A Further Algebraic Version of Cochran's Theorem and Matrix Partial Orderings*

Jerzy K. Baksalary
Department of Mathematics
Pedagogical University
PL-65-069 Zielona Góra, Poland
and
Department of Mathematical and Statistical Methods
Academy of Agriculture
PL-60-637 Poznań, Poland

and

Jan Hauke
Department of Mathematical and Statistical Methods
Academy of Agriculture
PL-60-637 Poznań, Poland

Submitted by George P. H. Styan

ABSTRACT

A new version of Cochran's theorem for rectangular matrices is established. Being oriented toward partial isometries, the new version parallels corresponding results concerned with arbitrary tripotent matrices and covers results concerned with Hermitian tripotent matrices. A discussion of a related new matrix partial ordering is also given.

1. INTRODUCTION

Let $\mathbf{x}$ be a $p\times 1$ random vector distributed normally with expectation $\mathrm{E}(\mathbf{x})=\mathbf{0}$ and with dispersion matrix $D(\mathbf{x})=\mathbf{I}_p$. Let $\mathbf{A}_1,\dots,\mathbf{A}_k$ be symmetric matrices with ranks $r_1,\dots,r_k$, respectively, let $q_i=\mathbf{x}'\mathbf{A}_i\mathbf{x}$, $i=1,\dots,k$, and

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suppose that $\Sigma q_i = \mathbf{x}'\mathbf{x}$. Then Theorem II of Cochran (1934) asserts that a necessary and sufficient condition for $q_1, \ldots, q_k$ to be independently distributed as chi-square variables is that $\Sigma r_i = p$.

This result attracted considerable attention in the literature and was generalized in several ways: (1) by allowing that $\mathrm{E}(\mathbf{x}) = \boldsymbol{\mu}$ and $D(\mathbf{x}) = \mathbf{V}$, where $\boldsymbol{\mu}$ is any $p \times 1$ vector and $\mathbf{V}$ is any $p \times p$ nonnegative definite matrix, (2) by considering general second-degree polynomials $\mathbf{x}'\mathbf{A}_i\mathbf{x} + 2\mathbf{a}_i'\mathbf{x} + \alpha_i$ instead of quadratic forms $\mathbf{x}'\mathbf{A}_i\mathbf{x}$, and (3) by considering linear combinations of independent chi-squares instead of pure chi-square variables. There are also numerous generalizations of Cochran's theorem from the algebraic standpoint, in which the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$ are allowed to be arbitrary complex matrices. We refer the reader to Scarowsky (1973), Khatri (1977, 1980), and Anderson and Styan (1982) for exhaustive discussions and bibliography concerning this problem.

Certain conditions involved in various versions of Cochran's theorem are expressible in terms of matrix partial orderings; see, e.g., Hartwig (1981), Baksalary and Hauke (1984, 1987), Hartwig and Styan (1986). The star partial ordering $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$, the minus partial ordering $\mathbf{K} \overset{-}{\leqslant} \mathbf{L}$, and the space preordering (or, in another terminology, quasiordering) $\mathbf{K} \overset{s}{\prec} \mathbf{L}$ in the set of $p \times q$ complex matrices are defined as

$$\mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Leftrightarrow \quad \mathbf{K}^*\mathbf{K} = \mathbf{K}^*\mathbf{L} \text{ and } \mathbf{K}\mathbf{K}^* = \mathbf{L}\mathbf{K}^*, \tag{1.1}$$

$$\mathbf{K} \overset{-}{\leqslant} \mathbf{L} \quad \Leftrightarrow \quad \mathbf{K}^-\mathbf{K} = \mathbf{K}^-\mathbf{L} \text{ and } \mathbf{K}\mathbf{K}^= = \mathbf{L}\mathbf{K}^= \quad \text{for some } \mathbf{K}^-, \mathbf{K}^= \in \mathbf{K}\{1\}, \tag{1.2}$$

$$\mathbf{K} \overset{s}{\prec} \mathbf{L} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{K}) \subseteq \mathscr{R}(\mathbf{L}) \text{ and } \mathscr{R}(\mathbf{K}^*) \subseteq \mathscr{R}(\mathbf{L}^*), \tag{1.3}$$

respectively, where $\mathbf{K}^*$ is the conjugate transpose of $\mathbf{K}, \mathbf{K}\{1\}$ in (1.2) denotes the set of (1)-inverses of $\mathbf{K}$ (i.e., the set of all $q \times p$ matrices $\mathbf{K}^-$ satisfying $\mathbf{K}\mathbf{K}^-\mathbf{K} = \mathbf{K}$), and $\mathscr{R}(\cdot)$ in (1.3) denotes the range of a matrix. The definition of the star ordering as in (1.1) was given by Drazin (1978). The definition of the minus ordering as in (1.2) is Hartwig and Styan's (1986, p. 146) version of the definition originally introduced by Hartwig (1980). It was shown by Hartwig that

$$\mathbf{K} \overset{-}{\leqslant} \mathbf{L} \quad \Leftrightarrow \quad \mathrm{r}(\mathbf{L} - \mathbf{K}) = \mathrm{r}(\mathbf{L}) - \mathrm{r}(\mathbf{K}), \tag{1.4}$$

where $\mathrm{r}(\cdot)$ stands for the rank of a matrix. Hence, according to Marsaglia and

