Parametric excitation of a piezoelectrically actuated system near Hopf bifurcation

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1. Introduction

Stability analysis is included as one of the mostly focused fields of studies in MEM and NEM devices. Having determined the stable and unstable nature of micro and nano structures, controlling the instability is of great significance. The governing differential equation of the most of the parametrically excited MEM and NEM devices lead in a Mathieu type ODE, whose stability analysis in linear damp-less form goes back to Mathieu (1868) in connection with the problem of vibrations of elliptic membrane [1]. In 2001 El-Dib [2] studied the nonlinear Mathieu equation and coupled resonance mechanism. The method of multiple scales was used to determine a third-order solution for a cubic nonlinear Mathieu equation. According to the achieved results the amplitude of the periodic coefficient of Mathieu equation plays a dual role in the stability of the nonlinear Mathieu equation. In 2002 Ramani et al. [3] worked on perturbation solution for secondary bifurcation in the quadratically-damped Mathieu equation which usually arises in parametrically actuated MEM and NEM devices. In 2003 Inspenger and Stepan [4] studied on the stability of the damped Mathieu equation with time delay using Hill’s infinite determinant.

This paper deals with investigation into the stability analysis for transverse motions of a cantilever micro-beam, which is axially loaded due to a voltage applied to the piezoelectric layers located on the lower and upper surfaces of the micro-beam. The piezoelectric layers are pinned to the open end of the micro-beam and not bonded to it through its length. Application of the DC and AC piezoelectric actuations creates steady and time varying axial forces. The equation of the motion is derived using variational principal, and discretized using modal expansion theorem. The differential equations of the discretized model are a set of Mathieu type ODEs, whose stability analysis is performed using Floquet theory for multiple degree of freedom systems. Considering first two eigen-functions in the modal expansion theorem leads in the prediction of flutter type of instability as a consequence of Hopf bifurcation, which is not seen in the reduced single degree of freedom system. The object of the present study is to passively control the flutter instability in the proposed model by applying AC voltage with suitable amplitude and frequency to the piezoelectric layers. The effect of various parameters on the stability of the structure, including damping coefficient, amplitude of the DC and AC voltages, and the frequency of the applied AC voltage is studied.

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In 2004 Rand et al. [6] investigated the parametric resonance of Hopf bifurcation; they applied perturbation techniques and numerical integration methods to investigate the behavior of the proposed model. In 2005 Mohanty [7] theoretically and experimentally investigated the dynamic stability of beams under parametric excitation; for ordinary beams Floquet’s theory and for sandwich beams modified Hsu’s method were applied. In 2006 Rhoads et al. [8] studied on generalized parametric resonance in electro-statically actuated micro-electromechanical oscillators. They used perturbation analysis to study the behavior of the proposed model. Their results include a wide array of interesting dynamical behavior, most of which can be attributed to the existence of nonlinear parametric excitation in their equation of motion. In 2006 Abouhazim et al. [9] investigated the damped cubic nonlinear quasi-periodic Mathieu equation, using perturbation method. They also investigated the effect of damping and nonlinearity on the resonant quasi-periodic motions of the quasi-periodic Mathieu equation. In 2007 Zhang and Meng [10] studied on the response and dynamics of the electro-statically actuated MEMS resonant sensors under two parameter and external excitations. They also performed stability analysis on the response using the multiple time scales method at steady-state conditions. In 2007 Zhu et al. [11] investigated the parametric resonance of coupled micro-electromechanical oscillators under periodically varying nonlinear coupling forces. They used harmonic balance method combined with Newton iteration method to find the steady state periodic solutions. Piezoelectric materials as a type of sensing or actuating devices nowadays are widely being used in the design and fabrication process of MEM and NEM systems including resonators [12]; if appropriately actuated, these materials can be used as a stabilizing device in MEM and NEM devices. Depending on the type of piezoelectric actuation, discretizing the governing equation of the motion corresponding to the piezo-electrically sandwiched micro-beams, may lead to linear or nonlinear Mathieu type ODEs of which stability analysis and control is of great interest. Rezazadeh et al. used piezoelectric actuation to control the pull-in voltage of a fixed-fixed and cantilever MEM actuators [13]. In 2008 Zamanian et al. [14] investigated the natural frequency and the deflection of a micro-beam subjected to combined electrostatic and piezoelectric actuations; they solved the governing nonlinear equation using Galerkin method. They showed that the pull-in instability, natural frequency and the deflection of the micro-beam not only do depend on the value of electrostatic actuation but also are functions of the location, thickness and applied voltage of the piezoelectric layers. In 2008 Mahmoodi and Jalili [15] experimentally studied on out of plane vibrations of a piezoelectrically actuated micro cantilever beam. They experimentally showed that there exist cubic and quadratic nonlinearities in the micro-cantilever. In 2009 Rezazadeh and Tahmassebi [16] studied the electromechanical behavior of micro-beams subjected to piezoelectric and electrostatic actuations; they used step by step linearization method to solve the governing nonlinear equation; the achieved results show that the trigger time of the micro switches could be controlled by applying appropriate voltage to piezoelectric layers. In 2009 Yang et al. [17] investigated dynamic stability in transverse parametric vibrations of an axially accelerating Timoshenko beam on simple supports. They used Galerkin method to discretize the governing equation into a finite set of ordinary differential equations and used the method of averaging to analyze the instability due to sub-harmonic combination resonance. In 2009 Chang et al. [18] studied on the vibration and stability of an axially moving beam. They used finite element method to solve the governing differential equations and used Floquet theory to investigate the stability. In 2009 Rezazadeh et al. [19] studied on the static and dynamic stabilities of a micro-beam with various boundary conditions actuated by a DC piezoelectric voltage. By presenting a mathematical formulation and numerical solution, they investigated the critical piezoelectric force to avoid instability in a cantilever micro-beam. In 2010 Bassiouni and Abdel-Khalil [20] investigated periodic solutions for a weakly damped nonlinear Mathieu equation using multiple scales perturbation technique. They also performed stability analysis on the governing equation and investigated the effects of various types of nonlinearities on the response of the system. In 2011 Azizi et al. [21] Studied on the stabilizing of pull-in instability of a fully clamped piezoelectrically sandwiched micro-beam, subjected to electrostatic actuation. They stabilized the pull-in instability by applying AC voltage with an appropriate amplitude and frequency to the piezoelectric layers; they applied Floquet theory to perform stability analysis on the governing differential equation. According to the presented literature stability analysis of the Mathieu type equations is of great importance.

