \mid F) = \dim B_i$ for each $i \in I$, since $B_i$ is a subalgebra of the finitely generated $F$-algebra $D$ (see, for example, [3, 2.3(b)]). This gives an alternative expression for $d$ as

$$d = \dim D - \max_i \{\dim B_i\}.$$ 

In the case $d = 0$, there exists $j \in I$ such that $t.d.(D \mid B_i) = 0$, that is, $D$ is algebraic over $B_j$. Hence $S^{-1}D$ is algebraic over the quotient field of $B_j$, and so is a field. Thus the result holds in this case.

From now on assume that $d > 1$, and that the result holds in any similar situation giving rise to a smaller value for $d$.

To prove equidimensionality, let $0 \subset P_1 \subset \cdots \subset P_c$ be a saturated chain of prime ideals in $S^{-1}D$ with $P_c$ a maximal ideal, and suppose initially that $c \neq 0$ (of course $\dim S^{-1}D$ is at most $d$, so is finite). Then $D \cap P_1$ is a prime ideal of height $1$ in $D$. Let $\overline{D} = D/(D \cap P_1)$, and let $\overline{S}, \overline{B}$ denote the images of $S, B_1$, respectively, in $\overline{D}$. Then $\overline{D}$ is a domain, finitely generated as an $F$-algebra, and of dimension $(\dim D - 1)$. Furthermore, each $B_i$ is mapped isomorphically onto $\overline{B_i}$ by the natural map, so $\dim \overline{B_i} = \dim B_i$; and $\overline{S}$ is the multiplicatively closed set generated by $\bigcup_i (\overline{B_i} \setminus 0)$. Hence $(S^{-1}D)/P_1 \cong (\overline{S})^{-1}(\overline{D})$ is an equidimensional Hilbert ring of dimension $d - 1$, by inductive hypothesis. In addition, the chain $0 \subset P_2/P_1 \subset \cdots \subset P_c/P_1$ of prime ideals in $(S^{-1}D)/P_1$ is saturated, with $P_c/P_1$ maximal, from which it follows
that $c = d$, which is precisely the property of equidimensionality. It therefore remains only to show that $c \neq 0$, in other words that $S^{-1}D$ contains a non-zero prime ideal.

Next suppose $P$ is a prime ideal in $S^{-1}D$, and $a$ is an element of $S^{-1}D$ with $a \notin P$. The Hilbert property requires the existence of a maximal ideal $M$ of $S^{-1}D$ such that $P \subseteq M$ and $a \notin M$. An argument similar to the above, applying the inductive hypothesis to $D/(D \cap P)$, will work provided $P \neq 0$. Moreover, in the case $P = 0$, it is sufficient to find a non-zero prime ideal $Q$ in $S^{-1}D$ with $a \notin Q$, for then the same argument may be applied once more, replacing $P$ by $Q$.

Now, after multiplying by a unit if necessary, we may assume $a \in D$. Also, the prime ideals of $S^{-1}D$ are in 1-1 correspondence with those prime ideals of $D$ which have zero intersection with each $B_i$. Thus the inductive step in the proof of the theorem, both for equidimensionality and for the Hilbert property, reduces to the following proposition.

**PROPOSITION 1.1.** Let $D, \{B_i, i \in I\}$ be as in Theorem 1, and suppose that $t.d.(D \mid B_i) > 0$ for all $i \in I$. Let $a$ be a non-zero element of $D$. Then there exists a non-zero prime ideal $Q$ of $D$ with $a \notin Q$ and $Q \cap B_i = 0$ for all $i \in I$.

Before proving the general case, let us dispose of the case $\dim D = 1$, that is, $t.d.(D \mid F) = 1$. Since $t.d.(D \mid B_i) > 0$ for all $i \in I$, it follows that each $B_i$ is algebraic over $F$, and so a field. But $D$ is not a field, and it clearly suffices to find a maximal ideal $Q$ of $D$ with $a \notin Q$. But this is immediate, since $D$ is a finitely generated $F$-algebra, and so a Hilbert ring.

From now on assume that $\dim D = t.d.(D \mid F) > 2$. Write $D$ as $F[x_1, \ldots, x_m]$ for some $x_1, \ldots, x_m \in D$. Let $\mathcal{A}$ be the polynomial algebra $F[X_1, \ldots, X_m]$ and let $\psi: \mathcal{A} \rightarrow D$ be the $F$-algebra epimorphism given by $\psi(X_i) = x_i$. Let $\theta: F^m+1 \rightarrow \mathcal{A}$ be the $F$-linear map

$$u = (u_0, \ldots, u_m) \mapsto u_0 + u_1 X_1 + \cdots + u_m X_m,$$

and let $\phi = (\psi \circ \theta): F^m+1 \rightarrow D$.