Styan (1974, p. 288) and Cline and Funderlic (1979, p. 195),

$$\mathbf{K} \overset{-}{\leqslant} \mathbf{L} \quad \Leftrightarrow \quad \mathbf{L}\mathbf{L}^{-}\mathbf{K} = \mathbf{K}\mathbf{L}^{=}\mathbf{L} = \mathbf{K}\mathbf{L}^{\equiv}\mathbf{K} = \mathbf{K}$$

$$\text{for some } \mathbf{L}^{-}, \mathbf{L}^{=}, \mathbf{L}^{\equiv} \in \mathbf{L}\{1\}. \quad (1.5)$$

An interesting property of the relations defined in (1.1), (1.2), and (1.3) is that

$$\mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} \overset{-}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} \overset{s}{<} \mathbf{L}. \quad (1.6)$$

Among algebraic versions of Cochran's theorem are those dealing with rectangular matrices; cf. Theorems 13 and 14 of Marsaglia and Styan (1974) and Theorem 1.2 of Anderson and Styan (1982). In this paper, a new such version is established. Involving conditions $\mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_i = \mathbf{A}_i$, $i = 1,\ldots,k$, and $\mathbf{A}\mathbf{A}^*\mathbf{A} = \mathbf{A}$, where $\mathbf{A} = \Sigma\mathbf{A}_i$, it may be viewed as oriented toward partial isometries. Consequently, it parallels corresponding results concerned with arbitrary tripotent matrices and covers results concerned with Hermitian tripotent matrices. It appears that combining certain two conditions that occur in the new version of Cochran's theorem leads to a new partial ordering. Some of its properties are discussed in the final section of the paper.

2. MAIN RESULT

In addition to the notation introduced in Section 1, $\text{tr}(\cdot)$ will denote the trace of a square matrix.

THEOREM 1. *Let $\mathbf{A}_1,\ldots,\mathbf{A}_k$ be $p \times q$ matrices, and let $\mathbf{A} = \Sigma\mathbf{A}_i$. Consider the following statements*:

(a) $\mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_i = \mathbf{A}_i$, $i = 1,\ldots,k$,
(b) $\mathbf{A}_i\mathbf{A}_j^* = \mathbf{0}$ *and* $\mathbf{A}_i^*\mathbf{A}_j = \mathbf{0}$, $i, j = 1,\ldots,k$, $i \neq j$,
(c) $\mathbf{A}\mathbf{A}^*\mathbf{A} = \mathbf{A}$,
(d) $\text{r}(\mathbf{A}) = \Sigma\text{r}(\mathbf{A}_i)$,
(e_0) $\mathbf{A}_i\mathbf{A}_i^*\mathbf{A} = \mathbf{A}_i$ *and* $\mathbf{A}\mathbf{A}_i^*\mathbf{A}_i = \mathbf{A}_i$, $i = 1,\ldots,k$,
(e_1) $\mathbf{A}_i\mathbf{A}^* = \mathbf{A}_i\mathbf{A}_i^*$ *and* $\mathbf{A}_i^*\mathbf{A} = \mathbf{A}_i^*\mathbf{A}_i$, $i = 1,\ldots,k$,
(e_2) $\mathbf{A}_i\mathbf{A}^* = \mathbf{A}\mathbf{A}_i^*$ *and* $\mathbf{A}_i^*\mathbf{A} = \mathbf{A}^*\mathbf{A}_i$, $i = 1,\ldots,k$,
(e_3) $\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_i$, $i = 1,\ldots,k$,
(e_4) $\mathscr{R}(\mathbf{A}_i) \subseteq \mathscr{R}(\mathbf{A})$ *and* $\mathscr{R}(\mathbf{A}_i^*) \subseteq \mathscr{R}(\mathbf{A}^*)$, $i = 1,\ldots,k$,
(f_1) $\text{tr}(\mathbf{A}_i\mathbf{A}^*) \geqslant \text{tr}(\mathbf{A}_i\mathbf{A}_i^*)$, $i = 1,\ldots,k$,
(f_2) $\text{tr}(\mathbf{A}\mathbf{A}^*) \geqslant \Sigma\,\text{tr}(\mathbf{A}_i\mathbf{A}_i^*)$,
(f_3) $\text{r}(\mathbf{A}_i) \geqslant \text{tr}(\mathbf{A}_i\mathbf{A}_i^*)$, $i = 1,\ldots,k$.