In this paper stability analysis is performed on the transverse motion of a cantilever micro-beam sandwiched by two piezoelectric layers located on the lower and upper surfaces of the micro-beam. The piezoelectric layers are pinned to the open end of micro beam end and not bonded to the micro-beam through the length of the micro-beam. In the modal expansion procedure first two modes are considered, which leads in the prediction of both divergence instability and flutter instability, which is the consequence of the Hopf (Andronov–Poincare–Hopf) bifurcation [22]. The flutter instability is possible in non-conservative systems and is essentially dynamical in nature and occurs when two eigen-forms coalesce. Flutter is thus possible only in systems with more than one degree of freedom. Of course it is worth to point out a kind of dynamical instability is possible in systems with one degree of freedom, which is known as stale flutter or galloping [23]. In the Andronov–Poincare–Hopf bifurcation system becomes unstable due to a pair of roots of the characteristic equation acquiring a same imaginary value and a positive real part. Typically in nonlinear systems structure in the post-critical behavior will exhibit a limit cycle (stable or unstable limit cycles) due to a complicated energy exchange but in linear systems (as in the structure proposed in the current article) limit cycles are no longer possible [23].

The stability analysis is performed, using Floquet theory for multiple degree of freedom systems. The stable and unstable regions of the problem are investigated, and the transition curves separating stable from unstable solution of the system in the parameter plane and in terms of amplitude and frequency of the internal parametric excitation are illustrated. It is tried to stabilize the flutter instability by applying a suitable AC voltage to the piezoelectric layers. The effect of damping
coefficient on the stability criteria is also investigated. According to the results using two terms of the eigen-function expansion leads in the results with interesting qualifications due to the existence of the Hopf bifurcation.

2. Modeling

As illustrated in Fig. 1 the studied model is an isotropic micro-beam of length \( l \), width \( a \), thickness \( h \), density \( \rho \) with Young’s modulus \( E \), sandwiched with piezoelectric layers having thickness \( h_p \), density \( \rho_p \) throughout the micro-beam length. The piezoelectric layers are pinned to the open end of micro-beam and not bonded to it through its length. The Young modulus of the piezoelectric layers is denoted by \( E_p \) and the equivalent piezoelectric coefficient is supposed to be \( \varepsilon_{31} \). The coordinate system as illustrated in Fig. 1, is attached to the middle of the left end of the micro-beam and \( x \) and \( z \) refer to the horizontal and vertical coordinates respectively.

