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The h-Laplace and q-Laplace transforms

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ABSTRACT

Starting with a general definition of the Laplace transform on arbitrary time scales, we specify the particular concepts of the h-Laplace and q-Laplace transforms. The convolution and inversion problems for these transforms are considered in some detail.

1. Introduction

The Laplace transform provides an effective method for solving linear differential equations with constant coefficients and certain integral equations, while the Z-transform, which can be considered as a discrete version of the Laplace transform, is well suitable for linear recurrence relations and certain summation equations. The theory of Laplace transforms on time scales, which is intended to unify and to generalize the continuous and discrete cases, was initiated by Hilger [12] and then developed by Peterson and the authors [4–6].

In general, when we solve an equation (differential or integral) by the method of integral transformations, we match the transforms that occur in it with a table of known transforms and we implicitly assume a uniqueness theorem, i.e., that no two different functions have the same transform. The uniqueness theorem is essential for the existence of a well-defined inverse transform. Another question is to find a formula for the inverse transform. These problems turned out to be difficult for the Laplace transform on arbitrary time scales. Therefore, we undertake in the present paper to elucidate them in the case of the particular time scales $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$.

In a recent paper [4], the authors have introduced the concept of convolution of two functions defined on an arbitrary time scale and proved the convolution theorem stating that the Laplace transform of the convolution of two functions equals the product of the Laplace transforms of the functions. In the present paper, we also illustrate this general result for the two interesting special cases of time scales $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$.

The paper is organized as follows. Following [4–6], we present in Section 2 a definition of the exponential function on an arbitrary time scale $\mathbb{T}$ and give some properties of the exponential function, whereas in Section 3 we present definitions of the Laplace transform and the convolution of two functions on an arbitrary time scale. In Sections 4 and 5, we specify the concepts from Sections 2 and 3 for the particular time scales $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = q^{\mathbb{N}_0}$, respectively. The inversion of the Laplace transform for these particular cases is considered more carefully.

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Note that the contents of Section 4 is closely connected to the classical Z-transform (for the Z-transform and its applications see [14]). Various versions of the q-Laplace transform and their applications were earlier considered in [1–3,8–10]. Our definition of the q-Laplace transform turns out to be different from those.

2. The exponential function

For the usual notions and notations connected to time scales calculus, we refer the reader to Appendix A at the end of this paper.

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Let $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$. A function $p : \mathbb{T} \to \mathbb{C}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{C}$ will be denoted by $\mathcal{R}$. The set $\mathcal{R}$ forms an Abelian group under the addition $\oplus$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all } t \in \mathbb{T}.$$

The additive inverse of p in this group is denoted by $\ominus p$, and it is given by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all } t \in \mathbb{T}.$$

We then define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) = \big(p \oplus (\ominus q)\big)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all } t \in \mathbb{T}.$$

It follows directly from the definitions that

$$p \ominus p = 0, \qquad \ominus(\ominus p) = p, \qquad \ominus(p \ominus q) = q \ominus p, \qquad \ominus(p \oplus q) = (\ominus p) \oplus (\ominus q).$$

The group $(\mathcal{R}, \oplus)$ is called the *regressive group*. Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem

$$y^{\Delta} = p(t)y, \qquad y(t_0) = 1 \tag{2.1}$$

has a unique solution on $\mathbb{T}$ (see [5, Theorem 5.8]).

Definition 2.1. If $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the unique solution of the initial value problem (2.1) is called the *exponential function* and denoted by $e_p(\cdot, t_0)$.

In the following theorem we collect some important properties of the exponential function. Their proofs can be found in [5, Theorems 2.36 and 2.39].

Theorem 2.2. *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = [1 + \mu(t)p(t)]e_p(t,s)$ *and* $e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(s)p(s)}$;
(iii) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(v) $e_p(t,s)e_q(t,s) = e_{p \oplus q}(t,s)$;
(vi) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t,s)$;
(vii) $(e_p(\cdot,s))^{\Delta}(t) = p(t)e_p(t,s)$ *and* $(e_p(s,\cdot))^{\Delta}(t) = -p(t)e_p(s,\sigma(t))$;
(viii) $(\frac{1}{e_p(\cdot,s)})^{\Delta}(t) = -\frac{p(t)}{e_p(\sigma(t),s)}$.

