Statistical convergence and statistical continuity on locally solid Riesz spaces

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A B S T R A C T

In this work, we introduce the concepts of statistical \( \tau \)-convergence, statistically \( \tau \)-Cauchy sequence and statistically \( \tau \)-bounded sequence in a locally solid Riesz space endowed with the topology \( \tau \), and investigate some properties of these concepts. We also examine the statistical \( \tau \)-continuity of a mapping defined on a locally solid Riesz space.

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1. Introduction

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [11] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [1]).

Recall that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is said to be a linear topology. A vector space equipped with a linear topology is called a topological vector space. A Riesz space equipped with a linear topology that has a base at zero consisting of solid sets is called a locally solid Riesz space [1].

As for the statistical convergence, it is a generalization of the ordinary convergence of a real sequence, which was first defined in 1951 by H. Fast [3] and H. Steinhaus [13] independently. This concept has been studied by many mathematicians up to date (see, for instance [4,5,10]). Finally, the notion of statistical convergence was defined in a more general topological space by G.D. Maio and Lj.D.R. Kočinac [9]. A study related to our current work in this area is the one by I.J. Maddox [8] in which the statistical convergence was introduced in a topological vector space by using pre-norms. In this work, we introduce the concept of statistical convergence and statistical continuity in a locally solid Riesz space and investigate certain their properties by using the mathematical tools of the theory of topological vector spaces.

2. Preliminaries

In this section, we recall some of the basic concepts related to Riesz spaces and statistical convergence, and we refer to [1,6,7,14] for more details.
Definition 2.1. Let \( L \) be a real vector space and let \( \leq \) be a partial order on this space. \( L \) is said to be an ordered vector space if it satisfies the following properties:

(i) If \( x, y \in L \) and \( y \leq x \), then \( y + z \leq x + z \) for each \( z \in L \).
(ii) If \( x, y \in L \) and \( y \leq x \), then \( \lambda y \leq \lambda x \) for each \( \lambda \geq 0 \).

If, in addition, \( L \) is a lattice with respect to the partial ordering, then \( L \) is said to be a Riesz space (or a vector lattice).

For an element \( x \) of a Riesz space \( L \) the positive part of \( x \) is defined by \( x^+ = x \lor 0 \), the negative part of \( x \) by \( x^- = (-x) \lor 0 \), and the absolute value of \( x \) by \( |x| = x \lor (-x) \), where \( 0 \) is the element zero of \( L \).

A subset \( S \) of a Riesz space \( L \) is said to be solid if \( y \in S \) and \( |x| \leq |y| \) imply that \( x \in S \).

A topology \( \tau \) on a real vector space \( L \) that makes the addition and the scalar multiplication continuous is said to be a linear topology, that is, the topology \( \tau \) makes the functions

\[
(x, y) \mapsto x + y \quad \text{(from } L \times L \text{ to } L),
\]

\[
(\lambda, x) \mapsto \lambda x \quad \text{(from } \mathbb{R} \times L \text{ to } L)
\]

continuous. In this case, the pair \((L, \tau)\) is called a topological vector space.

Every linear topology \( \tau \) on a vector space \( L \) has a base \( \mathcal{N} \) for the neighborhoods of \( 0 \) (zero) satisfying the following properties:

(a) Each \( V \in \mathcal{N} \) is a balanced set; that is, \( \lambda x \in V \) holds for all \( x \in V \) and every \( \lambda \in \mathbb{R} \) with \( |\lambda| \leq 1 \).
(b) Each \( V \in \mathcal{N} \) is an absorbing set; that is, for every \( x \in L \), there exists a \( \lambda > 0 \) such that \( \lambda x \in V \).
(c) For each \( V \in \mathcal{N} \) there exists some \( W \in \mathcal{N} \) with \( W + W \subseteq V \).

Definition 2.2. ([12]) A linear topology \( \tau \) on a Riesz space \( L \) is said to be locally solid if \( \tau \) has a base at zero consisting of solid sets. A locally solid Riesz space \((L, \tau)\) is a Riesz space \( L \) equipped with a locally solid topology \( \tau \).

In this paper, by the symbol \( \mathcal{N}_{sol} \) we will denote any base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology.