Since $D$ is a domain, the ideal $P = \ker \psi$ of $\mathcal{A}$ is prime, of height $m - \dim D \leq m - 2$, so Theorem 12 of [5] says that $P + A\theta(u)$ is a prime ideal of height $m - 1$ in $\mathcal{A}$, for all $u$ in the complement of some proper algebraic subvariety $V_0$ of $F^m+1$. In other words $D\phi(u)$ is a prime ideal of height 1 in $D$, for all $u \in F^m+1 \setminus V_0$. The proposition will be proved using this fact, together with the following Lemmas.

**Lem 1.2.** In the above notation, let $A$ be any $F$-subalgebra of $D$ such that $t.d.(D \mid A) \neq 0$. 
(i) The set $V = \{ u \in F^{m+1}, \phi(u) \text{ is algebraic over } A \}$ is a proper $F$-vector subspace of $F^{m+1}$ (and hence a proper algebraic subvariety).

(ii) For any $x \in D \setminus \phi(V)$, the sets

$$U = \{ f \in F, D(x-f) \cap A \neq 0 \}$$

and

$$U' = \{ f \in F, a \in D(x-f) \}$$

are finite, where $a \in D \setminus 0$.

**Lemma 1.3.** Let $F$ be an infinite field, let $r \geq 1$ be an integer, and let $\{ V_i, i \in I \}$ be a family of proper algebraic subvarieties of $F^r$, where $\text{Card}(I) < \text{Card}(F)$. Then $V = \bigcup_i V_i \neq F$.

**Proof of 1.2.** (i) Let $K, L$ denote the quotient fields of $A, D$, respectively, and let $K'$ denote the algebraic closure of $K$ in $L$. Then $K'$ is an $F$-vector subspace of $L$, and $V$ is the inverse image of $K'$ under the $F$-linear map

$$F^{m+1} \xrightarrow{\phi} D \hookrightarrow L.$$ 

Hence $V$ is an $F$-vector subspace of $F^{m+1}$. That $V$ is proper follows from the facts that $\phi(F^{m+1})$ generates $D$ as an $A$-algebra and that $D$ is not algebraic over $A$.

(ii) By definition of $V$, the element $x \in D \setminus \phi(V)$ is transcendental over $A$. Let $C$ denote the subring $A[x]$ of $D$, and let $\overline{K}$ denote the algebraic closure of the quotient field $K$ of $A$. For each $f \in F$, let $h_f$ denote the unique homomorphism $C \xrightarrow{\sim} \overline{K}$ such that $h_f(b) = b$ (for all $b \in A$) and $h_f(x) = f$. By [1, 5.23] there exists an element $g \in C \setminus 0$ such that whenever $h_f(g) \neq 0$, there is an extension $\overline{h}_f : D \xrightarrow{\sim} \overline{K}$ of $h_f$, with $\overline{h}_f(a) \neq 0$.

If $\overline{h}_f$ exists, then $a \notin \ker \overline{h}_f \ni D(x-f)$, so $f \notin U'$; and $D(x-f) \cap A \subseteq (\ker \overline{h}_f) \cap A = 0$, so $f \notin U$. Furthermore $h_f(g)$ can be written as $g(f)$, where $g = g(x) \in C = A[x] \subseteq \overline{K}[x]$ is regarded as a polynomial in the indeterminate $x$.

Since $g \neq 0$ in $\overline{K}[x]$, the sets $U$ and $U'$ are finite, being subsets of the set of roots (in $\overline{K}$) of $g$.

**Proof of 1.3.** We use induction on $r$. If $r = 1$ then each $V_i$ is a finite set, so $\text{Card}(V) < \text{Card}(F)$, whence $V \neq F$.

Suppose $r \geq 2$, and that the result holds for varieties in $F^{r-1}$. First note that each $V_i$ is a union of finitely many irreducible varieties, so we may replace $\{ V_i, i \in I \}$ by $\{ V'_j, j \in J \}$, where each $V'_j$ is a proper irreducible algebraic subvariety of $F^r$, $V = \bigcup_j V'_j$, and $\text{Card}(J) < \text{Card}(F)$. 