The governing equation of the transverse motion can be obtained by the minimization of the Lagrangian using Variational Principle. The potential energy includes the following terms [13,21]:

\[
U(t) = U_b + U_a = \frac{1}{2}(EI)_{sl} \int_0^l \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx + F_p \frac{1}{2} \frac{1}{(E)_{sl}} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx.
\]

Considering, \( \varepsilon_{piezo} = \varepsilon_{si} \) one will have:

\[
F_p = \frac{a\varepsilon_{31} V_{piezo} A_{piezo} E_{piezo}}{2A_{si} E_{si}} - \frac{A_{piezo} E_{piezo}}{A_{si} E_{si}}.
\]

where:

\[
V_{piezo} = (V_{dc} + V_{ac} \cos(2\omega t)),
\]

where \( l \) denotes the moment of inertia of the cross section about the horizontal axis passing through the center of the surface of the micro-beam cross section, \( w \) is the mid plane deflection, \( V_{ac} \), \( V_{dc} \), and \( \omega \) refer to the amplitude of the alternative and direct voltages and the frequency of the voltage applied to the piezoelectric layers respectively. The voltage \( V \) is applied to the upper and lower sides of the piezoelectric layers. The kinetic energy term can be expressed as follows:

\[
T = \frac{1}{2} \rho A \int_0^l \left( \frac{\partial w}{\partial t} \right)^2 dx.
\]

The Lagrangian is defined as:

\[
L(t) = T(t) - U_b - U_a = \frac{1}{2}(\rho A)_{sl} \int_0^l \left( \frac{\partial w}{\partial t} \right)^2 dx - (E)_{sl} \int_0^l \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx - F_p \frac{1}{2} \frac{1}{(E)_{sl}} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx.
\]

Extremizing the Hamiltonian and using Rayleigh’s dissipation function to exploit Lagrange equation for non-conservative systems, the dynamic equation of the motion inside a viscous environment will be obtained as follows [21]:

\[
(E)_{sl} \frac{\partial^2 w}{\partial x^2} + (\rho A)_{sl} \frac{\partial^2 w}{\partial t^2} + \bar{C} \frac{\partial w}{\partial t} - 2F_p \frac{\partial^2 w}{\partial x^2} = 0,
\]

where \( \bar{C} \) denotes the viscous damping per unit length of the micro-beam.

![Fig. 1. Schematic of a cantilever MEM actuator with piezoelectric actuation. (A) Front view, (B) side view.](Image)
3. Eigen value analysis

There is no exact solution to the governing equations for parametrically excited systems of second order differential equations with periodic coefficients [24]. An approximate solution of Eq. (7) is supposed to be in the form of:

\[ w = \sum_{j=1}^{n} \phi_j(x)q_j(t), \]

where \( \phi_j(x) \) and \( q_j(t) \) refer to the shape functions of a simple cantilever beam and time dependant amplitudes, respectively. Substituting Eq. (8) into Eq. (7) will result in:

\[ (EI)_{ii} \sum_{j=1}^{n} \phi_j''(x)q_j(t) + (\rho A)_{ii} \sum_{j=1}^{n} \phi_j(x)\ddot{q}_j(t) - F_P \sum_{j=1}^{n} \phi_j''(x)q_j(t) + \ddot{C} \sum_{j=1}^{n} \phi_j(x)\ddot{q}(t) = R(x, t). \]

Multiplying the obtained equation by \( \phi_i(x) \) and integrating the outcome over domain [0, l] one will have:

\[ (EI) \int_0^l \phi_i(x)\phi_j''(x)dx + (\rho A) \int_0^l \phi_i(x)\phi_j(x)dx - F_P \int_0^l \phi_i(x)\phi_j''(x)dx \]
\[ + \ddot{C} \int_0^l \phi_i(x)\phi_j(x)dx = 0. \]