A function \( f : A \to (L_2, \tau_2) \), from a subset \( A \) of a topological vector space \((L_1, \tau_1)\) to another topological vector space \((L_2, \tau_2)\) is called uniformly continuous if for every \( \tau_2 \)-neighborhood \( V \) of zero of \( L_2 \), there exists a \( \tau_1 \)-neighborhood \( W \) of \( 0 \) in \( L_1 \) such that \( f(x) - f(y) \in V \) whenever \( x, y \in A \) and \( x - y \in W \).

Now we recall some of the basic concepts related to statistical convergence.

Definition 2.3. ([2]) Let \( A \subseteq \mathbb{N} \) and \( A(n) := |\{1, \ldots, n\} \cap A| \), where \( | \cdot | \) denotes the cardinality of that set. If the limit \( \delta(A) := \lim_{n \to \infty} \frac{A(n)}{n} \) exists, then \( \delta(A) \) is called the asymptotic density of the set \( A \).

Definition 2.4. ([9]) A sequence \( (x_n) \) in a topological space \( X \) is said to be statistically convergent to \( x_0 \in X \), if for every neighborhood \( U \) of \( x_0 \), we have

\[
\delta(\{n \in \mathbb{N}: x_n \notin U\}) = 0.
\]

3. Statistical topological convergence in locally solid Riesz spaces

In this section, we introduce the concept of statistical topological convergence of a sequence in a locally solid Riesz space, and present some basic results.

Definition 3.1. Let \((L, \tau)\) be a locally solid Riesz space and \((x_n)\) be a sequence in \( L \). We say that \((x_n)\) is statistically \( \tau \)-convergent to \( x_0 \in L \) provided that, for every \( \tau \)-neighborhood \( U \) of zero,

\[
\lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n: x_k - x_0 \notin U\} \right| = 0
\]

holds. We denote this by \( st_{\tau} - \lim x_n = x_0 \) (or \( x_n \overset{st_{\tau}}{\longrightarrow} x_0 \), briefly).

Briefly, the sequence \((x_n)\) is statistically \( \tau \)-convergent to \( x_0 \in L \), if \( \delta(A_U) = 0 \) for each \( \tau \)-neighborhood \( U \) of zero, where \( A_U = \{n \in \mathbb{N}: x_n - x_0 \notin U\} \).
Example 3.1. Let us consider the locally solid Riesz space \((\mathbb{R}^2, \| \cdot \|)\) with the Euclidean norm \(\| \cdot \|\) and coordinatewise ordering. In this space, let us define a sequence \((x_n)\) by

\[
x_n = \begin{cases} 
(2 + \frac{1}{n}, 1 + \frac{3}{n}) & \text{for } n \neq k^2, \\
(5, 5) & \text{for } n = k^2
\end{cases}
\]

where \(k \in \mathbb{N}\). The family \(\mathcal{N}_{\text{sol}}\) of all \(V(\varepsilon)\) defined by

\[
V(\varepsilon) = \{ x \in \mathbb{R}^2 : \|x\| < \varepsilon \}
\]

where \(0 < \varepsilon \in \mathbb{R}\), constitutes a base at zero \((\theta = (0, 0))\). For \(x_0 = (2, 1)\), we have

\[
x_n - x_0 = \begin{cases} 
(\frac{1}{n}, \frac{3}{n}) & \text{for } n \neq k^2, \\
(3, 4) & \text{for } n = k^2.
\end{cases}
\]

For every \(\tau\)-neighborhood \(U\) of zero, there exists some \(V(\varepsilon) \in \mathcal{N}_{\text{sol}}\), \(\varepsilon > 0\) such that \(V(\varepsilon) \subseteq U\) and

\[
\{ n \in \mathbb{N} : x_n - x_0 \notin V(\varepsilon) \} = A \cup \{ 1, 4, 9, 16, \ldots, k^2, \ldots \}
\]

where \(A\) is a finite set. Then, we have

\[
\delta(\{ n \in \mathbb{N} : x_n - x_0 \notin U \}) = \delta(\{ n \in \mathbb{N} : x_n - x_0 \notin V(\varepsilon) \}) = \delta(A) + \delta(\{ 1, 4, 9, 16, \ldots, k^2, \ldots \}) = 0.
\]

Consequently, we have \(st_{\tau} - \lim x_n = (2, 1)\).

Definition 3.2. Let \((x_n)\) be a sequence in a locally solid Riesz space \((L, \tau)\). If there exists some \(\lambda > 0\) such that

\[
\delta(\{ n \in \mathbb{N} : \lambda x_n \notin U \}) = 0
\]

holds for every \(\tau\)-neighborhood \(U\) of zero, then we say that \((x_n)\) is statistically \(\tau\)-bounded.