Then

$$(\mathrm{d}),(\mathrm{e}_0) \Rightarrow (\mathrm{b}) \Leftrightarrow (\mathrm{e}_1) \Leftrightarrow (\mathrm{d}),(\mathrm{e}_2)$$
$$\Rightarrow (\mathrm{d}),(\mathrm{e}_3) \Rightarrow (\mathrm{e}_3),(\mathrm{e}_4), \quad (2.1)$$

$$(\mathrm{a}),(\mathrm{b}) \Leftrightarrow (\mathrm{b}),(\mathrm{c}), \quad (2.2)$$

and

$$(\mathrm{x}),(\mathrm{e}_0) \Leftrightarrow (\mathrm{d}),(\mathrm{e}_0) \Leftrightarrow (\mathrm{x}),(\mathrm{b}) \Leftrightarrow (\mathrm{x}),(\mathrm{e}_1)$$
$$\Leftrightarrow (\mathrm{x}),(\mathrm{d}),(\mathrm{e}_2) \Leftrightarrow (\mathrm{c}),(\mathrm{d}),(\mathrm{e}_3)$$
$$\Leftrightarrow (\mathrm{a}),(\mathrm{c}),(\mathrm{e}_3),(\mathrm{e}_4), \quad (2.3)$$

where each (x) *in* (2.3) *stands for either* (a) *or* (c). *Moreover,*

$$(\mathrm{a}),(\mathrm{b}) \Leftrightarrow (\mathrm{c}),(\mathrm{d}),(\mathrm{y}), \quad (2.4)$$

where (y) *stands for any of the conditions* $(\mathrm{f}_1),(\mathrm{f}_2),(\mathrm{f}_3)$.

Conditions (a), (b), (c), (e_0), (e_1), and (e_2) are modifications of the corresponding conditions for arbitrary square matrices appearing in Theorem 3.1 of Anderson and Styan (1982), while (f_1), (f_2), and (f_3) are analogous modifications of the conditions appearing in their Theorem 3.2. Condition (e_3) was considered by Rao, Mitra, and Bhimasankaram (1972) in the context of determining a matrix by its subclasses of (1)-inverses, while (e_4) is well known to be a necessary condition for both the star and the minus partial orderings; cf. (1.5). It seems that neither (e_3) nor (e_4) has hitherto been discussed in the context of Cochran's theorem.

Note that the conditions (b) and (e_2) express $*$-orthogonality and $*$-commutativity, respectively; cf. Hestenes (1961, Sections 3 and 4). Note also that some of the conditions in Theorem 1 admit geometrical interpretations: (b) means the orthogonality of $\mathscr{R}(\mathbf{A}_i)$ to $\mathscr{R}(\mathbf{A}_j)$ and $\mathscr{R}(\mathbf{A}_i^*)$ to $\mathscr{R}(\mathbf{A}_j^*)$ for $i, j = 1,\ldots,k$, $i \neq j$; (d) is equivalent to the direct-sum decomposition

$$\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{A}_1) \oplus \cdots \oplus \mathscr{R}(\mathbf{A}_k) \quad (2.5)$$

[cf. Styan and Takemura (1983, Lemma 6)]; and (f_1) and (f_2) are interpretable in terms of the inner products of the matrices involved.

The part (b) $\Rightarrow$ (d) of (2.1) was originally established by Marsaglia and Styan (1974, Theorem 14). In the same theorem, they proved that if (d) holds, then the first conditions in (b) and (e_2) are equivalent, which is partially related to the statement (b) $\Leftrightarrow$ (d),(e_2) included in (2.1).

Under the additional assumption that the matrices $\mathbf{A}_1,\dots,\mathbf{A}_k$ are all Hermitian, conditions (a) and (c) assert that $\mathbf{A}_i$ and $\mathbf{A}$, respectively, are tripotent. The remaining conditions modify then accordingly, and the result (2.4) becomes identical with Theorem 3.2 of Anderson and Styan (1982), which covers Theorem 3 of Luther (1965) and Theorem 2.2 of Tan (1975). Under the same additional assumption, the part (b) $\Leftrightarrow$ (d),(e_2) of (2.1) was established by Luther (1965, Theorem 1) and Taussky (1966, Theorem 2); see also Theorem 15 of Marsaglia and Styan (1974) for the corresponding result concerned with arbitrary square matrices. Moreover, the part (b),(c) $\Rightarrow$ (a),(d),(e_1) may be attributed to Khatri (1977, Lemma 10); see also the discussion of this implication by Anderson and Styan (1982, p. 15) in the case where the matrices involved are not necessarily Hermitian.