Suppose $H \subseteq F'$ is a hyperplane. If $H \subseteq V_j$ for some $j \in J$, then $H = V_j$, since $H$ is maximal among irreducible subvarieties of $F'$. Since there are $\text{Card}(F) > \text{Card}(J)$ hyperplanes in $F'$, we may choose one, $H_0$, say, such that $H_0 \notin V_j$ for all $j \in J$. In other words $H_0 \cap V_j$ is a proper algebraic subvariety of $H_0$ for all $j \in J$. Parametrizing $H_0$ as $F'^{-1}$ and applying the inductive hypothesis, we have

$$V \cap H_0 = \bigcup_j (V_j \cap H_0) \neq H_0,$$

whence $V \neq F'$, as claimed.

Proof of 1.1. By part (i) of Lemma 1.2 there exist proper $F$-vector subspaces $\{V_i, i \in I\}$ of $F^{m+1}$ such that $\phi(u)$ is transcendental over $B_i$ whenever $u \in F^{m+1} \setminus V_i$. Then $V = V_0 \cup (\bigcup_i V_i)$ is a proper subset of $F^{m+1}$, by Lemma 1.3 (where $V_0$ is the subvariety specified in the discussion following the statement of Proposition 1.1).

Fix $u = (u_0, u_1, \ldots, u_m) \in F^{m+1} \setminus V_i$, and define $x = \phi(u)$. Let $U_0$ be the set $\{f \in F, (u_0 - f, u_1, \ldots, u_m) \in V_0\}$. Then $U_0$ is a proper subvariety of $F$, and so is a finite set. By Lemma 1.2(ii) there are finite subsets $U_i (i \in I)$ and $U'$ of $F$ such that $D(x - f) \cap B_i = 0$ for $f \notin U_i$ (i.e., $f \notin U_i$), and $a \notin D(x - f)$ for $f \notin U'$.

Since $\text{Card}(F) > \text{Card}(I)$ and $F$ is infinite, the set $U = U_0 \cup U' \cup (\bigcup_i U_i)$ is a proper subset of $F$.

Fix $f \in F \setminus U$, and let $Q = D(x - f)$. Then $Q$ is a prime ideal of height 1 in $D$ (since $f \notin U_0$), with $Q \cap B_i = 0$ (since $f \notin U_i$) for each $i \in I$, and $a \notin Q$ (since $f \notin U'$).

2. Proof of Theorem 2

As in Section 1, it is clear that $S^{-1}D$ is Noetherian of dimension at most $d$ (since $S^{-1}D$ is a localization of $(B_1 \setminus 0)^{-1}D$), and that if $d = 0$ then $S^{-1}D$ is a field. So suppose that $d > 1$. A similar argument to that in Section 1 establishes that it suffices to show, given $a \in D \setminus 0$, that there exists a non-zero prime ideal $P$ in $D$ such that $a \notin P$ and $P \cap B_i = 0$, $1 \leq i \leq n$. So fix $a \in D \setminus 0$, and let $F$ denote the quotient field of $B$.

A simplified version of the argument presented in Section 1 is sufficient here to show that there exists $x \in D$ such that $x$ is transcendental over each $B_i$. Let $\{x_1, \ldots, x_m\}$ be a common generating set for $D$ as a $B_i$-algebra, $1 \leq i \leq n$; such a set certainly exists. For each $i = 1, \ldots, n$, let

$$V_i = \{(a_1, \ldots, a_m) \in F^m, a_1 x_1 + \cdots + a_m x_m \text{ is algebraic over } B_i\}.$$
Each $V_i$ is an $F$-subspace of $F^m$, and is in fact a proper subspace, since 
\{$x_1, \ldots, x_m$\} generates $D$ as a $B_i$-algebra and $D$ is not algebraic over $B_i$.
However, $F$ is an infinite field, so there exists $(a_1, \ldots, a_m) \in B^m$ such that 
$x := a_1 x_1 + \cdots + a_m x_m$ is transcendental over each $B_i$.

