Medians for weight metrics in the covering graphs of semilattices

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Abstract

We consider the undirected covering graph $G$ of a finite (meet) semilattice $X$ endowed with a lower valuation. More precisely, our main concerns are the lower valuations associated to a weighting of the join-irreducible elements of $X$ and the corresponding minimum path length metrics in $G$, which are frequently considered in the literature. Some results on the medians for such metrics are obtained, in relation with a lattice majority rule. Especially, these medians are characterized in the case where $X$ is distributive. The unanimity, or Pareto, property is also investigated for such medians.

1. Introduction

Given a metric space $(X, d)$ and a $p$-tuple (a $p$-profile, or simply a profile) $\pi = (x_1, \ldots, x_p)$ of elements of $X$, a median is any element $\mu$ of $X$ minimizing the remoteness $r(\mu) = \sum d(\mu, x_i)$. Medians are considered in various application fields like operation research, statistics, social choice and mathematical taxonomy. They may appear as optimal center values, as consensus objects, as maximum likelihood estimates of an unknown object or as results of a $p$-ary algebraic operation (see, for instance, [7, 13, 34, 41].

Here, we consider the case where the set $X$ is endowed with a (meet) semilattice structure $(X, \leq, \land)$, together with a strictly isotone real function $\nu$ on $X$. Then, the
undirected covering graph $G = (X, E)$ is valued by an edge length $l$ associated with $v$: for a pair $xy \in E$, $l(xy) = |v(x) - v(y)|$. The set $X$ is a metric space with the corresponding minimum path length metric, denoted as $d_v$ (see [32] for a survey on such metrics). Several examples of such a situation, mainly concerning mathematical taxonomy, will be recalled in the sequel (see also [12, 13, 14]). More generally, the search and the study of medians for minimum path length metrics in graphs or networks is a classical combinatorial optimization problem since Hakimi [19]; in the case of trees, the problem goes back to Jordan [22]. Several important contributions have been made by Hansen (see [20] for a recent one).

General definitions about lattices are recalled in Section 2. Several consensus elements, corresponding to majority or unanimity consensus rules, are associated with profiles in Section 3. This section ends with the recall of a fairly general relation between these consensus elements and the medians (Theorem 3.1): for a classical family of functions $v$ for which the metric $d_v$ has a simple expression, a majority rule provides, when it works, an upper bound for the medians.

In Section 4, we define a special class of functions, called weight valuations since they are associated with weightings of the join-irreducible elements of $X$. To consider such functions is always possible, and quite natural in many cases. For instance, they include as a special case the extensively studied problem (since Régnier [36]) of the research of central partitions with the symmetric difference metric. We give in Proposition 4.2 an algebraic property of medians for weight metrics (the metrics induced by weight valuations), leading to improved bounds (Corollary 4.3). In Section 5, we consider the case of a distributive semilattice. It includes, especially, the case of a median semilattice, that is the case where $G$ is a median graph. When $X$ is distributive, the weight valuations directly generalize valuations in lattices (Proposition 5.2); they are characterized by the fact that the bounds obtained in Section 4 are reached as often as possible (Theorem 5.5). Conversely, these properties of the medians characterize valuations on distributive semilattices. These results generalize some previously known ones on distributive lattices or median semilattices; they are also related with some recent results of Bandelt [5] and Barthélémy and Constantin [10]. In Section 6, weight metrics in distributive semilattices are characterized in terms of valuation, of weights, of edge lengths and of medians (Theorem 6.2).

In many domains of application (social choice, mathematical taxonomy), it is often asked to an aggregation procedure to satisfy a unanimity (or Pareto) property: roughly, when the elements of a profile all agree on some point, so must do an acceptable consensus object. It is already known that, with the symmetric difference metric, median partitions have the Pareto property [36]. In Section 7, several counterexamples, illustrating the fact that the unanimity property for medians is far from being general, are given. It is established that the case of partitions is an isolated good one.

2. Definitions about lattices

General information on lattices may be found in the Birkhoff book [16]; we just recall the main definitions and give the terminology used in the sequel. Given a set $X$, (partially) ordered by $\leq$, a real function $v$ on $X$ is strictly isotone if $x < y$ implies
\( v(x) < v(y) \). The ordered set \( X \) is a meet semilattice (or, simply, a semilattice in the sequel) if any pair \( x, y \) of elements of \( X \) admits a greatest lower bound (g.l.b.) denoted \( x \wedge y \), the meet of \( x \) and \( y \); one has \( x \wedge y \leq x, y \) and \( x \wedge y \geq z \) for all \( z \) such that \( z \leq x, y \). Here, it is always assumed that \( X \) is finite. Then, any subset \( Y \) of \( X \) has also a g.l.b., denoted \( \bigwedge Y \); if, moreover, \( Y \) has an upper bound, then it has also a least upper bound (l.u.b.) \( \bigvee Y \). The smallest element \( \bigwedge X \) of \( X \) is denoted as \( 0 \) (italic). The semilattice \( X \) is a lattice if it has largest element \( U = \bigvee X \); then, \( \bigvee Y \) exists for any subset \( Y \). We adopt the usual convention \( \bigvee 0 = 0 \), while \( \bigwedge 0 \) is equal to \( U \) in the case of a lattice and not defined otherwise. Several examples of finite (but possibly with a large size) lattices and semilattices will be considered in Sections 4–7.

The covering relation \( < \) associated with \( X \) is defined as usual: for all \( x, y, z \in X \):

\[
x < y \quad \text{if and only if} \quad x \leq y \text{ and } x < z < y \text{ imply } x = z.
\]

The undirected covering graph \( G \) is the undirected graph corresponding to this (directed) covering relation: an unordered pair \( xy \) of elements of \( X \) is an element of the edge set \( E \) of \( G \) if and only if \( x < y \) or \( y < x \).

The rank \( h(x) \) of an element \( x \in X \) is the minimum number of edges in a path of \( G \) between \( 0 \) and \( x \). The semilattice \( X \) is ranked if \( x < y \) implies \( h(y) = h(x) + 1 \). In this case, the metric \( d_{h} \) associated with the rank function \( h \) is the minimum path length metric in the unvalued graph \( G \). It will be denoted \( d \) and called the lattice metric on \( X \).