The version of Cochran's theorem for arbitrary tripotent matrices, given by Anderson and Styan (1982, Theorem 3.1), becomes comparable with Theorem 1 above when both these results are related to Hermitian matrices. Anderson and Styan proved, in particular, that

$$(a),(b) \Leftrightarrow (c),(d),(z), \tag{2.6}$$

where (z) stands for any of the conditions (e_0), (e_1), or (e_2). It is seen that (2.3) covers (2.6), indicating in addition the possibility of interchanging (a) with (c).

We conclude this section by pointing out that certain implications in (2.1) cannot be reversed. The matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

show that (b) does not imply (e_0); the matrices

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{A}_2 = (1/\sqrt{2})\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.7}$$

show that (d),(e_3) do not imply (b); and the matrices

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

show that $(e_3),(e_4)$ do not imply (d). The example (2.7) also shows that (c) neither can be replaced by (a) in the triplet $(c),(d),(e_3)$ nor can be deleted from the quadruplet $(a),(c),(e_3),(e_4)$ in (2.3). It would be interesting to know whether (a) can be deleted from the latter.

3. PROOF OF THE MAIN RESULT

Theorem 13 of Marsaglia and Styan (1974) asserts that (d) holds if and only if, for some (1)-inverse $\mathbf{A}^-$ of $\mathbf{A}$,

$$\mathbf{A}_i\mathbf{A}^-\mathbf{A}_i = \mathbf{A}_i, \qquad i = 1,\dots,k, \tag{3.1}$$

and

$$\mathbf{A}_i\mathbf{A}^-\mathbf{A}_j = \mathbf{0}, \qquad i,j = 1,\dots,k, i \neq j, \tag{3.2}$$

which implies that

$$\mathbf{A}\mathbf{A}^-\mathbf{A}_i = \mathbf{A}_i \text{ and } \mathbf{A}_i\mathbf{A}^-\mathbf{A} = \mathbf{A}_i, \qquad i = 1,\dots,k. \tag{3.3}$$

[Actually, the fact that (d) implies (3.3) may be seen directly from the geometrical versions of these two conditions given in (2.5) and statement (e_4) of Theorem 1, respectively.] Consequently, postmultiplying the former equality in (e_0) by $\mathbf{A}^-\mathbf{A}_j$ and premultiplying the latter equality there by $\mathbf{A}_j\mathbf{A}^-$ yields $\mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_j = \mathbf{0}$ and $\mathbf{A}_j\mathbf{A}_i^*\mathbf{A}_i = \mathbf{0}$, and hence (b) follows on account of $\mathscr{R}(\mathbf{A}_i\mathbf{A}_i^*) = \mathscr{R}(\mathbf{A}_i)$ and $\mathscr{R}(\mathbf{A}_i^*\mathbf{A}_i) = \mathscr{R}(\mathbf{A}_i^*)$. The part (b) $\Rightarrow (e_1)$ is clear. According to (1.1), condition (e_1) actually states that every $\mathbf{A}_i$ is below $\mathbf{A}$ with respect to the star ordering. In view of Theorem 2 of Hartwig and Styan (1986), this is equivalent to the minus-ordering condition

$$\mathbf{A}_i \overset{-}{\leqslant} \mathbf{A}, \qquad i = 1,\dots,k, \tag{3.4}$$

accompanied by (e_2). Then Theorem 1 of Hartwig (1981), stating in particular that (3.4) is equivalent to (d), completes the proof that $(e_1) \Leftrightarrow (d),(e_2)$. Postmultiplying now (3.2) by $\mathbf{A}^*$ and utilizing (e_2) along with (3.3) yields

$$\mathbf{0} = \mathbf{A}_i\mathbf{A}^-\mathbf{A}_j\mathbf{A}^* = \mathbf{A}_i\mathbf{A}^-\mathbf{A}\mathbf{A}_j^* = \mathbf{A}_i\mathbf{A}_j^*.$$

The latter equality in (b) follows similarly, thus establishing the part (d), (e_2) $\Rightarrow$ (b). Finally, (3.1) and the first equality in (e_2) imply that

$$\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i\mathbf{A}^-\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i\mathbf{A}^-\mathbf{A}\mathbf{A}_i^*\mathbf{A}_i = \mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_i,$$

which is (e_3). In view of the known implication (d) $\Rightarrow$ (e_4) [cf. (1.6)], the proof of (2.1) is complete.