Definition 3.3. Let \((x_n)\) be a sequence in a locally solid Riesz space \((L, \tau)\). If there exists some \(k \in \mathbb{N}\) such that

\[
\delta(\{ n \in \mathbb{N} : x_n - x_k \notin U \}) = 0
\]

holds for every \(\tau\)-neighborhood \(U\) of zero, then we say that \((x_n)\) is statistically \(\tau\)-Cauchy.

Theorem 3.1. Let \((L, \tau)\) be a Hausdorff locally solid Riesz space, \((x_n)\) and \((y_n)\) be two sequences in \(L\). Then the following hold:

(a) If \(st_{\tau} - \lim x_n = x_1\) and \(st_{\tau} - \lim x_n = x_2\), then \(x_1 = x_2\).

(b) If \(st_{\tau} - \lim x_n = x\), then \(st_{\tau} - \lim \alpha x_n = \alpha x\) for each \(\alpha \in \mathbb{R}\).

(c) If \(st_{\tau} - \lim x_n = x\) and \(st_{\tau} - \lim y_n = y\), then \(st_{\tau} - \lim (x_n + y_n) = x + y\).

Proof. (a) Let \(U\) be any \(\tau\)-neighborhood of zero. Then, there exists a \(V \in \mathcal{N}_{\text{sol}}\) such that \(V \subseteq U\). Choose a \(W \in \mathcal{N}_{\text{sol}}\) such that \(W + W \subseteq V\). If \(st_{\tau} - \lim x_n = x_1\) and \(st_{\tau} - \lim x_n = x_2\), then we have \(\delta(K_1) = \delta(K_2) = 1\), where

\[
K_1 = \{ n \in \mathbb{N} : x_n - x_1 \in W \}
\]

and

\[
K_2 = \{ n \in \mathbb{N} : x_n - x_2 \in W \}.
\]

Now let \(K = K_1 \cap K_2\). Then, we have

\[
x_1 - x_2 = x_1 - x_1 + x_1 - x_2 \in W + W \subseteq V \subseteq U
\]

for every \(n \in K\). Hence for every \(\tau\)-neighborhood \(U\) of zero we have \(x_1 - x_2 \in U\). Since \((L, \tau)\) is Hausdorff, the intersection of all \(\tau\)-neighborhoods \(U\) of zero is the singleton \(\{\theta\}\). Thus we get \(x_1 - x_2 = \theta\), i.e., \(x_1 = x_2\).

(b) Let \(st_{\tau} - \lim x_n = x\). Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{\text{sol}}\) such that \(V \subseteq U\). Since \(st_{\tau} - \lim x_n = x\), we have

\[
\delta(\{ n \in \mathbb{N} : x_n - x \in V \}) = 1.
\]
Let \(|\alpha| \leq 1\). Since \(V\) is balanced, \(x_n - x \in V\) implies that \(\alpha(x_n - x) \in V\). Hence we have

\[
\{n \in \mathbb{N} : x_n - x \in V\} \subseteq \{n \in \mathbb{N} : \alpha x_n - \alpha x \in V\} \subseteq \{n \in \mathbb{N} : \alpha x_n - \alpha x \in U\}.
\]

Thus we get

\[
\delta\left(\{n \in \mathbb{N} : \alpha x_n - \alpha x \in U\}\right) = 1
\]

for each \(\tau\)-neighborhood \(U\) of zero. Now let \(|\alpha| > 1\) and \(|\alpha|\) be the smallest integer greater than or equal to \(|\alpha|\). There exists a \(W \in \mathcal{N}_{\text{sol}}\) such that \(|\alpha| W \subseteq V\). Since \(st_\tau - \lim x_n = x\), we have \(\delta(K) = 1\) where

\[K = \{n \in \mathbb{N} : x_n - x \in W\}.
\]

Then we have

\[
|\alpha x - \alpha x_n| = |\alpha| |x - x_n| \leq |\alpha| |x - x_n| W \subseteq V \subseteq U,
\]

for each \(n \in K\). Since the set \(V\) is solid, we have \(\alpha x - \alpha x_n \in V\) and \(\alpha x - \alpha x_n \in U\) for each \(n \in K\). Thus we get

\[
\delta\left(\{n \in \mathbb{N} : \alpha x_n - \alpha x \in U\}\right) = 1
\]

for each \(\tau\)-neighborhood \(U\) of zero. Hence \(st_\tau - \lim x_n = \alpha x\) for every \(\alpha \in \mathbb{R}\).