An element \( j \in X \) is said to be join-irreducible if \( Y \in X \) and \( \bigvee Y = j \) implies \( j \in Y \). Equivalently, \( j \) is join-irreducible if it covers a unique element, the predecessor of \( j \), denoted as \( j^{*} \), of \( X \). It follows from the convention \( \bigvee 0 = 0 \) and from the definition above that the \( 0 \) element is not a join-irreducible. The set of all the join-irreducibles of \( X \) is denoted as \( J \). A join-irreducible element \( j \) is an atom if \( j^{*} = 0 \). The semilattice \( X \) is atomic if all its join-irreducibles are atoms.

For \( x \in X \), set \( J(x) = \{ j \in J : j \leq x \} \). Then, in all cases, \( x = \bigvee J(x) \) is a canonical, and useful, expression of \( x \) as the join of a subset of \( J \). The map \( x \mapsto J(x) \) from \( X \) into the Boolean lattice \( (\mathcal{P}(J), \cup, \cap) \) has the following three properties:

- (J1) \( J(x) = J(y) \) implies \( x = y \),
- (J2) \( J(x \wedge y) = J(x) \cap J(y) \),
- (J3) when \( x \vee y \) exists, \( J(x \vee y) \supseteq J(x) \cup J(y) \).

These notions dualize to meet-irreducible elements, but, if \( X \) is not a lattice, \( m \) is meet-irreducible if either it is covered by a unique element \( m^{*} \) of \( X \) or it is maximal. Let \( J' \) be the set of all the meet-irreducible elements of \( X \) and, for any \( x \in X \), set \( J'(x) = \{ m \in J' : x \leq m \} \). Then \( x = \bigwedge J'(x) \) is a canonical expression of \( x \) as the meet of a subset of \( J' \). One has \( J'(x) = J'(y) \) implies \( x = y \), \( J'(x \wedge y) \supseteq J'(x) \cup J'(y) \) and, when \( x \vee y \) exists, \( J'(x \vee y) = J'(x) \cap J'(y) \).

3. Two approaches for the aggregation problem

Let \( \pi = (x_{1}, \ldots, x_{p}) \) be a profile of elements of a semilattice \( X \), with \( P = \{ 1, \ldots, p \} \) a set of indices. For the aggregation of \( \pi \) into a unique element, the metric approach leads to the medians already defined. Another way is the algebraic construction of consensus elements. For \( j \in J \), let \( s(j) = |\{ i \in P : j \leq x_{i} \}| \) be the score of the join-
irreducible $j$; if $s(j) > p/2$ (respectively $s(j) < p/2$, $s(j) = p/2$, $s(j) = p$), $j$ is said to be a majority (respectively minority, balancing, unanimity) join-irreducible (for the profile $\pi$). Then:

$$c(\pi) = \bigvee \{j \in J: s(j) > p/2\},$$

$$b(\pi) = \bigvee \{j \in J: s(j) \geq p/2\},$$

and

$$u(\pi) = \bigvee \{j \in J: s(j) = p\} = \bigwedge_{i \in P} x_i.$$

Among these three elements, only $u(\pi)$ always exists; it corresponds to the so-called unanimity rule. The aggregation functions $b$ and $c$ constitute lattice formalizations of majority rules. When $X$ is not a lattice, they are generally not defined for all profiles. When $b(\pi)$ exists, so does $c(\pi)$, with the relation $c(\pi) \leq b(\pi)$ (notice that $c(\pi) = b(\pi)$ when $p$ is odd). These rules have often an alternative expression as lattice polynomials; for instance, when it is defined, the expression $\chi(\pi) = \bigvee \{\bigwedge_{i \in P} x_i: I \subseteq P, |I| > p/2\}$ is equal to $c(\pi)$ (for a proof, see, e.g., [25]). The median semilattices constitute a particularly interesting class: they are exactly the distributive semilattices for which $\chi(\pi)$ exists for all profiles [6].

We dually define the dual score $s'(m)$ of a meet-irreducible element $m \in J'$ by

$$s'(m) = \{i \in P: x_i \leq m\}.$$ Three dual majority rule elements are

$$c'(\pi) = \bigwedge \{m \in J': s'(m) > p/2\},$$

$$b'(\pi) = \bigwedge \{m \in J': s'(m) \geq p/2\},$$

and

$$u'(\pi) = \bigwedge \{m \in J': s'(m) = p\} = \bigvee_{i \in P} x_i.$$

When $X$ is not a lattice, the existence of $c'(\pi)$ (respectively $b'(\pi)$) is equivalent to the fact that the set $\{m \in J': s'(m) > p/2\}$ (respectively the set $\{m \in J': s'(m) \geq p/2\}$) is not empty; this existence implies that $b(\pi)$ (respectively $c(\pi)$) exists and satisfies $b(\pi) \leq c'(\pi)$ (respectively $c(\pi) \leq b'(\pi)$). The dual unanimity rule element $u'(\pi)$ is, when it exists, an upper bound of all the majority rule ones.

Fig. 1 shows a (ranked) meet semilattice with nine elements. In this example, $J = \{1, 2, 3, 4, 5, 7\}$ and $J' = \{2, 3, 5, 6, 7, 8\}$. For the 4-profile $\pi = (2, 3, 5, 7)$, one has, on one hand, $s(2) = s(3) = s(4) = s(5) = s(7) = 1$ and $s(1) = 2$, and, on the other hand, $s'(2) = s'(3) = s'(5) = s'(7) = 1$, $s'(6) = 2$ and $s'(8) = 3$. So, $u'(\pi)$ does not exist, $u(\pi) = c(\pi) = \bigvee \phi = 0$, $b(\pi) = 1$, $c'(\pi) = 8$ and $b'(\pi) = 6$. Here, the four majority rule consensus elements exist and are all distinct.

The following theorem gives metric and median characterizations of the functions $v$ satisfying condition (i), which are called lower valuations. For the well-known equivalence of conditions (i) and (ii), we refer to the book of Birkhoff [16] in the case of lattices, and to papers of Barthelemy [8] and Monjardet [32] for the more general cases of semilattices and other ordered sets. The equivalence of conditions (i) and (iii) is shown in a previous paper [26]; it provides a general relation between the metric
and the algebraic approaches of consensus which is our startpoint for the next developments.