Note that if $\mathbb{T} = \mathbb{R}$, then we have for all $t \in \mathbb{T}$ that $\sigma(t) = t$, $\mu(t) = 0$, and $y^{\Delta} = y'$ is the usual derivative. Therefore, in this case for any function $p : \mathbb{T} \to \mathbb{C}$ and $t, t_0 \in \mathbb{T}$ we have

$$e_p(t, t_0) = \exp\left\{ \int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau \right\}.$$

In particular, for any complex constant α, we have

$$e_\alpha(t, t_0) = e^{\alpha(t - t_0)}.$$

3. The Laplace transform

Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$. Below we assume that z is a complex constant which is regressive, that is, $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$. Then also $\ominus z \in \mathcal{R}$ and therefore $e_{\ominus z}$ is well defined on $\mathbb{T}$.

Definition 3.1. Suppose $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is a locally Δ-integrable function, i.e., it is Δ-integrable over each compact subinterval of $[t_0, \infty)_{\mathbb{T}}$. Then the *Laplace transform* of x is defined by

$$\mathcal{L}\{x\}(z) = \int_{t_0}^{\infty} x(t) e_{\ominus z}\big(\sigma(t), t_0\big) \Delta t \quad \text{for } z \in \mathcal{D}\{x\}, \tag{3.1}$$

where $\mathcal{D}\{x\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper Δ-integral exists.

The following two concepts are introduced and investigated by the authors in [4].

Definition 3.2. For a given function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, its *shift* (or *delay*) $\hat{f}(t, s)$ is defined as the solution of the problem

$$\begin{aligned} &\hat{f}^{\Delta_t}\big(t, \sigma(s)\big) = -\hat{f}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T},\ t \geqslant s \geqslant t_0, \\ &\hat{f}(t, t_0) = f(t), \quad t \in \mathbb{T},\ t \geqslant t_0. \end{aligned} \tag{3.2}$$

Definition 3.3. For given functions $f, g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, their *convolution* $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \hat{f}\big(t, \sigma(s)\big) g(s) \Delta s, \quad t \in \mathbb{T},\ t \geqslant t_0, \tag{3.3}$$

where $\hat{f}$ is the shift of f introduced in Definition 3.2.

Note that if $\mathbb{T} = \mathbb{R}$, then $\ominus z = -z$, $\sigma(t) = t$,

$$e_{\ominus z}\big(\sigma(t), t_0\big) = e^{-z(t - t_0)},$$

and (3.1) with $t_0 = 0$ becomes the usual Laplace transform

$$\mathcal{L}\{x\}(z) = \int_0^{\infty} x(t) e^{-zt}\, \mathrm{d}t.$$

The problem (3.2) takes the form

$$\frac{\partial \hat{f}(t, s)}{\partial t} = -\frac{\partial \hat{f}(t, s)}{\partial s}, \qquad \hat{f}(t, t_0) = f(t),$$

and its unique solution is $\hat{f}(t, s) = f(t - s + t_0)$. Therefore the definition (3.3) with $t_0 = 0$ takes in this case the form

$$(f * g)(t) = \int_0^{t} f(t - s) g(s)\, \mathrm{d}s.$$

4. The *h*-Laplace transform

In this section we consider the time scale

$$\mathbb{T} = h\mathbb{Z} = \{hk\colon\ k \in \mathbb{Z}\},$$

where h is a fixed positive real number and $\mathbb{Z}$ denotes the set of all integers. Then we have

$$\sigma(t) = t + h \quad \text{and} \quad \mu(t) = h.$$

For a function $f : h\mathbb{Z} \to \mathbb{C}$ we have

$$f^{\Delta}(t) = \frac{f(t + h) - f(t)}{h} \quad \text{for all } t \in h\mathbb{Z}.$$