This can be transformed as follows:

\[ \sum_{j=1}^{n} \ddot{q}_j(t)K_{ij} + \sum_{j=1}^{n} \ddot{q}_j(t)M_{ij} + \sum_{j=1}^{n} \ddot{q}(t)C_{ij} = 0, \]

where:

\[ K_{ij} = (EI) \int_0^l \phi_i(x)\phi_j''(x)dx - F_P \int_0^l \phi_i''(x)\phi_j(x)dx. \]
\[ M_{ij} = (\rho A) \int_0^l \phi_i(x)\phi_j(x)dx, \]
\[ C_{ij} = \ddot{C} \int_0^l \phi_i(x)\phi_j(x)dx. \]

For a cantilever micro-beam the first two eigen-functions are given as follows [25]:

\[ \phi_1(x) = -1.0178 \left( \cos \left( 4.73 \frac{x}{l} \right) \right) - \cosh \left( 4.73 \frac{x}{l} \right) \left[ \sin \left( 4.73 \frac{x}{l} \right) - \sinh \left( 4.73 \frac{x}{l} \right) \right], \]

\[ \phi_2(x) = -0.9992 \left( \cos \left( 7.85 \frac{x}{l} \right) \right) - \cosh \left( 7.85 \frac{x}{l} \right) \left[ \sin \left( 7.85 \frac{x}{l} \right) - \sinh \left( 7.85 \frac{x}{l} \right) \right]. \]

To determine the first two natural frequencies of the micro-beam it is supposed that there exists neither AC voltage nor viscous damping. So the solution to Eq. (11) is supposed to be in the form \( q_j(t) = Aje^{ijt} \). The eigen values will be obtained solving the following equation:

\[ \text{Det}(K - Ms) = 0. \]

Figs. 2 and 3 illustrate the eigen values of the system with respect to various applied DC voltages. According to the results, as the applied voltage increases the eigen values corresponding to the first and the second modes of the micro-beam increase and decrease, respectively, this behavior is continued up to a definite DC voltage where the mentioned eigen values coalesce, this point is regarded as flutter instability, which is due to the modal interaction and denoted as Hopf bifurcation. In the present model the voltage denoting the flutter instability is 6.6 (V), which is in a good agreement with that achieved from the critical voltage corresponding to the critical load of the Beck problem denoted as [19,23]:

\[ P^c = 20.05 \frac{(EI)_{ii}}{l^2}. \]

Increasing the amount of the applied DC voltage up to 32.8 (V), leads in divergence instability.
4. Stability analysis

As mentioned flutter instability is due to the modal interaction, and the discretized model will cease to predict this type of instability, unless at least two terms of the eigen-function expansion, in the modal expansion process are considered. In the present study the effects of first two eigen functions in the modal expansion procedure are considered, so that one can predict both flutter and divergence instabilities in the response of the structure. So considering two terms of the modal expansion (8) and dropping the rest, and applying the transformation $x = s$ Eq. (11) will take the following form:

$$
\frac{d^2 q_1}{ds^2} = -\frac{M_2 K_{22} - M_2 K_{11}}{\sigma^2 (M_{11} M_{22} - M_{12} M_{21})} q_1 + \frac{M_2 \tilde{C}_{12} M_2 \tilde{C}_{11}}{\sigma^2 (M_{11} M_{22} - M_{12} M_{21})} \frac{dq_1}{d\tau},
$$

$$
\frac{d^2 q_2}{ds^2} = -\frac{M_1 K_{11} - M_2 K_{11}}{\sigma^2 (M_{11} M_{22} - M_{12} M_{21})} q_2 + \frac{M_1 \tilde{C}_{11} M_2 \tilde{C}_{11}}{\sigma^2 (M_{11} M_{22} - M_{12} M_{21})} \frac{dq_2}{d\tau}.
$$

As relations (12) and (16) express the stiffness components include steady and time varying components. Expressing Eq. (16) in phase space is possible by the defining components of vector $S$ as:

$$
S_1 = q_1, \quad S_2 = \frac{dq_1}{d\tau},
$$

$$
S_3 = q_2, \quad S_4 = \frac{dq_2}{d\tau}.
$$
where $S_i$ components of $S$ represent the phase space variables. The corresponding set of ODEs in the phase space will be as follows:

$$\frac{dS}{dt} = \Pi S,$$

where components of matrix $\Pi$ are as following:

$$\Pi_{11} = 0, \quad \Pi_{12} = 1, \quad \Pi_{13} = 0, \quad \Pi_{14} = 0,$$

$$\Pi_{21} = \frac{M_{12}K_{21} - M_{22}K_{11}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})}, \quad \Pi_{22} = \frac{M_{12}\tilde{C}_{21} - M_{22}\tilde{C}_{11}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})},$$