(c) Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{\text{sol}}\) such that \(V \subseteq U\). Let us choose another \(W \in \mathcal{N}_{\text{sol}}\) such that \(W + W \subseteq V\). Since \(st_\tau - \lim x_n = x\) and \(st_\tau - \lim y_n = y\), we have \(\delta(K_1) = \delta(K_2) = 1\) where

\[K_1 = \{n \in \mathbb{N} : x_n - x \in W\}
\]

and

\[K_2 = \{n \in \mathbb{N} : y_n - y \in W\}.
\]

Now let \(K = K_1 \cap K_2\). Hence we have \(\delta(K) = 1\) and

\[
(x_n + y_n) - (x + y) = (x_n - x) + (y_n - y) \in W + W \subseteq V \subseteq U
\]

for each \(n \in K\). Thus we get

\[
\delta\left(\{n \in \mathbb{N} : (x_n + y_n) - (x + y) \in U\}\right) = 1.
\]

Since \(U\) is arbitrary, we have \(st_\tau - \lim(x_n + y_n) = x + y\). \(\square\)

**Theorem 3.2.** If a sequence \((x_n)\) in a locally solid Riesz space \((L, \tau)\) is statistically \(\tau\)-convergent, then it is statistically \(\tau\)-bounded.

**Proof.** Let \((x_n)\) be statistically \(\tau\)-convergent to the point \(x_0 \in L\). Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{\text{sol}}\) such that \(V \subseteq U\). Let us choose \(W \in \mathcal{N}_{\text{sol}}\) such that \(W + W \subseteq V\). Since \(x_n \xrightarrow{st-\tau} x_0\), we have \(\delta(K) = 0\), where \(K = \{n \in \mathbb{N} : x_n - x_0 \notin W\}\). Since \(W\) is absorbing, there exists a \(\mu > 0\) such that \(\mu x_0 \in W\). Let \(\lambda\) be such that \(\lambda \leq 1\) and \(\lambda \leq \mu\). Since \(W\) is solid and \(|\lambda x_0| \leq |\mu x_0|\), we have \(\lambda x_0 \in W\). Since \(W\) is balanced, \(x_n - x_0 \in W\) implies that \(\lambda(x_n - x_0) \in W\). Then we have

\[
\lambda x_n = \lambda(x_n - x_0) + \lambda x_0 \in W + W \subseteq V \subseteq U
\]

for each \(n \in \mathbb{N} \setminus K\), and thus we get

\[
\delta\left(\{n \in \mathbb{N} : \lambda x_n \notin U\}\right) = 0.
\]

Consequently, \((x_n)\) is statistically \(\tau\)-bounded. \(\square\)

**Theorem 3.3.** If a sequence \((x_n)\) in a locally solid Riesz space \((L, \tau)\) is statistically \(\tau\)-convergent, then it is statistically \(\tau\)-Cauchy.

**Proof.** Let \((x_n)\) be statistically \(\tau\)-convergent to the point \(x_0 \in L\) and \(U\) an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{\text{sol}}\) such that \(V \subseteq U\). Let us choose \(W \in \mathcal{N}_{\text{sol}}\) such that \(W + W \subseteq V\). Since \(x_n \xrightarrow{st-\tau} x_0\), we have \(\delta(K) = 0\), where \(K = \{n \in \mathbb{N} : x_n - x_0 \notin W\}\). We have

\[
x_n - x_k = x_n - x_0 + x_0 - x_k \in W + W \subseteq V \subseteq U
\]
for all \( k, n \in \mathbb{N} \setminus K \), and thus
\[
\{n \in \mathbb{N} : x_n - x_k \notin U\} \subseteq K.
\]
Hence there exists some \( k \in \mathbb{N} \) such that
\[
\delta\left(\{n \in \mathbb{N} : x_n - x_k \notin U\}\right) = 0
\]
holds for each \( \tau \)-neighborhood \( U \) of zero, which proves that \( (x_n) \) is statistically \( \tau \)-Cauchy. \( \square \)

**Theorem 4.3.** Let \((L, \tau)\) be a locally solid Riesz space. Let \((x_n)\), \((y_n)\) and \((z_n)\) be three sequences in \( L \) such that \( x_n \leq y_n \leq z_n \) for each \( n \in \mathbb{N} \). If \( \lim \tau - \lim x_n = \lim \tau - \lim z_n = a \) holds, then \( \lim \tau - \lim y_n = a \).