Therefore for any complex number z, the initial value problem

$$y^{\Delta} = zy, \quad t \in \mathbb{T}, \qquad y(t_0) = 1$$

takes the form

$$y(t+h) = (1+hz)y(t), \quad t \in h\mathbb{Z}, \qquad y(t_0) = 1.$$

Hence $e_z(t, t_0)$ has (for $z \neq -1/h$) the form

$$e_z(t, t_0) = (1+hz)^{\frac{t-t_0}{h}} \quad \text{for all } t \in h\mathbb{Z}.$$

Next, we have

$$\ominus z = -\frac{z}{1+\mu(t)z} = -\frac{z}{1+hz}$$

so that the initial value problem

$$y^{\Delta} = (\ominus z)(t)y, \quad t \in \mathbb{T}, \qquad y(t_0) = 1$$

takes the form

$$f(t+h) = \frac{1}{1+hz}y(t), \quad t \in h\mathbb{Z}, \qquad y(t_0) = 1.$$

Hence $e_{\ominus z}(t, t_0)$ has (again for $z \neq -1/h$) the form

$$e_{\ominus z}(t, t_0) = (1+hz)^{-\frac{t-t_0}{h}} \quad \text{for all } t \in h\mathbb{Z}.$$

Consequently, for any function $x : [t_0, \infty)_{h\mathbb{Z}} \to \mathbb{C}$, its Laplace transform $\tilde{x}(z)$ has, according to (3.1), the form

$$\begin{aligned}
\tilde{x}(z) = \mathcal{L}\{x\}(z) &= h \sum_{t \in [t_0,\infty)_{h\mathbb{Z}}} x(t)(1+hz)^{-\frac{t+h-t_0}{h}} \\
&= h \sum_{t \in [0,\infty)_{h\mathbb{Z}}} x(t+t_0)(1+hz)^{-\frac{t+h}{h}} \\
&= h \sum_{k=0}^{\infty} \frac{x(kh+k_0h)}{(1+hz)^{k+1}} = \frac{h}{1+hz} \sum_{k=0}^{\infty} \frac{x(kh+k_0h)}{(1+hz)^k},
\end{aligned}$$

where we have put $k_0 = t_0/h$ so that $k_0 \in \mathbb{Z}$. It is clear from the latter formula that the case $t_0 \neq 0$ does not essentially differ from the case $t_0 = 0$. Thus we arrive at the following definition.

Definition 4.1. If $x : h\mathbb{N}_0 \to \mathbb{C}$ is a function, then its *h-Laplace transform* is defined by

$$\tilde{x}(z) = \mathcal{L}\{x\}(z) = \frac{h}{1+hz} \sum_{k=0}^{\infty} \frac{x(kh)}{(1+hz)^k} \tag{4.1}$$

for those values of $z \neq -1/h$ for which this series converges.

Setting

$$h_* = -\frac{1}{h}, \tag{4.2}$$

we can rewrite the formula (4.1) in the form

$$\tilde{x}(z) = \mathcal{L}\{x\}(z) = \frac{1}{z-h_*} \sum_{k=0}^{\infty} \frac{x(kh)}{h^k(z-h_*)^k}. \tag{4.3}$$

Remark 4.2. The classical Z-transform of a sequence $\{x(k)\}_{k=0}^{\infty}$ is defined by (see [14])

$$\mathcal{Z}\{x\}(z) = \sum_{k=0}^{\infty} \frac{x(k)}{z^k}.$$

Our definition (4.3) shows that in the case $h = 1$ we have

$$\mathcal{L}\{x\}(z) = \frac{\mathcal{Z}\{x\}(z+1)}{z+1}.$$

Let us now set

$$R = \limsup_{k\to\infty} \sqrt[k]{\left|x(kh)\right|}. \tag{4.4}$$

Note that the number R in general may depend on h. If $0 < R < \infty$, then the series (4.3) converges in the region $|z - h_*| > R/h$ and diverges for $|z - h_*| < R/h$. If $R = 0$, then the series (4.3) converges everywhere with the possible exception of the point $z = h_*$. Finally, if $R = \infty$, then the series (4.3) diverges everywhere.