To establish (2.2) observe that (a) and (b) clearly entail (c). Further, premultiplying (c) written in the form $\Sigma\mathbf{A}_j\mathbf{A}^*\mathbf{A} = \Sigma\mathbf{A}_j$ by $\mathbf{A}_i^*$ and utilizing (b) yields $\mathbf{A}_i^*\mathbf{A}_i\mathbf{A}^*\mathbf{A} = \mathbf{A}_i^*\mathbf{A}_i$. Applying (b) again leads to $\mathbf{A}_i^*\mathbf{A}_i\mathbf{A}_i^*\mathbf{A}_i = \mathbf{A}_i^*\mathbf{A}_i$, and hence (a) follows by the left-hand cancellation rule; cf. Marsaglia and Styan (1974, Theorem 2).

For the proof of (2.3) first notice that if (a) holds, then premultiplying and postmultiplying, respectively, the two equalities in (e_0) by $\mathbf{A}_i^*$ leads to (e_1). Since the star ordering implies the corresponding minus ordering [cf. (1.6)], it follows that (a),(e_0) $\Rightarrow$ (3.4). If (c) holds, then $\mathbf{A}^*$ is a (1)-inverse of $\mathbf{A}$. Postmultiplying the former condition in (e_0) by $\mathbf{A}_i^*$ and applying the left-hand cancellation rule gives $\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i$. Moreover, it is clear that (e_0) entails (e_4). In view of (1.5), this shows that also (c),(e_0) $\Rightarrow$ (3.4), and hence the relation

$$(\mathrm{x}),(\mathrm{e}_0) \Rightarrow (\mathrm{d}) \tag{3.5}$$

follows by Theorem 1 of Hartwig (1981). Further, the implication

$$(\mathrm{d}),(\mathrm{e}_0) \Rightarrow (\mathrm{a}) \tag{3.6}$$

is easily obtainable by postmultiplying the former equality in (e_0) by $\mathbf{A}^-\mathbf{A}_i$ and utilizing (3.1) and (3.3). Combining now (e_3) and (3.1), the latter with $\mathbf{A}^-$ replaced by $\mathbf{A}^*$, shows that

$$(\mathrm{c}),(\mathrm{d}),(\mathrm{e}_3) \Rightarrow (\mathrm{a}). \tag{3.7}$$

From (c) with (e_4) and (a) with (e_3) we get

$$\mathbf{A}\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i\mathbf{A}^*\mathbf{A} = \mathbf{A}_i\mathbf{A}^*\mathbf{A}_i = \mathbf{A}_i.$$

Consequently, it follows that (a), (c), (e_3), and (e_4) entail

$$(\mathbf{A}_i\mathbf{A}_i^*\mathbf{A} - \mathbf{A}_i)(\mathbf{A}_i\mathbf{A}_i^*\mathbf{A} - \mathbf{A}_i)^* = \mathbf{0},$$

which is clearly equivalent to the former equality in (e_0). The latter equality is obtainable similarly, and therefore

$$(a),(c),(e_3),(e_4) \Rightarrow (e_0). \tag{3.8}$$

Combining (2.1) and (2.2) with (3.5), (3.6), (3.7), and (3.8) concludes the proof of (2.3).

If (3.1) holds with $\mathbf{A}^-$ replaced by $\mathbf{A}^*$, then $\mathbf{A}_i\mathbf{A}^*$ is clearly idempotent. Hence $r(\mathbf{A}_i) = \operatorname{tr}(\mathbf{A}_i\mathbf{A}^*)$, and thus $(c),(d),(f_1) \Leftrightarrow (c),(d),(f_3)$. Using in addition (3.3) with $\mathbf{A}^*$ in place of $\mathbf{A}^-$ shows that, under (c) and (d),

$$(f_1) \quad \Leftrightarrow \quad \operatorname{tr}(\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i\mathbf{A}^*) \geqslant \operatorname{tr}(\mathbf{A}_i\mathbf{A}^*\mathbf{A}\mathbf{A}_i^*), \quad i=1,\dots,k, \tag{3.9}$$

and

$$(f_2) \quad \Leftrightarrow \quad \sum \operatorname{tr}(\mathbf{A}_i\mathbf{A}^*\mathbf{A}_i\mathbf{A}^*) \geqslant \sum \operatorname{tr}(\mathbf{A}_i\mathbf{A}^*\mathbf{A}\mathbf{A}_i^*). \tag{3.10}$$

But a result given by Graybill (1969, p. 235) asserts that for any square matrix $\mathbf{B}$ we have $\operatorname{tr}(\mathbf{B}^*\mathbf{B}) \geqslant \operatorname{tr}(\mathbf{B}^2)$, with equality if and only if $\mathbf{B} = \mathbf{B}^*$; see also Lemma 3.2 in Anderson and Styan (1982). Consequently, the relations (3.9) and (3.10) show that, under (c) and (d), any of the conditions $(f_1),(f_2),(f_3)$ is equivalent to (e_2). In view of (2.3), this completes the proof.