**Theorem 3.1.** If $X$ is a meet semilattice and $v$ is a strictly isotone real function on $X$, the following three conditions are equivalent:

1. For all $x, y \in X$ such that $x \lor y$ exists, the following inequality (1) holds:

   $$v(x) + v(y) \leq v(x \lor y) + v(x \land y).$$

2. The minimum path length metric $d_v$ on the covering graph $G$ is given by:

   $$d_v(x, y) = v(x) + v(y) - 2v(x \land y).$$

3. For any profile $\pi$ such that $c'(\pi)$ exists and for any $\pi$-median $\mu$ with regard to the metric $d_v$, the inequality $\mu \leq c'(\pi)$ holds.

Let us complete the example of Fig. 1 by considering the function $v$ such that $v(2) = 4$, and $v$ is equal to the rank function otherwise. We let the reader to verify that $v$ is a lower valuation, and that the medians are $0$ and $1$, with a remoteness equal to 7. These medians have $c'(\pi) = 8$ as an upper bound.

A semilattice $X$ is (lower) semimodular if, for every $x, y \in X$ such that $x \lor y$ exists, $x < x \lor y$ and $y < x \lor y$ imply $x \land y < x$ and $x \land y < y$; a semimodular semilattice is ranked (the ranked semilattice of Fig. 1 is not semimodular). If $X$ is a lattice, the upper semimodularity is dually defined. If $X$ is a ranked semilattice, its rank function is a lower valuation if and only if it is semimodular (see [32]); then, Theorem 3.1 applies to its lattice metric.

4. **Weight valuations**

Now we consider a special class of lower valuations. A real function $v$ on $X$ is said to be a weight valuation if there exists a real strictly positive mapping $w$ defined on $J$ such that, for any $x \in X$, $v(x) = \sum_{j \in J} w(j)$. By a usual convention, $v(\emptyset) = 0$.

**Proposition 4.1.** A weight valuation $v$ on a meet semilattice is a lower valuation. The distance $d_v$ is given, for all $x, y \in X$, by:

$$d_v(x, y) = \sum_{j \in J} w(j).$$
Proof. The properties (J2) and (J3) of Section 2 directly imply that a weight valuation satisfies property (i) of Theorem 3.1. The expression for \( d_v \) is then derived from (J2) and condition (ii) in the same theorem. □

So, Theorem 3.1 applies to weight valuations. The importance of this class of functions is emphasized in the two following remarks:

(1) Any finite meet semilattice admits weight valuations. By definition, these lower valuations, and the corresponding metrics, are particularly easy to obtain.

(2) Especially, if we take the weights \( w(j) \) uniformly equal to 1, we find the following metric \( \delta \) as \( d_v \): for all \( x, y \in X \), \( \delta(x, y) = |J(x) \Delta J(y)| \). This symmetric difference metric is very natural when the elements \( x \) of \( X \) are canonically represented by the sets \( J(x) \).

Barthélémy [9] has axiomatically characterized this metric for several types of lattices or semilattices of binary relations, including those of orders, preorders and equivalences.

The most known case of a lattice where the medians for the metric \( \delta \) have been extensively studied is the lattice of all the partitions of a finite set \( A \). Many heuristics or exact algorithms have been proposed ([2, 18, 27, 36, 37, 38, 40] among others).

Wakabayashi has shown the problem of finding median partitions using metric \( \delta \) (the Régnier problem) to be NP-hard (for recent survey on complexity results and recent advances on the research of medians for the metric \( \delta \), see [2, 21]).

The lattice of partitions is geometric, i.e., atomic and (upper) semimodular. The atoms are the partitions where just one pair of elements is linked. The partition lattice is a special case of the geometric lattices associated with cycle matroids of graphs (cf., e.g., [1, pp. 54 and 259]).

If \( \pi = (x_1, \ldots, x_p) \) is a profile of \( X \), the remoteness function \( r(x) \) associated with a weight valuation is given, according to Proposition 4.1, by:

\[
r(x) = \sum_{J \in J(x)} (p - s(j))w(j) + \sum_{j \in J, J \notin J(x)} s(j)w(j).
\]

Let us associate the constant \( \rho(\pi) = \sum_{j \in J} s(j)w(j) \) with the profile \( \pi \). An alternative expression for the remoteness is:

\[
r(x) = \rho(\pi) - \sum_{j \in J(x)} (2s(j) - p)w(j).
\]

Then, the quantity \( (2s(j) - p)w(j) \) may be seen as the contribution of the element \( j \) of \( J(x) \) to the remoteness of \( x \). This contribution is negative if \( s(j) > p/2 \) (that is to say \( j \) is a majority join-irreducible), positive if \( s(j) < p/2 \) (\( j \) is a minority one), and null if \( s(j) = p/2 \) (\( j \) is a balancing one). In order to reduce the remoteness, starting from an element \( x \), it is desirable to add majority join-irreducibles and to remove minority ones in \( J(x) \). The difficulty of doing so is due to the condition that the resulting set of irreducibles must be equal to \( J(z) \) for some \( z \in X \). Nevertheless, these considerations lead to the following Proposition 4.2. For all \( x \in X \), we set \( x_c = \bigvee \{ j \in J(x): s(j) > p/2 \} \) and, similarly, \( x_b = \bigvee \{ j \in J(x): s(j) < p/2 \} \). So, \( x_c \) and \( x_b \) depend on \( x \) and on the profile \( \pi \), with \( x_c \leq x_b \leq x \). If \( c(\pi) \) (respectively \( b(\pi) \)) exists, then \( x_c = x \wedge c(\pi) \) (respectively \( x_b = x \wedge b(\pi) \)).
Proposition 4.2. Let $X$ be a finite meet semilattice endowed with a weight metric $d_v$. For any profile $\pi$ and for any median $\mu$ of $\pi$, the equality $\mu = \mu_b$ holds; moreover, there exists a median $\mu$ of $\pi$ such that the equality $\mu = \mu_c$ holds.

Proof. Assume $\mu$ is a median of $\pi$ such that $\mu_b < \mu$. Then $J(\mu_b) \subset J(\mu)$. The elements of $J(\mu) \setminus J(\mu_b)$ are all minority ones and, so, $r(\mu_b) < r(\mu)$, a contradiction. This proves the first part.

For the second part, we start from a median $\mu$ such that $\mu_c < \mu_b = \mu$. We similarly observe that, in such a case, the elements of $J(\mu) \setminus J(\mu_c)$ are only minority and balancing join-irreducibles; in fact, only balancing ones since $\mu$ is a median. Then, $r(\mu_c) = r(\mu)$ and $\mu_c$ is also a median. \]

In algebraic terms, Proposition 4.2 states that the medians are $\vee$-generated by majority and balancing join-irreducibles. An immediate consequence is:

Corollary 4.3. Let $X$ be a finite meet semilattice endowed with a weight metric $d_v$.