Let us present some useful properties of the h-Laplace transform.

Theorem 4.3 *(Shifting Theorem). If*

$$\mathcal{L}\big\{x(kh)\big\}(z) = \tilde{x}(z) \quad \textit{for } |z - h_*| > A,$$

then, for the same values of z,

$$\mathcal{L}\big\{x(kh + h)\big\}(z) = (1 + hz)\tilde{x}(z) - hx(0) \tag{4.5}$$

and

$$\mathcal{L}\big\{x(kh + 2h)\big\}(z) = (1 + hz)^2\tilde{x}(z) - h(1 + hz)x(0) - hx(h). \tag{4.6}$$

Proof. Indeed,

$$\begin{aligned}
\mathcal{L}\big\{x(kh + h)\big\}(z) &= \frac{1}{z - h_*}\sum_{k=0}^{\infty}\frac{x(kh + h)}{h^k(z - h_*)^k} = \frac{1}{z - h_*}\sum_{k=1}^{\infty}\frac{x(kh)}{h^{k-1}(z - h_*)^{k-1}} \\
&= h\sum_{k=1}^{\infty}\frac{x(kh)}{h^k(z - h_*)^k} = h\sum_{k=0}^{\infty}\frac{x(kh)}{h^k(z - h_*)^k} - hx(0) \\
&= h(z - h_*)\tilde{x}(z) - hx(0) = (hz + 1)\tilde{x}(z) - hx(0)
\end{aligned}$$

so that (4.5) holds. Formula (4.6) is obtained by applying (4.5) twice. □

Theorem 4.4 *(Initial Value and Final Value Theorem). We have the following*:

(a) *If $\tilde{x}(z)$ exists for $|z - h_*| > A$, then*

$$x(0) = \lim_{z\to\infty}\big\{z\tilde{x}(z)\big\}.$$

(b) *If $\tilde{x}(z)$ exists for $|z - h_*| > h^{-1}$ and $z\tilde{x}(z)$ is analytic at $z = 0$, then*

$$\lim_{k\to\infty} x(kh) = \lim_{z\to 0}\big\{z\tilde{x}(z)\big\}.$$

Proof. Part (a) follows immediately from the definition (4.1) of the h-Laplace transform. To prove (b), consider

$$\begin{aligned}
\mathcal{L}\big\{x(kh + h) - x(kh)\big\}(z) &= \frac{1}{z - h_*}\sum_{k=0}^{\infty}\frac{x(kh + h) - x(kh)}{h^k(z - h_*)^k} \\
&= \frac{h}{1 + hz}\sum_{k=0}^{\infty}\frac{x(kh + h) - x(kh)}{(1 + hz)^k}.
\end{aligned}$$

By using the shifting theorem (Theorem 4.3), we have

$$\begin{aligned}
\mathcal{L}\big\{x(kh + h) - x(kh)\big\}(z) &= \mathcal{L}\big\{x(kh + h)\big\}(z) - \mathcal{L}\big\{x(kh)\big\}(z) \\
&= (1 + hz)\tilde{x}(z) - hx(0) - \tilde{x}(z) = hz\tilde{x}(z) - hx(0).
\end{aligned}$$

Therefore

$$(1+hz)\big[z\tilde{x}(z)-x(0)\big]=\sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}.$$

Hence, for any $n\in\mathbb{N}_0$,

$$(1+hz)\big[z\tilde{x}(z)-x(0)\big]-\sum_{k=0}^{n}\frac{x(kh+h)-x(kh)}{(1+hz)^k}=\sum_{k=n+1}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}.$$