**Proof.** Let \( U \) be an arbitrary \( \tau \)-neighborhood of zero. Then there exists a \( V \in \mathcal{N}_{\text{sol}} \) such that \( V \subseteq U \). Let us choose \( W \in \mathcal{N}_{\text{sol}} \) such that \( W + W \subseteq V \). Since \( \lim \tau - \lim x_n = \lim \tau - \lim z_n = a \), the sets \( K_1 = \{n \in \mathbb{N} : x_n - a \in W\} \) and \( K_2 = \{n \in \mathbb{N} : z_n - a \in W\} \) have asymptotic density 1. Now let \( K = K_1 \cap K_2 \). Then we have
\[
x_n - a \leq y_n - a \leq z_n - a,
\]
for each \( n \in K \). Since \( V \) is solid,
\[
y_n - a \in V \subseteq U
\]
for each \( n \in K \). Thus we have
\[
\delta\left(\{n \in \mathbb{N} : y_n - a \notin U\}\right) = 1
\]
for each \( \tau \)-neighborhood \( U \) of zero. Hence we get \( \lim \tau - \lim y_n = a \). \( \square \)

4. Statistical continuity in locally solid Riesz spaces

**Definition 4.1.** Let \((L_1, \tau_1)\) and \((L_2, \tau_2)\) be locally solid Riesz spaces and \( A \subseteq L_1 \). A function \( f : A \to L_2 \) is said to be statistically continuous at a point \( x_0 \in A \) if \( x_n \xrightarrow{\tau_1} x_0 \) in \( A \) implies \( f(x_n) \xrightarrow{\tau_2} f(x_0) \) in \( L_2 \).

**Theorem 4.1.** If a function \( f : L_1 \to L_2 \) between two locally solid Riesz spaces is uniformly continuous, then \( f \) is statistically continuous.

**Proof.** Let the function \( f : (L_1, \tau_1) \to (L_2, \tau_2) \) be uniformly continuous and \( x_n \xrightarrow{\tau_1} x_0 \) holds in \( L_1 \). Let us denote the zeros of \( L_1 \) and \( L_2 \) by \( \theta_1 \) and \( \theta_2 \), respectively. Let \( V \) be an arbitrary \( \tau_2 \)-neighborhood of \( \theta_2 \). Since \( f \) is uniformly continuous, there exists some \( \tau_1 \)-neighborhood \( W \) of \( \theta_1 \) such that
\[
x - y \in W \Rightarrow f(x) - f(y) \in V.
\]
(1)
Since \( x_n \xrightarrow{\tau_1} x_0 \), we have \( \delta(K) = 1 \), where \( K = \{n \in \mathbb{N} : x_n - x_0 \in W\} \). By (1), we have
\[
f(x_n) - f(x_0) \in V
\]
for each \( n \in K \). Then we get
\[
K \subseteq \{n \in \mathbb{N} : f(x_n) - f(x_0) \in V\},
\]
and hence
\[
\delta\left(\{n \in \mathbb{N} : f(x_n) - f(x_0) \in V\}\right) = 1.
\]
Thus we have \( f(x_n) \xrightarrow{\tau_2} f(x_0) \), which shows that \( f \) is statistically continuous. \( \square \)

**Theorem 4.2.** Let \((L, \tau)\) be a locally solid Riesz space. Then the mappings

(a) \((L, \tau) \times (L, \tau) \to (L, \tau); (x, y) \mapsto x \lor y\),
(b) \((L, \tau) \times (L, \tau) \to (L, \tau); (x, y) \mapsto x \land y\).
(c) \((L, \tau) \rightarrow (L, \tau); x \mapsto |x|\),
(d) \((L, \tau) \rightarrow (L, \tau); x \mapsto x^-
(e) \((L, \tau) \rightarrow (L, \tau); x \mapsto x^+

are all statistically continuous.