For any profile $\pi$ such that $b(\pi)$ exists and for any median $\mu$ of $\pi$, the inequality $\mu \leq b(\mu)$ holds.

For any profile $\pi$ such that $c(\pi)$ exists and for any median $\mu$ of $\pi$, there exists a median $\mu_0$ of $\pi$ such that: (i) $\mu_0 \leq c(\pi)$ holds, (ii) $\mu_0 \leq \mu$ and (iii) all the elements of the lattice interval $[\mu_0, \mu]$ are medians.

This corollary gives a large extension to a result already known in the case of partitions endowed with the metric $\delta$ \cite{30, 35}. Because of the inequalities $c(\pi) \leq b(\pi) \leq c'(\pi)$, recalled in Section 3, its bound for medians is often an improvement of the one of Theorem 3.1. This improvement may be important, as the following example shows: $X$ is the lattice of all the partitions of a set $A$ with $n$ elements ($n \geq 3$) and $\pi$ is the $(2^{n-1} - 1)$-profile of all the partitions of $A$ into two classes (bipartitions). Since $\pi$ has an odd number of elements, one has $c(\pi) = b(\pi)$ and $c'(\pi) = b'(\pi)$. The meet-irreducible elements $m$ of $X$ are the bipartitions, and $s'(m)$ is uniformly equal to 1. So, the bound $c'(\pi)$ of Theorem 3.1 is the degenerate partition $\bigcup \emptyset = U$ with one class, providing no information on the medians. It is not difficult to see that the score $s(j)$ of any atom partition is uniformly equal to $2^{n-2} - n + 2$, which is less than half the number of elements of $\pi$; so, $c(\pi) = \sqrt{\emptyset} = 0$: by Corollary 4.3, the $0$-partition (the partition into $n$ classes) is the unique median of the above profile for any weight metric on $X$.

5. Valuations in distributive semilattices

In this section, we deal with some well-known types of lattices and their straightforward extensions to semilattices.

First, the modular lattices are those satisfying, for all $x, y, z \in X$ such that $x \leq z$, the equality $(x \lor y) \land z = x \lor (y \land z)$. In fact, other equivalent characterizations are to retain here: a lattice is modular (i) if it is lower and upper semimodular; or (ii) if it has
no sublattice isomorphic to the pentagon $N_5$ (Fig. 2(a)); or (iii) if it admits a valuation, that is to say a strictly isotone real function $v$ such that, for all $x, y \in X$,

$$v(x \lor y) + v(x \land y) = v(x) + v(y).$$

Such a valuation is provided by the rank function of $X$. The modularity property extends to semilattices: a semilattice $X$ is said to be modular if, for any $x \in X$, the principal ideal $\downarrow x = \{ y \in X : y \leq x \}$ is a modular lattice. If $X$ is a semilattice, we say that a strictly isotone real function $v$ is a valuation if it satisfies (6) for all $x, y$ such that $x \lor y$ exists. Obviously, a semilattice which has a valuation is modular.

A lattice $X$ is distributive if it satisfies the distributivity laws: for all $x, y, z \in X$,

$$(x \lor y) \land z = (x \land z) \lor (y \land z) \quad \text{or, equivalently,} \quad (x \land y) \lor z = (x \lor z) \land (y \lor z).$$

Distributive lattices are exactly the modular lattices which have no sublattice isomorphic to $M_3$ (Fig. 2(b)), where $t' = x \lor y = x \lor z = y \lor z$ and $t = x \land y = x \land z = y \land z$. So, Fig. 2 gives the proscribed two sublattices in a distributive lattice.

A semilattice $X$ is said to be distributive if, for any $x \in X$, the principal ideal $\downarrow x$ is a distributive lattice. A distributive semilattice $X$ is a median semilattice if, moreover, for all $x, y, z \in X$, $x \lor y \lor z$ exists as soon as $x \lor y$, $x \lor z$ and $y \lor z$ all exist. Distributive and median semilattices have been introduced by Sholander [39]. The trees and, generally, the so-called median graphs are the undirected covering graphs of median semilattices [3, 4]. We first recall some characteristic properties of distributive semilattices:

**Lemma 5.1.** For a meet semilattice $X$, the following three conditions are equivalent:

(i) $X$ is distributive.

(ii) For all $j \in J$ and $Y \subseteq X$, the inequality $j \leq \bigvee Y$ implies that there exists some $y \in Y$ such that $j \leq y$.

(iii) The map $x \mapsto J(x)$ from $X$ into the Boolean lattice $(\mathcal{P}(J), \cup, \cap)$ preserves all the joins existing in $X$.

**Proof.** (i) implies (ii): $j \leq \bigvee Y$ implies $j = j \land (\bigvee Y) = \bigvee \{ j \land y : y \in Y \}$ (the last equality by the distributivity hypothesis); since $j$ is join-irreducible, there is some $y \in Y$ such that $j = j \land y$, that is $j \leq y$.

(ii) implies (iii): let $x, y \in X$ such that $x \lor y$ exists, and $j \in J(x \lor y)$. By (ii), $j \leq x$ or $j \leq y$ holds; so, $J(x \lor y) \subseteq J(x) \cup J(y)$. So, the inequality (13) of Section 2 becomes the equality $J(x \lor y) = J(x) \cup J(y)$.

(iii) implies (i): by (J1), (J2) and (iii), the map $x \mapsto J(x)$ is, for all $y \in X$, an isomorphism between the lattice $\downarrow y$ and a sublattice of $\mathcal{P}(J)$. Then, $\downarrow y$ is a distributive lattice for all $y \in X$ and, so, $X$ is a distributive semilattice. $\square$

Fig. 2. (a) $N_5$. (b) $M_3$. 

![Fig. 2](image_url)
Our purpose is to extend to the distributive semilattices some well-known properties of valuations in distributive lattices (see [16]). First, the weight valuations are exactly the valuations:

**Proposition 5.2.** Let $X$ be a finite distributive semilattice and $v$ a real function on $X$ such that $v(0) = 0$. The following three conditions are equivalent:

(i) $v$ is a weight valuation.