$$\Pi_{23} = \frac{M_{12}K_{22} - M_{22}K_{12}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})}, \quad \Pi_{24} = \frac{M_{12}\tilde{C}_{22} - M_{22}\tilde{C}_{12}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})},$$

$$\Pi_{31} = 0, \quad \Pi_{32} = 0, \quad \Pi_{33} = 0, \quad \Pi_{34} = 1,$$

$$\Pi_{41} = \frac{M_{12}K_{11} - M_{11}K_{21}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})}, \quad \Pi_{42} = \frac{M_{12}\tilde{C}_{11} - M_{11}\tilde{C}_{21}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})},$$

$$\Pi_{43} = \frac{M_{12}K_{12} - M_{11}K_{22}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})}, \quad \Pi_{44} = \frac{M_{12}\tilde{C}_{12} - M_{11}\tilde{C}_{22}}{\omega^2(M_{11}M_{22} - M_{12}M_{21})}.$$

To determine the eigen-values and consequently Floquet exponents of Eq. (17), fundamental set of solutions of Eq. (17) are numerically solved and the corresponding Floquet exponents are derived.

### 5. Results and discussions

The values of the parameters involved in the problem are given in Table 1:

The results of the stability analysis are given in Figs. 4–10 to illustrate the stable and unstable (shaded) regions in the plane of the excitation frequency and the amplitude of $V_{ac}$.

As the results claim, increasing the amount of the applied DC voltage enlarges the vastity of the unstable region. This is because of the bending stiffness hardening and softening nature of the positive DC voltage for the first and second mode of the micro-beam, respectively. Furthermore the more the amplitude of the AC voltage increases the more does the tendency of the system to experience unstable response increases. This is reasonable due to the growth of the amplitude of the time varying component of the stiffness. In Fig. 8 the applied DC voltage is equal to that in which flutter instability occurs, so almost everywhere in the plane of the amplitude of $V_{ac}$, and the excitation frequency is unstable, except those of high excitation frequency and low amplitude of $V_{ac}$; this is considerably important, because one may excite the micro-beam with a

| Table 1 Geometrical and material properties of the micro-beam and piezoelectric layers. |
|-------------------------------|-------------------------------|
| **Micro-beam** | **Piezoelectric layer** |
| Length | 250 $\mu$m | 250 $\mu$m |
| Width | 50 $\mu$m | 50 $\mu$m |
| Height | 3 $\mu$m | 0.01 $\mu$m |
| Young’s modulus | 169.61 GPa | 76.6 GPa |
| Density | 2331 kg/m$^3$ | 7500 kg/m$^3$ |
| $\varepsilon_{31}$ | $\frac{34}{1000}$ | $\frac{33}{1000}$ |
| $\tilde{C}$ | $-9.29$ $[9]$ |

![Fig. 4a.](image-url) $V_{dc} = 0.00$ (V), $\tilde{C} = 0.00$. 


Fig. 4b. $V_{dc} = 0.00$ (V), $\tilde{C} = 0.05$.

Fig. 4c. $V_{dc} = 0.00$ (V), $\tilde{C} = 0.10$.

Fig. 5a. $V_{dc} = 2.00$ (V), $\tilde{C} = 0.00$.

Fig. 5b. $V_{dc} = 2.00$ (V), $\tilde{C} = 0.05$.
Fig. 5c. $V_{dc} = 2.00\, (V)$, $\tilde{C} = 0.10$.

Fig. 6a. $V_{dc} = 4.00\, (V)$, $\tilde{C} = 0.00$.

Fig. 6b. $V_{dc} = 4.00\, (V)$, $\tilde{C} = 0.05$.

Fig. 6c. $V_{dc} = 4.00\, (V)$, $\tilde{C} = 0.10$. 
Fig. 7a. $V_{dc} = 6.00 \text{ (V)}, \ddot{C} = 0.00$.

Fig. 7b. $V_{dc} = 6.00 \text{ (V)}, \ddot{C} = 0.05$.

Fig. 7c. $V_{dc} = 6.00 \text{ (V)}, \ddot{C} = 0.10$.