4. A NEW PARTIAL ORDERING FOR MATRICES

As already pointed out, certain conditions involved in Theorem 1 admit interpretations in terms of matrix partial orderings. In view of (1.1), the equalities in (e_1) may be reworded as $\mathbf{A}_i \overset{*}{\leqslant} \mathbf{A}$, $i = 1,\dots,k$, whereas in view of Theorem 1 of Hartwig (1981), condition (d) is equivalent to $\mathbf{A}_i \overset{-}{\leqslant} \mathbf{A}$, $i = 1,\dots,k$. We now prove that also conditions $(e_3),(e_4)$ define a matrix partial ordering and show how it is related to the minus ordering.

DEFINITION. Let $\mathbf{K}$ and $\mathbf{L}$ be $p \times q$ complex matrices. We define that $\mathbf{K} \overset{s}{\preccurlyeq} \mathbf{L}$ if $\mathbf{K} \prec \mathbf{L}$ and $\mathbf{K}\mathbf{L}^*\mathbf{K} = \mathbf{K}\mathbf{K}^*\mathbf{K}$.

THEOREM 2. *The relation* $\preccurlyeq$ *is a partial ordering of the set of complex* $p \times q$ *matrices. Moreover,*

$$\mathbf{K} \preccurlyeq \mathbf{L} \quad \Leftrightarrow \quad \mathbf{K}^+ \overset{-}{\leqslant} \mathbf{L}^+, \tag{4.1}$$

where the plus superscript denotes the Moore-Penrose inverse of a matrix.

Proof. It is obvious that the relation $\preccurlyeq$ is reflexive. Further, Rao, Mitra, and Bhimasankaram (1972, Lemma 2) proved that

$$\mathbf{KL^*K} = \mathbf{KK^*K} \text{ and } \mathbf{LK^*L} = \mathbf{LL^*L} \quad \Rightarrow \quad \mathbf{K} = \mathbf{L}.$$

[Notice that the proof of antisymmetry becomes trivial when we additionally utilize $\mathscr{R}(\mathbf{K}) = \mathscr{R}(\mathbf{L})$.] The space preordering $\mathbf{K} \overset{s}{\prec} \mathbf{L}$ has been added to the equality $\mathbf{KL^*K} = \mathbf{KK^*K}$ to make the relation considered transitive. If $\mathbf{K} \preccurlyeq \mathbf{L}$ and $\mathbf{L} \preccurlyeq \mathbf{M}$, then it is clear that $\mathbf{K} \overset{s}{\prec} \mathbf{M}$ and, since $\mathbf{K} = \mathbf{LCL}$ for some matrix $\mathbf{C}$,

$$\mathbf{KM^*K} = \mathbf{LCLM^*LCL} = \mathbf{LCLL^*LCL} = \mathbf{KL^*K} = \mathbf{KK^*K}.$$

Finally, $\mathbf{K^*LK^*} = \mathbf{K^*KK^*}$ is equivalent to $\mathbf{K^+LK^+} = \mathbf{K}^+$, i.e.,

$$\mathbf{K}^+(\mathbf{L}^+)^+\mathbf{K}^+ = \mathbf{K}^+. \tag{4.2}$$

In view of (1.5), combining (4.2) with $\mathbf{K}^+ \overset{s}{\prec} \mathbf{L}^+$ establishes (4.1). ■

Rewriting $\mathbf{KL^*K} = \mathbf{KK^*K}$ in the form $\mathbf{K^*(L-K)K^*} = \mathbf{0}$, and the equations on the right-hand side of (1.1) in the forms $\mathbf{(L-K)K^*} = \mathbf{0}$ and $\mathbf{K^*(L-K)} = \mathbf{0}$, yields

$$\mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} \preccurlyeq \mathbf{L} \quad \Rightarrow \quad \mathbf{K} \overset{s}{\prec} \mathbf{L} \tag{4.3}$$

and shows the extent to which the new condition is weaker than (1.1). On the other hand, consider the matrices