(ii) $v$ is a valuation.

(iii) $v$ is strictly isotone; the associated metric $d_v$ is given by formula (2) and, for all $x, y \in X$ such that $x \vee y$ exists, by:

$$d_v(x, y) = 2v(x \vee y) - v(x) - v(y) = v(x \vee y) - v(x \wedge y).$$

**Proof.** (i) implies (ii) is an immediate consequence of the equalities $J(x \wedge y) = J(x) \cap J(y)$ and $J(x \vee y) = J(x) \cup J(y)$. If $v$ satisfies (ii), the first expression of $d_v$ in (iii) follows from Theorem 3.1; the other ones are straightforward with equality (6).

In order to derive (i) from (iii), we construct the weight function $w$ on $J$ by induction on the rank of the elements of $X$. First, $v(0) = 0 = \sum_{j \in J} w(j)$. Let $z$ be such that $w(j)$ is known for all $j < z$, and assume that the equality $v(x) = \sum_{j \in J(x)} w(j)$ is true for all $x < z$. If $z = x \vee y$, where both $x$ and $y$ are distinct of $z$, then we obtain, using Lemma 5.1, the following equalities:

$$v(z) = v(x \vee y) = v(x) + v(y) - v(x \wedge y)$$

$$= \sum_{j \in J(x \vee y)} w(j) + \sum_{j \in J(x)} w(j) - \sum_{j \in J(x) \cap J(y)} w(j) = \sum_{j \in J(x) \cup J(y)} w(j)$$

$$= \sum_{j \in J(z)} w(j).$$

If $z \in J$, then $z$ covers a unique element $z^*$. Since $J(z) = J(z^*) \cup \{z\}$, we have just to set $w(z) = v(z) - v(z^*)$, a positive quantity since $v$ is strictly isotone. So, the equality $v(z) = \sum_{j \in J(z)} w(j)$ is true for $z$ in all cases. Finally, $v$ is a weight valuation. □

In a distributive semilattice, the rank function $h$ satisfies the above condition (ii). In fact, the lattice metric $\delta$ and the symmetric difference metric $\delta^\prime$ are the same. Let us give some examples of valuations in distributive semilattices.

1. Tree orders with a $0$-element constitute one of the basic families of median semilattices. In this case, all the elements, except the $0$, are join-irreducible. Any length $l$ on the edges of the covering graph corresponds with a valuation: setting $w(x) = l(x^* x)$ for all $x \neq 0$, one gets the valuation defined by $v(x) = d(0, x)$, for all $x$.

2. Let $A$ be a finite set and $\mathcal{A}$ an antichain of the Boolean lattice $\mathcal{P}(A)$; $F, F' \in \mathcal{A}$ implies $F \neq F'$. Then, the ideal $I(\mathcal{A}) = \{B \subseteq A: B \subseteq F$ for some $F \in \mathcal{A}\}$, endowed with set inclusion, is a distributive semilattice, generally not a median one. For instance, with an integer $k < |A|$, the set $\mathcal{P}_{(k)}(A) = \{B \subseteq A: |B| \leq k\}$ is a distributive semilattice. Any strictly positive real function $w$ on $A$ is a weight function and provides a valuation on $I(\mathcal{A})$. It is easy to see that if $A$ and $\mathcal{A}$ are respectively the set of the vertices and the set of the maximal cliques of a given graph, then $I(\mathcal{A})$ is a median...
semilattice. Conversely, a consequence of a Gilmore theorem (see [15, p. 396]) is that if \( I(A) \) is median, then \( A \) is the set of the maximal cliques of some graph on \( A \).

(3) The following median semilattice belongs to the type described just above. Let \( A \) be a finite set; a hierarchy (classification tree) on \( A \) is a set \( H \) of subsets of \( A \) such that: \( A \in H; \emptyset \notin H; \) for all \( a \in A \), \( \{a\} \in H; \) for all \( C, C' \in H, C \cap C' \in \{\emptyset, C, C'\} \). The set \( \mathcal{H} \) of all the hierarchies on \( A \), ordered by set inclusion in \( \mathcal{P}(\mathcal{P}(A)) \), is a median semilattice \((\mathcal{H}, \subseteq, \cap)\) [12, 23]. The evaluations on \( \mathcal{H} \) are the functions \( v(H) = \sum_{C \in H} w(C) \), where \( w \) is a real positive weighting of all the possible untrivial clusters, that is the subsets of \( A \) with at least two and at most \(|A| - 1\) elements. The corresponding weight metric is then \( d_v(H, H') = \sum_{C \in H \triangle H'} w(C) \). Several metrics of this type are described in [24]. Other median semilattices of classification models are presented in [12].

Now we characterize medians for weight metrics in distributive semilattices; the first part of the following result was already given in [26]; (the other results of this section are new) and, before, by Barthélémy and Janowitz [11, Proposition 18] in the case of the lattice metric on a median semilattice.

**Proposition 5.3.** Let \( d \) be a weight metric on a distributive semilattice \( X \) and \( \pi \) be a profile of \( X \). The set \( M \) of all the medians of \( \pi \) is then as follows:

If the majority rule element \( c(\pi) \) exists, \( M \) is the set of all the elements \( \mu \) such that \( \mu = c(\pi) \lor (\sqrt{K}) \), for some set \( K \) of balancing join-irreducibles; especially, if \( p \) is odd, \( c(\pi) \) is the unique median.

If \( c(\pi) \) does not exist, \( M \) is the set of all the elements \( \mu \) such that \( \mu = (\sqrt{K'}) \lor (\sqrt{K}) \), where: (i) \( K' \) is a set of majority join-irreducibles such that \( \sqrt{K'} \) exists and maximizes the quantity \( \sum_{j \in K'} (2s(j) - p)w(j) \); (ii) \( K \) is as above.

**Proof.** Let \( \pi \) be a profile of elements of \( X \) such that \( c(\pi) \) exists. We first notice that, by Lemma 5.1, \( j_0 \in J \) and \( j_0 \leq \sqrt{\{ j \in J: s(j) > p/2 \}} = c(\pi) \) implies there exists some majority join-irreducible \( j_1 \) such that \( j_0 \leq j_1 \); then, one has \( s(j_0) \geq s(j_1) \) and, so, \( j_0 \) is a majority join-irreducible. So, \( J(c(\pi)) \) contains only majority elements. By Corollary 4.3, there exists a median \( \mu \) such that \( \mu \leq c(\pi) \); if \( \mu < c(\pi) \), then \( J(\mu) \subset J(c(\pi)) \) and, according to formula (5) of Section 4, \( \mu \) cannot be a median. So, \( \mu = c(\pi) \). The possible other medians correspond to some additions of balancing join-irreducible elements to \( J(\mu) \), which do not change the remoteness.