**Proof.**
(a) Let \(st_{x \times y} \lim (x_n, y_n) = (x, y)\) and \(U\) be an arbitrary \(\tau\)-neighborhood of zero in \(L\). Then there exists a \(V \in \mathcal{N}_{sol}\) such that \(V \subseteq U\). Let us choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq V\). Since \(st_{x \times y} \lim (x_n, y_n) = (x, y)\), we have \(\delta(K) = 1\), where
\[
K = \{n \in \mathbb{N}: (x_n - x, y_n - y) \in W \times W\}.
\]

Also we have
\[
|x_n \vee y_n - x \vee y| \leqslant |x_n - x| + |y_n - y| \in W + W \subseteq V
\]
for every \(n \in K\). Since \(V\) is solid, we have \(x_n \vee y_n - x \vee y \in V\) for every \(n \in K\). Then we get
\[
\{n \in \mathbb{N}: x_n \vee y_n - x \vee y \in U\} \supseteq K,
\]
and thus
\[
\delta(\{n \in \mathbb{N}: x_n \vee y_n - x \vee y \in U\}) = 1.
\]
Consequently, we have \(st_{x \times y} \lim x_n \vee y_n = x \vee y\).

(b) Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{sol}\) such that \(V \subseteq U\). Let us choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq V\). Let \(st_{x \times y} \lim (x_n, y_n) = (x, y)\). Then we have \(\delta(K) = 1\) where
\[
K = \{n \in \mathbb{N}: (x_n - x, y_n - y) \in W \times W\}.
\]

Also we have
\[
|x_n \wedge y_n - x \wedge y| = |(-x_n) \vee (-y_n) + (-x) \vee (-y)| = |(-x) - (-x_n)| + |(-y) - (-y_n)| = |x_n - x| + |y_n - y| \in W + W \subseteq V
\]
for every \(n \in K\). Since \(V\) is solid, we have \(x_n \wedge y_n - x \wedge y \in V\) for every \(n \in K\). Then we get
\[
\delta(\{n \in \mathbb{N}: x_n \wedge y_n - x \wedge y \in U\}) = 1.
\]
Hence \(st_{x \times y} \lim x_n \wedge y_n = x \wedge y\).

(c) Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{sol}\) such that \(V \subseteq U\). Let us choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq V\). Let \(st_{x \times y} \lim x_n = x \in L\). Then we have \(\delta(K) = 1\), where
\[
K = \{n \in \mathbb{N}: x_n - x \in W\}.
\]
We have
\[
||x_n| - |x|| = |x_n \vee (-x_n) - x \vee (-x)| \leqslant |x_n - x| + \left|(-x_n) - (-x)\right| \in W + W \subseteq V
\]
for every \(n \in K\). Since \(V\) is solid, we have \(|x_n| - |x| \in V\) for every \(n \in K\). Then we get
\[
\delta(\{n \in \mathbb{N}: |x_n| - |x| \in U\}) = 1.
\]
Hence \(st_{x \times y} \lim |x_n| = |x|\).

(d) Let \(U\) be an arbitrary \(\tau\)-neighborhood of zero. Then there exists a \(V \in \mathcal{N}_{sol}\) such that \(V \subseteq U\). Let \(st_{x \times y} \lim x_n = x\). Then we have \(\delta(K) = 1\), where
\[
K = \{n \in \mathbb{N}: x_n - x \in V\}.
\]
Also we have
\[
|x_n^- - x^-| = |(-x_n) \vee 0 - (-x) \vee 0| \leqslant |(-x_n) - (-x)| + |0 - 0| = |x - x_n| \in V
\]
for every $n \in K$. Since $V$ is solid, we have $x_n^- - x^- \in V$ for every $n \in K$. Then we get
\[ \delta\left(\{n \in \mathbb{N}: x_n^- - x^- \in U\}\right) = 1, \]
and thus $\operatorname{st}_\tau \lim x_n^- = x^-$. 

(e) Let $U$ be an arbitrary $\tau$-neighborhood of zero. Then there exists a $V \in \mathcal{N}_{\text{sol}}$ such that $V \subseteq U$. Let $\operatorname{st}_\tau \lim x_n = x$. Then we have $\delta(K) = 1$, where
\[ K = \{n \in \mathbb{N}: x_n - x \in V\}. \]
We also have
\[ \left|x_n^+ - x^+\right| = \left|(x_n \vee 0) - (x \vee 0)\right| \leq |x_n - x| + |0 - 0| \]
\[ = |x_n - x| \in V \]
for every $n \in K$. Since $V$ is solid, we have $x_n^+ - x^+ \in V$ for every $n \in K$. Then we get
\[ \delta\left(\{n \in \mathbb{N}: x_n^+ - x^+ \in U\}\right) = 1, \]
and thus $\operatorname{st}_\tau \lim x_n^+ = x^+$. \hfill \Box

References