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 1 & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \tag{4.4}$$

and notice that if $l_{12} = l_{22} = 1$, $l_{21} = 0$, then $\mathbf{K} \preccurlyeq \mathbf{L}$ but $\mathrm{r}(\mathbf{L}-\mathbf{K}) > \mathrm{r}(\mathbf{L}) - \mathrm{r}(\mathbf{K})$, and that if $l_{12} = 0$, $l_{21} = -1$, $l_{22} = 1$, then $\mathbf{K} \overset{-}{\leqslant} \mathbf{L}$ but $\mathbf{KL^*K} \neq \mathbf{KK^*K}$. Hence it follows that neither of the implications $\mathbf{K} \preccurlyeq \mathbf{L} \Rightarrow \mathbf{K} \overset{-}{\leqslant} \mathbf{L}$ and $\mathbf{K} \overset{-}{\leqslant} \mathbf{L} \Rightarrow \mathbf{K} \preccurlyeq \mathbf{L}$ is valid in general, and therefore the chains (1.6) and (4.3) cannot be dovetailed.

From (1.1) and (1.4) it is easily seen that $\mathbf{K} \overset{*}{\leqslant} \mathbf{L} \Leftrightarrow \mathbf{L} - \mathbf{K} \overset{*}{\leqslant} \mathbf{L}$ and $\mathbf{K} \overset{-}{\leqslant} \mathbf{L} \Leftrightarrow \mathbf{L} - \mathbf{K} \overset{-}{\leqslant} \mathbf{L}$. It appears that such a property is not valid for $\mathbf{K} \preccurlyeq \mathbf{L}$. A counterexample may be obtained from (4.4) by taking $l_{12} = l_{22} = 1$, $l_{21} = 0$. In this context, it seems noteworthy to recall the observation by Hartwig and Styan (1986, p. 154) that even if the relations $\mathbf{K} \overset{-}{\leqslant} \mathbf{L}$, $\mathbf{K} \preccurlyeq \mathbf{L}$, and $\mathbf{L} - \mathbf{K} \preccurlyeq \mathbf{L}$ hold simultaneously, then not necessarily $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$.

In view of the above, a natural question is what extra conditions must be added to $\mathbf{K} \preccurlyeq \mathbf{L}$ in order to get $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$. On account of (4.1) and the equivalence $\mathbf{K} \overset{*}{\leqslant} \mathbf{L} \Leftrightarrow \mathbf{K}^{+} \overset{*}{\leqslant} \mathbf{L}^{+}$, this may be answered directly by utilizing Theorem 2 of Hartwig and Styan (1986) and the theorem of Baksalary (1986). Notice that some of the conditions so obtained, viz. (1) $\mathbf{KL}^{+} = (\mathbf{KL}^{+})^{*}$ and $\mathbf{L}^{+}\mathbf{K} = (\mathbf{L}^{+}\mathbf{K})^{*}$, (2) $\mathbf{K}^{+}\mathbf{L} = (\mathbf{K}^{+}\mathbf{L})^{*}$ and $\mathbf{LK}^{+} = (\mathbf{LK}^{+})^{*}$, (3) $(\mathbf{L} - \mathbf{K})^{+} = \mathbf{L}^{+} - \mathbf{K}^{+}$, (4) $(\mathbf{L} - [\gamma/(\gamma - 1)]\mathbf{K})^{+} = \mathbf{L}^{+} - \gamma\mathbf{K}^{+}$ for some nonzero $\gamma \neq 1$, (5) $\mathbf{L}^{+}\mathbf{KL}^{+} = \mathbf{K}^{+}$, and (6) $\mathbf{LK}^{+}\mathbf{L} = \mathbf{K}$, assure that both $\mathbf{K} \overset{-}{\leqslant} \mathbf{L}$ and $\mathbf{K} \preccurlyeq \mathbf{L}$ are strengthened to the star ordering $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$.

It is known [cf. Theorem 3 of Drazin (1978), Lemma 2 of Hartwig and Spindelböck (1983), and Theorem 2.1 of Baksalary, Pukelsheim, and Styan (1989)] that if $\mathbf{L}$ is a partial isometry, a contraction, an orthogonal projector, or an idempotent, then every $\mathbf{K}$ satisfying $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$ inherits the same property, i.e.,

$$\mathbf{L}^{+} = \mathbf{L}^{*} \text{ and } \mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K}^{+} = \mathbf{K}^{*}, \tag{4.5}$$

$$\|\mathbf{L}\|_2 \leqslant 1 \text{ and } \mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \|\mathbf{K}\|_2 \leqslant 1, \tag{4.6}$$

$$\mathbf{L} = \mathbf{LL}^{*} \text{ and } \mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} = \mathbf{KK}^{*}, \tag{4.7}$$

$$\mathbf{L} = \mathbf{L}^{2} \text{ and } \mathbf{K} \overset{*}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} = \mathbf{K}^{2}, \tag{4.8}$$

where $\|\cdot\|_2$ in (4.6) denotes the spectral norm of a matrix. Extending the result (3.14) of Hartwig and Styan (1986), Baksalary and Hauke (1987) pointed out that (4.8) may be strengthened to the form

$$\mathbf{L} = \mathbf{L}^{2} \text{ and } \mathbf{K} \overset{-}{\leqslant} \mathbf{L} \quad \Rightarrow \quad \mathbf{K} = \mathbf{K}^{2},$$

and showed that the minus ordering does not suffice in (4.5), (4.6), and (4.7).