If \( c(\pi) \) does not exist, it may be similarly observed (using Proposition 4.2 instead of Corollary 4.3) that there is a median \( \mu \) such that \( J(\mu) \) is a maximal set of majority join-irreducibles admitting a join. According to formula (5) again, the remoteness of \( \mu \) is minimized when the quantity \( \sum_{j \in J(\mu)} (2s(j) - p)w(j) \) is maximal.

As an illustration, let us consider a profile \( \pi \) of subsets of \( A \) in the semilattice \( \mathcal{P}(k)(A) \) of example (2) above. Let \( C = \{ a \in A: s(a) > p/2 \} \); \( C \) is a subset of \( A \) but not necessarily an element of \( \mathcal{P}(k)(A) \). If \( |C| \leq k \), then \( c(\pi) = C \) and the medians are the subsets \( B \) of \( A \) such that \( C \subseteq B \subseteq \{ a \in A: s(a) > p/2 \} \) and \( |B| < k \). If \( |C| > k \), they are all the subsets \( B \) of \( C \) such that \( |B| = k \) and \( \sum_{a \in B} (2s(a) - p)w(a) \) is maximal; they may be obtained by a greedy procedure.
Another illustration, corresponding to example (3), is the Margush and McMorris [28] characterization of median n-trees. They show that the majority rule in the set \( H \) of all the hierarchies on \( A \) gives a median for the symmetric difference metric \( \delta \); in fact, this is true in all median semilattices [11, Proposition 18]. Proposition 5.3 extends this result to all the weight metrics.

From a general point of view, one might say that obtaining medians is easy when \( c(\pi) \) exists, as it is the case in median semilattices. But, when \( X \) is not median, there are profiles \( \pi \) such that \( c(\pi) \) does not exist; then, the search of a maximally weighted set of majority join-irreducibles may be a more difficult problem.

Proposition 5.4 hereunder is a converse of Proposition 5.3; the two results assemble into Theorem 5.5.

Proposition 5.4. Let \( v \) be a strictly isotone real function on a semilattice \( X \) such that, for any profile \( \pi \in X^p \) with odd \( p \) for which the majority rule element \( c(\pi) \) exists, \( c(\pi) \) is a median of \( \pi \) for the metric \( d_v \). Then, \( v \) is a valuation and \( X \) is distributive.

Proof. We first prove that \( v \) is a valuation. Let \( x, y \in X \) be such that \( t = x \vee y \) exists. Set \( z = x \wedge y \) and \( \alpha = v(x) - v(z) = d(x, z) \), \( \beta = v(t) - v(x) = d(x, t) \) and \( \epsilon = v(t) + v(z) - v(x) - v(y) \); then, \( d(t, y) = v(t) - v(y) = \alpha - \epsilon \) and \( d(y, z) = v(y) - v(z) = \beta + \epsilon \) (Fig. 3).

Assume \( \epsilon > 0 \); then, we have \( d(x, y) \leq \alpha + \beta - \epsilon \) (consider the path through \( t \) in the figure). We find an odd profile \( \pi \in X^p \) with \( p = 2q + 1 \) elements such that \( c(\pi) \) is not a median as follows: take \( q \) elements equal to \( x \), \( q \) others equal to \( y \) and one equal to \( z \). Then, a join-irreducible is a majority one if and only if it belongs to \( J(z) \); so, \( z = c(\pi) \). With \( r(z) = q\alpha + q(\beta + \epsilon) \) and \( r(x) = \alpha + q(\alpha + \beta - \epsilon) \), we get \( r(z) - r(x) \geq 2q\epsilon - \alpha \) and, for large enough \( q \), \( z \) is not a median.

The case \( \epsilon < 0 \) is similar, with a profile \( \pi \) where \( q \) elements are taken equal to \( x \), \( q \) others equal to \( y \) and one equal to \( t \). For such a profile, a join-irreducible is a majority one if and only if it belongs to \( J(x) \cup J(y) \). Then, \( t = \sqrt{J(x) \cup J(y)} = c(\pi) \) and \( r(t) - r(x) \geq 2q\epsilon - \beta \). For large enough \( q \), \( t \) is not a median. Finally, \( \epsilon = 0 \).

So, \( v \) is a valuation, and, as a consequence, \( X \) is modular. Assume that \( X \) has a sublattice isomorphic to \( M_3 \), as in Fig. 2(b). Set \( \lambda = v(t') - v(x) \); using equality (6), we successively obtain \( \lambda = v(y) - v(t) = v(z) - v(t) = v(t') - v(y) = v(t') - v(z) = v(x) - v(t) \). Then, \( \pi = (x, x, x, y, y, y, z, z, z, t', t') \) is a 11-profile for which \( c(\pi) = t \), and \( r(t) = 13\lambda \), while \( r(t') = 9\lambda \).

Theorem 5.5. Let \( v \) be a strictly isotone real function on a semilattice \( X \). Then, \( c(\pi) \) is, when it exists, a median of any odd profile for the metric \( d_v \) if and only if \( X \) is distributive and \( v \) is a valuation on \( X \).

Fig. 3.
6. Weight metrics on distributive semilattices

Let \( x y z t \) be a quadrilateralon of \( X \), that is \( x \) and \( y \) cover \( z \) and \( t \) covers \( x \) and \( y \). Obviously, all the 4-cycles in \( G \) belong to this type. Consider the edge length \( l \) on the edges of \( G \) associated as in Section 1 with a strictly isotone real function \( v \) on \( X \). If \( X \) is distributive and \( v \) is a valuation, the equality \( l(xy) = l(zt) \) for all 4-cycles \( xyzt \) of \( G \) is a consequence of (6). In fact, this condition on 4-cycles characterizes the valuations:

**Proposition 6.1.** Let \( l \) be an edge length satisfying the equality \( l(xy) = l(zt) \) for any 4-cycle \( xyzt \) of the undirected covering graph \( G \) of a distributive semilattice \( X \). Then, there exists a valuation \( v \) on \( X \) such that \( l(xy) = |v(x) - v(y)| \) for all the edges \( xy \) of \( G \).