Next,

$$\begin{aligned}\sum_{k=0}^{n}\frac{x(kh+h)-x(kh)}{(1+hz)^k}&=\sum_{k=0}^{n}\frac{x(kh+h)}{(1+hz)^k}-\sum_{k=0}^{n}\frac{x(kh)}{(1+hz)^k}\\&=-x(0)+\sum_{j=1}^{n}x(jh)\left[\frac{1}{(1+hz)^{j-1}}-\frac{1}{(1+hz)^j}\right]+\frac{x(nh+h)}{(1+hz)^n}\\&=-x(0)+hz\sum_{j=1}^{n}\frac{x(jh)}{(1+hz)^j}+\frac{x(nh+h)}{(1+hz)^n}.\end{aligned}$$

Thus

$$\begin{aligned}&(1+hz)\big[z\tilde{x}(z)-x(0)\big]+x(0)-\frac{x(nh+h)}{(1+hz)^n}-hz\sum_{j=1}^{n}\frac{x(jh)}{(1+hz)^j}\\&\quad=\sum_{k=n+1}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}.\end{aligned}\tag{4.7}$$

Now choosing a sufficiently large value of $n\in\mathbb{N}_0$, we can make the absolute value of the right-hand side of (4.7) less than an arbitrarily given $\varepsilon>0$, uniformly with respect to z in a small neighbourhood of $z=0$. Then we pass to the limit in (4.7) as $z\to 0$. These reasonings complete the proof of part (b). □

For the case $\mathbb{T}=h\mathbb{Z}$, the *shifting problem* (3.2) takes the form

$$\hat{f}(t+h,s+h)=\hat{f}(t,s),\quad t,s\in h\mathbb{Z},\ t\geqslant s\geqslant t_0,$$
$$\hat{f}(t,t_0)=f(t),\quad t\in h\mathbb{Z},\ t\geqslant t_0,$$

where $f:[t_0,\infty)_{h\mathbb{Z}}\to\mathbb{C}$ is a given function. The unique solution of this problem is

$$\hat{f}(t,s)=f(t-s+t_0).$$

Therefore, for given functions $f,g:[t_0,\infty)_{h\mathbb{Z}}\to\mathbb{C}$, their convolution $f*g$ is, according to (3.3),

$$(f*g)(t)=h\sum_{s\in[t_0,t)_{h\mathbb{Z}}}f(t-s-h+t_0)g(s)\quad\text{for } t\in h\mathbb{Z},\ t\geqslant t_0.$$

In the case $t_0=0$, this formula yields the following definition.

Definition 4.5. For given functions $f,g:h\mathbb{N}_0\to\mathbb{C}$, their *convolution* $f*g$ is defined by

$$(f*g)(t)=h\sum_{s\in[0,t)_{h\mathbb{N}_0}}f(t-s-h)g(s)\quad\text{for } t\in h\mathbb{N}_0,$$

i.e.,

$$(f*g)(kh)=h\sum_{m=0}^{k-1}f(kh-mh-h)g(mh)\quad\text{for } k\in\mathbb{N}_0,$$

where the value of the sum for $k=0$ is understood to be zero and $\mathbb{N}_0$ denotes the set of all nonnegative integers.

It is easy to see that the convolution product is commutative and associative (i.e., $f * g = g * f$ and $(f * g) * u = f * (g * u)$).

Theorem 4.6 *(Convolution Theorem). If $\mathcal{L}\{f\}(z)$ exists for $|z - h_*| > A$ and $\mathcal{L}\{g\}(z)$ exists for $|z - h_*| > B$, then*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) \quad \text{for } |z - h_*| > \max\{A, B\}.$$

Proof. For $|z - h_*| > \max\{A, B\}$, we have

$$\begin{aligned}\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) &= \frac{1}{z-h_*}\left\{\sum_{j=0}^{\infty}\frac{f(jh)}{h^j(z-h_*)^j}\right\}\frac{1}{z-h_*}\left\{\sum_{m=0}^{\infty}\frac{g(mh)}{h^m(z-h_*)^m}\right\}\\ &= \frac{h}{z-h_*}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{f(jh)g(mh)}{h^{j+m+1}(z-h_*)^{j+m+1}},\end{aligned}$$

and if we put $j + m + 1 = k$, then we obtain

$$\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) = \frac{h}{z-h_*}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\frac{f(jh)g(kh-jh-h)}{h^k(z-h_*)^k}.$$