Fig. 8a. $V_{dc} = 6.63 \text{ (V)}, \ddot{C} = 0.00$. 
suitable amplitude and frequency of piezoelectric actuation, to stabilize the instability. As reasonable increasing the value of the damping coefficient lessens the amplitude of the motion of the micro-beam so has a stabilizing effect in the behavior of the structure. As obvious, the transition curve in Fig. 8 asymptotically lies on the horizontal axis as the excitation frequency reduces; This is due to the unstable nature of the flutter point; however applying 'AC' voltage even with a small amplitude but a high enough frequency results in the stability of the micro-beam. Fig. 9 illustrates the results of the stability analysis for DC voltage, more than that in which flutter instability occurs, as clear one can stabilize the unstable behavior, in a strip with high enough excitation frequency and a specified range of $V_{ac}$; this strip enlarges as the damping coefficient increases. Fig. 10 illustrates the results of stability analysis for the applied DC voltage equal with that of divergence instability in which the eigen values are all imaginary with no real part which leads in the exponentially growth of the amplitude of the motion. As clear no stable region is seen in the specified domain of excitation frequency and the amplitude of $V_{ac}$. Fig. 11 illustrates the results of the stability analysis using Floquet theory for single degree of freedom system, considering the fundamental
eigen function in the modal expansion procedure. Comparing the results, reveals that considering one mode in the eigen
function expansion, not only ceases to predict the flutter type of instability but also DC voltages lower than that which
corresponds to flutter instability, leads in strict full results. In other words a point may be regarded to stable region however
it is predicted to be unstable in one mode expansion. Figs. 12–22 illustrate the time histories and the corresponding phase
portrait of the highlighted points (A), (B)...

Figs. 12–14 correspond to the time histories and the corresponding phase portraits of point (A) with specified damping
coefficients illustrated in Fig. 5. As clear increasing the damping coefficient to a high enough limit stabilizes the response of
point (A), which in damp-less form was located in unstable region. The results are in good agreement with what stability
analysis predicts.

Fig. 9b. $V_a = 7.0 \, (V), \, \bar{c} = 0.05$.

Fig. 9c. $V_a = 7.0 \, (V), \, \bar{c} = 0.10$.

Fig. 10a. $V_a = 33.0 \, (V), \, \bar{c} = 0.00$. 
Figs. 15–20 correspond to the time histories and the phase portraits of points (B) and (C) illustrated in Fig. 8. Point (B) in Fig. 8, which refers to the flutter instability, is located in unstable region however, point (C) refers to the same amplitude of $V_{ac}$ but a high enough excitation frequency to stabilize the unstable behavior of the structure. As logical increasing the damping coefficient, decrease the amplitude of the response, which leads in the stability of point (B) without the necessity of increasing excitation frequency.

As Fig. 9c illustrates, points (D) and (E) are located in stable and unstable regions. So one expects unbounded and bounded responses as Figs. 21 and 22, claim respectively. The instability of point (E) can be stabilized by altering the amplitude of the applied AC voltage to an arbitrary value included in the range of the strip in the same excitation frequency.
Fig. 12a. Point (A): $\omega = 1.50 \times 10^3$ (rad/s), $V_{ac} = 6$ (V), $\ddot{C} = 0.0$.

Fig. 12b. Point (A): $\omega = 1.50 \times 10^3$ (rad/s), $V_{ac} = 6$ (V), $\ddot{C} = 0.0$.

Fig. 13a. Point (A): $\omega = 1.50 \times 10^3$ (rad/s), $V_{ac} = 6$ (V), $\ddot{C} = 0.05$. 
Fig. 23. Illustrates the phase portrait of the end tip point of the micro-beam with various applied DC voltages and without any AC voltages. The figure includes the bifurcation from stable center to unstable spiral in the DC voltage equal to that of flutter instability. For more clarity the initial conditions are chosen so that to excite only the fundamental mode.

Fig. 13b. Point (A): $\omega = 1.50 \times 10^5 \text{ (rad/s)}$, $V_{ac} = 6 \text{ (V)}$, $\tilde{C} = 0.05$.

Fig. 14a. Point (A): $\omega = 1.50 \times 10^5 \text{ (rad/s)}$, $V_{ac} = 6 \text{ (V)}$, $\tilde{C} = 0.1$.

Fig. 14b. Point (A): $\omega = 1.50 \times 10^5 \text{ (rad/s)}$, $V_{ac} = 6 \text{ (V)}$, $\tilde{C} = 0.1$.
Fig. 15a. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{C} = 0.0$.

Fig. 15b. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{C} = 0.0$.