Fix $i$ such that $2 \leq i \leq n$. Extend $B_i[x]$ to a finite polynomial extension 
$C^{(i)} := B_i[x, y, \ldots, z]$ (with $x, y, \ldots, z$ algebraically independent over $B_i$) such that 
$D$ is algebraic over $C^{(i)}$. Since $D$ is a finitely generated $C^{(i)}$-algebra, there exists 
$c_i \in C^{(i)} \setminus 0$ such that $C^{(i)} c_i \subseteq D c_i$ is an integral extension. Let 
e = ac_2 \cdots c_n$, so $e \neq 0$. Then there exists $f \in B_i[x] \setminus 0$ such that any prime 
ideal in $B_i[x]$ which excludes $f$ lifts to a prime ideal in $D$ which excludes $e$ (cf. [1, 5.23]). Consider $f$ as a polynomial over $B_i$, and pick any element 
b \in B which is not a root of $f$ (such an element exists since $B$ is infinite). The 
ideal $(x - b) B_i[x]$ is the kernel of the retraction $B_i[x] \rightarrow B_i$ which maps $x$ to $b$, and so is a prime ideal of $B_i[x]$ which excludes $f$. (Note also that for 
similar reasons
\[(x - b) B_i[x]) \cap B_i = 0, \quad 1 \leq i \leq n.\]
Hence there exists a prime ideal $P$ in $D$ such that $e \notin P$ and 
P \cap B_i[x] = (x - b) B_i[x].$ Without loss of generality, we may shrink $P$ so that it is 
minimal over $(x - b) D$. We now show that it is also the case that 
P \cap B_i[x] = (x - b) B_i[x], \quad 2 \leq i \leq n.
It then follows that 
P \cap B_i = ((x - b) B_i[x]) \cap B_i = 0, \quad 1 \leq i \leq n.
Moreover $a \notin P$, since $e \notin P$.
Fix $i$ such that $2 \leq i \leq n$. Now $C^{(i)}$ is integrally closed, since $B_i$ is, and 
hence $C^{(i)} c_i$ is also integrally closed. Moreover $C^{(i)} c_i \subseteq D c_i$ is an integral 
extension, and $P$ survives in $D c_i$ (for $c_i \notin P$, since $e \notin P$). The extension 
$C^{(i)} c_i \subseteq D c_i$ satisfies “Going down,” by the Cohen–Seidenberg Theorem 
[1, 5.16]. The extension $B_i[x] \subseteq C^{(i)} c_i$ also satisfies “Going down,” since it is 
a composite of flat extensions, and so a flat extension. Hence the prime ideal 
P \cap B_i[x] is minimal over the ideal $(x - b) B_i[x]$, which is itself prime, and 
so these ideals are in fact equal.

3. Remarks

1. Suppose further, in the situation of Theorem 1, that $I$ is finite but 
that $B$ is infinite. We remark that in this case, Theorem 1 can be deduced 
from Theorem 2. The argument at the beginning of Section 1 shows (since $I$ 
is now finite) that we may suppose that $B$ is a field, which we rename $F$, and
that $D$ is an affine domain over $F$. Let $\overline{D}$ denote the integral closure of $D$, $\overline{B}_i$ the integral closure of $B_i$, $1 \leq i \leq n$, and $\mathcal{S}$ the multiplicatively closed set in $\overline{D}$ generated by $\bigcup_i (\overline{B}_i \setminus \{0\})$. As in [4, Sect. 1] it follows that $S^{-1}D \subseteq \mathcal{S}^{-1}\overline{D}$ is an integral extension; moreover, $\overline{D}$ is an affine domain over $F$. The hypotheses of Theorem 2 apply to $\overline{D}$ together with the $\overline{B}_i$, and it is clear that for each $i$

\[
t.d.(D | B_i) = t.d.(\overline{D} | \overline{B}_i).
\]

So in the situation where $d \geq 1$, let $a \in D \setminus 0$. On considering $a \in \overline{D} \setminus 0$, Theorem 2 provides us with a non-zero prime ideal $P$ of $\mathcal{S}^{-1}\overline{D}$ such that $a \notin P$. Let $Q = P \cap S^{-1}D$. Since the extension $S^{-1}D \subseteq \mathcal{S}^{-1}\overline{D}$ is integral and is composed of domains, $Q \neq 0$; moreover $a \notin Q$, so Theorem 1 follows (see the first part of Section 1).

2. Suppose that we are in the situation of Theorem 1, with $B$ and $I$ both finite, and with $d \geq 1$ (the case $d = 0$ being trivial). Then $B$ is a field, which we rename $F$, and we consider the effect of applying the functor $\overline{F} \otimes -$ where $\overline{F}$ is the algebraic closure of $F$ and the tensor product is taken over $F$. Consider $\overline{F} \otimes D$ and $\overline{F} \otimes B_i$ $(1 \leq i \leq n)$ where, without loss of generality, we take $\overline{F} \otimes B_i \subseteq \overline{F} \otimes D$. Then $\overline{F} = \overline{F} \otimes F = \overline{F} \otimes (\bigcap_i (\overline{F} \otimes B_i))$. Now $D \subseteq \overline{F} \otimes D$ is an integral, faithfully flat extension (cf. [4, Sect. 1]), and $\overline{F} \otimes D$ is a finitely generated $\overline{F}$-algebra. Let $a \in D \setminus 0$, and let $P$ be a minimal prime ideal in $\overline{F} \otimes D$. Then $P \cap D = 0$ by "Going down." For $i = 1, \ldots, n$, let $P_i = P \cap (\overline{F} \otimes B_i)$ and consider $\overline{D} = \overline{F} \otimes D/P$ and $\overline{B}_i = \overline{F} \otimes B_i/P_i$ where we take $\overline{B}_i \subseteq \overline{D}$ and $D \subseteq \overline{D}$. Then $\overline{F} \subseteq \bigcap_i \overline{B}_i$ in $\overline{D}$, and $\overline{D}$ is an affine domain over $\overline{F}$. The hypotheses of Theorem 1 apply to $D$ together with the $\overline{B}_i$. Note that $D \subseteq \overline{D}$ is an integral extension of domains, and that $B_i \subseteq \overline{B}_i$ is also an integral extension; hence $t.d.(\overline{D} | \overline{B}_i) = t.d.(\overline{D} | \overline{B}_i)$.