**Proof.** Let \( x \) and \( y \) be two elements of \( X \) such that \( x \preceq y \) and let \( k = h(y) - h(x) \) be the difference between the ranks of \( x \) and \( y \). An increasing path \( I \) between \( x \) and \( y \) is a sequence \( x = z_0, z_1, z_2, \ldots, z_k = y \) where \( z_i \) covers \( z_{i-1} \), for \( i = 1, \ldots, k \). We first show that any increasing path \( I' = (x, z_1', z_2', \ldots, z_k', y) \) between \( x \) and \( y \) has the same length (according to \( l \)) as \( I \). By the condition on 4-cycles, it is true for \( k = 2 \); assume it is true for all the rank differences inferior to \( k \). The equality is again true if \( z_k' = z_k ; \) if not, because of the semimodularity, \( z_k' \) and \( z_k \) cover an element \( z \), such that \( z \geq x \). Set \( \alpha = l(z z_{k-1}) = l(z_{k-1} y), \beta = l(z z_k) = l(z_k y) \) and let \( \gamma \) be the common length of all the increasing paths between \( x \) and \( z \). Then, the common length of all the increasing paths between \( x \) and \( z_{k-1} \) is \( \gamma + \alpha \), the common length of all the increasing paths between \( x \) and \( z_{k-1} \) is \( \gamma + \beta \) and \( I \) and \( I' \) have the same length \( \alpha + \beta + \gamma \).

Then, we may take, for all \( x \in X \), the common length of all the increasing paths between 0 and \( x \) as \( v(x) \). By a result of Barthélemy [8] on semimodular posets, a necessary and sufficient condition for \( v \) being a lower valuation is \( v(x) + v(y) \leq v(z) + v(t) \) for all quadrilaterons \( xyzt \) of \( X \); this is true by the 4-cycle hypothesis. When \( x \cup y \) exists, the dual result is similarly obtained in the distributive lattice \( \downarrow (x \cup y) \). So, \( v \) satisfies condition (iii) of Proposition 5.2; it is a valuation on \( X \). 

Finally, the previous results (Propositions 5.1, 5.2, 5.3 and 6.1) lead to characterizations of weight metrics on distributive semilattices in terms, successively, of valuation, of weights, of medians and of edge lengths:

**Theorem 6.2.** Let \( X \) be a distributive semilattice and \( d \) a real function on \( X^2 \). The following four conditions are equivalent:

(i) \( d = d_v \) for a valuation \( v \) on \( X \).

(ii) \( d \) is a weight metric.

(iii) \( d = d_v \) for a real strictly isotone function \( v \) on \( X \), and, for any profile \( \pi \in X^p \) with odd \( p \) such that the majority rule element \( c(\pi) \) exists, \( c(\pi) \) is a median of \( \pi \).

(iv) \( d \) is the geodesic metric in the covering graph \( G \) of \( X \) endowed with an edge length \( l \) such that, for any 4-cycle \( xyzt \) of \( G \), \( l(xy) = l(zt) \).

We end this section with some comments on this theorem. The implication (i) \( \Rightarrow \) (iii) has been given by Monjardet [31] in the case of distributive lattices. The equivalence...
of (iii) and (iv) has a large intersection with a result of Bandelt [5]: in a median network (a network is median if its underlying graph is a median graph and if the length function $l$ on its edges satisfies the condition (iv) of Theorem 6.2), the median vertices are the same as those of the unvalued underlying graph. Condition (iii) completes a median characterization of valuations in modular lattices given in [26] and a median characterization of distributive lattices in [25]. Condition (iv) is related with a parallelism relation $R$ on the edges of a median graph recently studied by Barthélémy and Constantin [10]: in a 4-cycle $xyzt$, the edges $xy$ and $zt$ are parallel; $R$ is the transitive closure of the set of such parallel pairs. This definition extends directly to covering graphs of lower distributive semilattices. Then, weight metrics exactly correspond with real positive lengths which are constant on each parallelism class.

7. Medians and the unanimity property

According to the previous results, including those recalled in Theorem 3.1, the metric medians are related with majority rules in many cases. Then, it is tempting to try to complete these relations by introducing other algebraic bounds, for instance, in order to find lower bounds when the majority rules give only upper ones. The unanimity element $u(\pi)$ is the most obvious candidate for weak, but not trivial, such bounds. In fact, we present, in this section, several cases of a profile $\pi$ in a semilattice $X$ endowed with a lower valuation $v$ where the inequality $u(\pi) \leq \mu$ is not satisfied by a median $\mu$.

The following notations for subsets and partitions of a given set $A$ will be used: a subset will be denoted $123$ instead of $\{1, 2, 3\}$; given two subsets $B$ and $C$ of $A$, their symmetric difference is $B \Delta C = (B \cup C) \setminus (B \cap C)$. A partition will be denoted, for instance, $/22/34/$ instead of ($\{\{1, 2\}, \{3, 4\}\}$); a partition where only the elements 1 and 2 are linked (the other classes being singletons) will be denoted as $/2/$. Our first counterexample concerns a lower valuation (not a weight valuation) in a distributive lattice:

Counterexample 1. Let $\pi = (123, 124, 134)$ be a 3-profile in the Boolean lattice $X = 2^{\{1, 2, 3, 4\}}$ endowed with the lower valuation $v$ given by: $v(\emptyset) = 0$; $v(1) = 2$; $v(2) = v(3) = v(4) = 6$; $v(12) = v(13) = v(14) = 8$; $v(23) = v(24) = v(34) = 13$; $v(123) = v(124) = v(134) = 30$; $v(234) = 20$; $v(1234) = 55$.

The remoteness function is then: $r(\emptyset) = 90$; $r(1) = r(2) = r(3) = r(4) = 84$; $r(12) = r(13) = r(14) = 78$; $r(23) = r(24) = r(34) = 79$; $r(123) = r(124) = r(134) = 88$; $r(234) = 72$; $r(1234) = 75$.

So, the unique median is $\mu = 234$, which satisfies $\mu \leq c(\pi) = 1234$, but is not superior to $u(\pi) = 1$. This example has two interesting features: the structure of $X$, a Boolean lattice, is particularly strong; the profile has only three elements, the minimum number for which the unanimity condition may be not satisfied.