Fig. 16a. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{C} = 0.05$. 
Fig. 16b. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{c} = 0.05$.

Fig. 17a. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{c} = 0.1$.

Fig. 17b. Point (B): $\omega = 1.50 \times 10^5$ (rad/s), $V_w = 0.2$ (V), $\tilde{c} = 0.1$. 
Fig. 18a. Point (C): $\omega = 2.0 \times 10^5$ (rad/s), $V_{ac} = 0.2$ (V), $\tilde{C} = 0.0$.

Fig. 18b. Point (C): $\omega = 2.0 \times 10^5$ (rad/s), $V_{ac} = 0.2$ (V), $\tilde{C} = 0.0$.

Fig. 19a. Point (C): $\omega = 2.0 \times 10^5$ (rad/s), $V_{ac} = 0.2$ (V), $\tilde{C} = 0.05$. 
Fig. 19b. Point (C): \( \omega = 2.0 \times 10^5 \) (rad/s), \( V_w = 0.2 \) (V), \( \ddot{\gamma} = 0.05 \).

Fig. 20a. Point (C): \( \omega = 2.0 \times 10^5 \) (rad/s), \( V_w = 0.2 \) (V), \( \ddot{\gamma} = 0.1 \).

Fig. 20b. Point (C): \( \omega = 2.0 \times 10^5 \) (rad/s), \( V_w = 0.2 \) (V), \( \ddot{\gamma} = 0.1 \).
**Fig. 21a.** Point (D): $\omega = 2.0 \times 10^5 \text{ (rad/s)}, V_w = 0.9 \text{ (V)}, \tilde{C} = 0.1.$

**Fig. 21b.** Point (D): $\omega = 2.0 \times 10^5 \text{ (rad/s)}, V_w = 0.9 \text{ (V)}, \tilde{C} = 0.1.$

**Fig. 22a.** Point (E): $\omega = 2.0 \times 10^5 \text{ (rad/s)}, V_w = 0.9 \text{ (V)}, \tilde{C} = 0.1.$
Fig. 24 illustrates the phase portrait of the end tip point of the micro beam with various applied DC voltages. The amplitude of the AC voltage is supposed to be zero. In this figure, all the trajectories are excited with the same initial conditions, which leads to the excitation of the first two frequencies of the system.

Fig. 22b. Point (E): $\omega = 2.0 \times 10^5 \text{ (rad/s)}$, $V_{ac} = 0.75 \text{ (V)}$, $\tilde{C} = 0.1$.

Fig. 23. Phase portrait of the end tip point of the micro beam.

Fig. 24. Phase portrait of the end tip point of the micro beam.
6. Conclusion

In this paper the stability analysis was performed for transverse motions of a cantilever micro-beam when is axially loaded by steady and time varying axial forces created by application of AC and DC voltages applied to the piezoelectric layers. The governing differential equation of the motion was obtained using Hamiltonian principal and developed to non-conservative systems applying Rayleigh's dissipation function. The partial differential equation of the motion was discretized to set of ODEs using modal expansion theorem and numerically solved by the application of the Runge–Kutta method. The stability analysis was performed using Floquet theory for multiple degree of freedom systems. Depended on the sign of the applied DC voltage to the piezoelectric layers, tensile or compressive axial load is created in the micro-beam, whose effect was seen in the equivalent stiffness matrix of the discretized model. Application of AC voltage to the piezoelectric layers resulted in the time varying elements in the stiffness matrix. Although the excitation frequency range of the AC voltage was so that the fundamental mode dominated the response, we considered the effects of first two modes in the modal expansion theorem; this was due to the fact that flutter instability which is a common failure in mechanical structures, could not be predicted in a single degree of freedom model. According to the results of applying positive DC voltage to the piezoelectric layers resulted in the increase and decrease of the fundamental and second frequency of the system. This behavior was continued up to a critical voltage in which the first two frequencies of the system coalesce, which leads in the creation of the imaginary part in the eigen values of the system. This is denoted as flutter instability or Hopf bifurcation, in which the amplitude of the motion growths boundlessly. It was shown that one can passively control the flutter instability by the application AC voltage with definite amplitude and a high enough excitation frequency. It was observed that the damping coefficient has a stabilizing effect in the response of the structure; this is due to the fact that damping coefficient limits the amplitude of the response. According to the results in Hopf bifurcation the stable centre trajectories transform to unstable spirals.

References