Let $a \in D \setminus 0$, and consider $a \in \overline{D} \setminus 0$. Then Theorem 1, applied to $\overline{D}$, provides us with a non-zero prime ideal $Q$ of $\overline{D}$ such that $a \notin Q$ and $Q \cap \overline{B}_i = 0$, $1 \leq i \leq n$. Let $Q$ be the preimage of $\overline{Q}$ in $\overline{F} \otimes D$, and let $Q' = Q \cap D$. As before, $Q'$ is a non-zero prime ideal in $D$ since the extension $D \subseteq \overline{F} \otimes D$ is integral; clearly $a \notin Q'$ and $Q' \cap B_i = 0$, $1 \leq i \leq n$ (since $P \cap D = 0$).

Hence Theorem 1 extends to cover the case where $B$ and $I$ are finite.

3. Suppose that we are now in the situation of Theorem 2 except that $B$ is now finite, and so a field, which again we rename $F$. Suppose further that $F$ is algebraically closed in the quotient field of $D$. We again consider the effect of applying $\overline{F} \otimes -$ . Using [8, Corollary 2, p. 198] and [2, Proposition 19, p. 318], and arguing as in Remark 2, one sees that the consequences of Theorem 2 descend from the domain $\overline{F} \otimes D$ (with subrings $\overline{F} \otimes B_i$, $1 \leq i \leq n$) to $D$ (with subrings $B_i$, $1 \leq i \leq n$).
4. Consider a variant of Theorem 2, where we delete hypotheses (i) and (iii), but add the new hypothesis that $D$ be integrally closed. For $i = 1, \ldots, n$, let $\bar{B}_i$ denote the integral closure of $B_i$ in $D$. Then each $\bar{B}_i$ is the integral closure of $B_i$ in the quotient field of $D$, and it follows easily that $\bar{B} := \bigcap_i \bar{B}_i$, if finite, is algebraically closed in the quotient field of $D$. Let $\bar{S}$ be the multiplicatively closed set generated by $\bigcup_i (\bar{B}_i \setminus 0)$. Noting that $\text{t.d.}(D | B_i) = \text{t.d.}(D | \bar{B}_i)$, and considering Remark 3 if $B$ is finite, the consequences of Theorem 2 hold for $\bar{S}^{-1}D$, and as in Remark 1, these consequences descend to $S^{-1}D$.

This variant of Theorem 2 again generalizes Trung's result, and again (as in Remark 1) one can deduce from it the particular case of Theorem 1 where $I$ (only) is assumed finite, and $B$ may be finite or infinite.

We can also deduce, from the two theorems, variants to cover the situation where we no longer suppose $D$ to be a domain, but suppose only that each $B_i$ is a domain. For if $a \in D \setminus 0$ and if $P$ is a prime ideal in $D$ such that $a \notin P$ and $P \cap B_i = 0$ for each $i$, we may pass to the domain $D/P$ and apply the relevant theorem there. The exact statements of the results are rather messy, so they are not given explicitly.

5. Finally, we are left with the open question as to what can be said about the situation in Remark 3 if we no longer suppose that $B$ is algebraically closed in the quotient field of $D$.

Note added in proof. M. Nagata has also independently obtained a proof of the conjecture of O'Carroll and Qureshi. The second author has obtained an answer to the open question in Remark 5; details will appear elsewhere.

References


As K. A. Brown has pointed out, the argument in Remark 3 can be extended to the situation where the algebraic closure of $F$ in the quotient field of $D$ is a finite extension of $F$. One tensors by a suitable infinite subfield of $\bar{F}$, and the argument proceeds as before.