In the case of a weight metric in a lower distributive semilattice, it is an immediate consequence of Proposition 5.3 that the unanimity property is satisfied by the
medians of any profile \( \pi \) such that \( c(\pi) \) exists. But this property may vanish when \( c(\pi) \) does not exist:

**Counterexample 2.** Consider the set \( A = \{1, 2, 3, 4, 5, 2', 3', 4', 5', 2'', 3'', 4'', 5''\} \) and the ideal \( X \) of \( \mathcal{P}(A) \) defined by: a subset \( B \) of \( A \) belongs to \( X \) if either \( I \not\in B \) or \( |B| \leq 9 \). Let \( \pi = (123452'3'4'5', 123452''3''4''5'', 12'3'4'5'2''3''4''5'') \). All the elements of \( A \) are majority join-irreducibles and, so, \( c(\pi) \) does not exist in \( X \). According to the second part of Proposition 5.3, any median is a maximal element of \( X \). With the symmetric difference metric \( \delta \), the remoteness of the maximal subsets containing \( I \) (which all have nine elements) is 16. The unique median is the subset \( \mu = 23452'3'4'5'2''3''4''5'' \) whose remoteness is 15; it is not superior to \( u(\pi) = I \).

Régnier [36] has shown the median procedure to satisfy the unanimity property in the lattice of partitions endowed with the metric \( \delta \). So, for an odd profile \( \pi \) of the lattice of partitions, consider two medians \( \mu \) and \( \mu' \) of \( \pi \), respectively, the metrics \( \delta \) and \( \delta' \). We have always the following relations between consensus partitions: \( u(\pi) \leq \mu \leq c(\pi) \leq \mu' \), by, successively, the Régnier result, Corollary 4.3 and a result in [25]. In this last reference, a question, seemingly still open, is asked: does \( \mu' \leq u'(\pi) \) always hold?

The following two counterexamples show that the lower bound \( u(\pi) \) for median partitions and metric \( \delta \) is by no means as general as the upper bound \( b(\pi) \) of Section 4: the unanimity condition may be no longer valid when the atom partitions are unequally weighted, or when the partition lattice is replaced by another one of the same geometric type.

**Counterexample 3.** Consider the profile \( \pi = (x_1, x_2, x_3) \) of partitions of the set \( A = \{1, 2, 3, 4\} \) defined by \( x_1 = /123/4/ \), \( x_2 = /1/234/ \) and \( x_3 = /1234/ \). Set \( w(/12/) = w(/14/) = w(/34/) = 10 \) and \( w(/13/) = w(/23/) = w(/24/) = 1 \). The unique median is \( \mu = /12/34/ \) which is not superior to \( u(\pi) = /1/234/ \).

**Counterexample 4.** Consider the undirected graph \( G = (A, J) \) with 12 vertices and 38 edges of Fig. 4. One has \( A = \{1, I', 2, 3, 4, 2', 3', 4', 6, 2'', 3'', 4'', 6''\} \); the vertices \( 2, 3, 4, 6, 2', 3', 4' \) generate a \( K_{5,5} \) complete bipartite subgraph, the other edges being \( 1I', 12', 1'2 \), and, for all \( i \in \{2, 3, 4, 6\} \), \( 1i \) and \( 1'i \). The join-irreducible elements of the geometric lattice \( X \) associated with the cycle matroid of \( G \) are the edges of \( G \).

Consider the following 3-profile \( \pi = (x_1, x_2, x_3) \) of closed subsets of \( X \): \( x_1 \) and \( x_2 \) are respectively the sets of edges of the subgraphs \( G_1 \) and \( G_2 \) of Fig. 5 and \( x_3 = J \). Then, the set of all the majority join-irreducible edges is \( x_1 \cup x_2 \) and the edge \( I' \) is the only unanimity one.

For the search of the medians for the symmetric difference metric \( \delta \), we use the considerations of Section 4: for instance, if a closed set \( x \) may be obtained from another one \( y \) by the addition of only majority edges, then \( r(x) < r(y) \). The results of this research may be summarized as follows: the remoteness of the set of edges of the 4-clique \( I'22' \) is 46 and is minimum among the closed sets containing simultaneously \( I' \), at least one edge \( 1i \) and at least one edge \( 1'i \). The remoteness of \( x_1 \) is 43 and is minimum among the closed sets containing \( I' \) and no edge of the \( 1'i \) type; the case of
8. Conclusion

The covering graphs of semilattices include trees, cubes and other median graphs. They are frequently encountered in real world problems, as graphs of elementary transforms on combinatorial objects: partitions, preorders, orders, weak orders and classification trees of several types (see [33] for other examples). The weight metrics, especially the symmetric difference one, are frequently natural in such structures.

The properties of the medians obtained in Sections 4 and 5 may sometimes serve to make easier the obtention of medians, since they allow to restrain the domain where the medians may be searched. Especially, medians for weight metrics in distributive semilattices are completely characterized in Section 5.

Our results also provide theoretic information on the median procedure, which may be useful to decide, when addressing a specific aggregation problem, whether the research of medians is a good method, or a method that needs improvement, or an inadequate one. For such an appreciation, the relations with the majority procedure are interesting properties. The examples of Section 7 are also important: they show that, in many cases, the median procedure does not satisfy the inequality $u(\pi) \leq \mu$, which is the lattice formalization of the frequently requested unanimity (or Pareto)
condition: when all the elements of the profile agree on some point, so does a good consensus object \( x \).

Among the many uses of medians briefly mentioned in the introduction above, two types may be distinguished. In many situations, the minimization of the remoteness is enough to justify the use of the median procedure. This is generally the case, for instance, in location problems or when a probabilistic model is used, such that the median is the maximum likelihood estimator of an unknown true object (for such models, see [41] in social choice and [29] in aggregation of classifications; according to Young, this aspect of the median procedure goes back to [17]). In these cases, to have not the unanimity property may be of minor importance.

On the other hand, this property is generally required in situations where the purpose is to summarize all the profiles \( \pi \) by a unique representative element. Then, if we know that the bare median procedure has not the unanimity property, it is possible to consider a modified problem of the type: minimize the remoteness, subject to additional constraints implied by the unanimity property. One may also turn to other approaches, for instance the algebraic rules of Section 3 or those (sometimes the same) axiomatically characterized by Monjardet [33] or by Barthélemy and Janowitz [11].

References