It appears that $\mathbf{K} \preccurlyeq \mathbf{L}$ cannot be used to replace $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$ in either of the statements (4.5), (4.7), and (4.8), a common counterexample being the matrices

$$\mathbf{K} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, $\mathbf{K} \preccurlyeq \mathbf{L}$ can be used to replace $\mathbf{K} \overset{*}{\leqslant} \mathbf{L}$ in (4.6). Actually, we may establish an even stronger result, given in Theorem 3 below. Moreover, although the idempotency is not inherited under the $\preccurlyeq$-ordering in the set of all complex matrices, it is inherited in the subset of Hermitian matrices. More precisely,

$$\mathbf{K}^* = \mathbf{K} \preccurlyeq \mathbf{L} = \mathbf{L}\mathbf{L}^* \quad \Rightarrow \quad \mathbf{K}\mathbf{K}^* = \mathbf{K} \overset{*}{\leqslant} \mathbf{L}, \tag{4.9}$$

which is an analogue to the result (3.14) of Hartwig and Styan (1986) referring to the minus partial ordering. The implication (4.9) follows by noting (i) that if $\mathbf{L} = \mathbf{L}\mathbf{L}^*$ (i.e., $\mathbf{L}$ is an orthogonal projector) and $\mathbf{K} = \mathbf{K}^*$, then $\mathbf{K} \overset{s}{<} \mathbf{L}$ is equivalent to $\mathbf{L}\mathbf{K} = \mathbf{K}$, and hence $\mathbf{K}\mathbf{L}^*\mathbf{K} = \mathbf{K}\mathbf{K}^*\mathbf{K}$ reduces to $\mathbf{K}^2 = \mathbf{K}^3$ and further to $\mathbf{K} = \mathbf{K}^2$, and (ii) that in the set of orthogonal projectors, the star ordering is equivalent to the space preordering; cf., e.g., Hartwig and Styan (1987, Theorem 5.8).

Theorem 3. *If a $p \times q$ complex matrix $\mathbf{L}$ is a contraction, then every $p \times q$ matrix $\mathbf{K}$ satisfying $\mathbf{K}\mathbf{L}^*\mathbf{K} = \mathbf{K}\mathbf{K}^*\mathbf{K}$ is also a contraction.*

Proof. Let $\mathbf{K} = \mathbf{U}\mathbf{D}\mathbf{V}^*$ be a singular-value decomposition of $\mathbf{K}$, with unitary matrices $\mathbf{U}$ and $\mathbf{V}$ of orders p and q, respectively, and with $p \times q$ matrix $\mathbf{D}$ such that $d_{ij} = 0$ for all $i \neq j$, $d_{11} > 0, \dots, d_{rr} > 0, d_{r+1,r+1} = 0, \dots, d_{ss} = 0$, where $r = \mathrm{r}(\mathbf{K})$ and $s = \min(p, q)$. Then the condition $\mathbf{K}\mathbf{L}^*\mathbf{K} = \mathbf{K}\mathbf{K}^*\mathbf{K}$ is equivalent to

$$\mathbf{D}\mathbf{V}^*\mathbf{L}^*\mathbf{U}\mathbf{D} = \mathbf{D}\mathbf{D}^{\mathrm{T}}\mathbf{D}, \tag{4.10}$$

where $\mathbf{D}^{\mathrm{T}}$ is the transpose of $\mathbf{D}$. Since the spectral norm is the matrix norm corresponding to (or, in another terminology, compatible with) the Euclidean vector norm, it is multiplicative; cf., e.g., Ben-Israel and Greville (1974, pp.

34–35). Moreover, it is unitarily invariant, and therefore a consequence of (4.10) and the assumption that $\|\mathbf{L}\|_2 \leqslant 1$ is

$$\|\mathbf{K}\|_2^3 = \|\mathbf{D}\|_2^3 \leqslant \|\mathbf{D}\|_2^2\|\mathbf{V}^*\mathbf{L}^*\mathbf{U}\|_2 = \|\mathbf{D}\|_2^2\|\mathbf{L}\|_2 \leqslant \|\mathbf{K}\|_2^2,$$

and hence $\|\mathbf{K}\|_2 \leqslant 1$, as desired. ■

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