Interchanging the order of summation yields

$$\begin{aligned}\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) &= \frac{1}{z-h_*}\sum_{k=1}^{\infty}\left\{h\sum_{j=0}^{k-1}f(jh)g(kh-jh-h)\right\}\frac{1}{h^k(z-h_*)^k}\\ &= \frac{1}{z-h_*}\sum_{k=1}^{\infty}\left\{h\sum_{m=0}^{k-1}f(kh-mh-h)g(mh)\right\}\frac{1}{h^k(z-h_*)^k}\\ &= \frac{1}{z-h_*}\sum_{k=0}^{\infty}\left\{h\sum_{m=0}^{k-1}f(kh-mh-h)g(mh)\right\}\frac{1}{h^k(z-h_*)^k}\\ &= \mathcal{L}\{f * g\}(z).\end{aligned}$$

This completes the proof. □

Now we consider the inverse problem, i.e., given $\tilde{x}(z)$, find $x(t)$. For the existence of a well-defined inverse transform, the uniqueness property must hold: If there are two functions x and y for which $\tilde{x}(z) \equiv \tilde{y}(z)$, then we must have $x(t) \equiv y(t)$. Equivalently, if $\tilde{x}(z) \equiv 0$, then we must have $x(t) \equiv 0$.

Let $x : h\mathbb{N}_0 \to \mathbb{C}$ be a function and R be defined by (4.4) and h_* be defined by (4.2). Suppose $R < \infty$.

Lemma 4.7. *For each $A > R/h$, the series* (4.3) *converges uniformly in the region $|z - h_*| \geqslant A$.*

Proof. It follows from $A > R/h$ that there exists $\varepsilon > 0$ such that

$$A > \frac{R+\varepsilon}{h}.$$

Next, for this ε we can find by (4.4) a positive integer m such that

$$\left|x(kh)\right| \leqslant (R+\varepsilon)^k \quad \text{for all } k \geqslant m.$$

Then for $|z - h_*| \geqslant A$ we have

$$\begin{aligned}\left|\sum_{k=m}^{\infty}\frac{x(kh)}{h^k(z-h_*)^k}\right| &\leqslant \sum_{k=m}^{\infty}\frac{|x(kh)|}{h^k|z-h_*|^k}\\ &\leqslant \sum_{k=m}^{\infty}\frac{(R+\varepsilon)^k}{h^kA^k} = \left(1-\frac{R+\varepsilon}{hA}\right)^{-1}\left(\frac{R+\varepsilon}{hA}\right)^m\\ &\to 0 \quad \text{as } m \to \infty.\end{aligned}$$

This means that the series (4.3) converges uniformly in the region $|z - h_*| \geqslant A$. □

Theorem 4.8 *(Uniqueness Theorem). Let $\tilde{x}(z)$ be defined by* (4.3)*. If $\tilde{x}(z) \equiv 0$ for $|z - h_*| > R/h$, then $x(t) \equiv 0$ for $t \in h\mathbb{N}_0$.*

Proof. By the assumption, we have

$$x(0) + \frac{x(h)}{h(z - h_*)} + \frac{x(2h)}{h^2(z - h_*)^2} + \cdots \equiv 0 \quad \text{for } |z - h_*| > \frac{R}{h}. \tag{4.8}$$

Passing in (4.8) to the limit as $|z| \to \infty$ (we can take a term-by-term limit in (4.8) due to the uniform convergence proved in Lemma 4.7), we get $x(0) = 0$. Now we multiply the remaining equation (4.8) by $z - h_*$ and pass to the limit as $|z| \to \infty$ to obtain $x(h) = 0$. Repeating this procedure we find $x(0) = x(h) = x(2h) = \cdots = 0$. □

Theorem 4.9. *Let $\tilde{x}(z)$ be defined by* (4.3) *and A be a real number such that $A > R/h$. Then*

$$x(kh) = \frac{h^k}{2\pi i} \int_\Gamma (z - h_*)^k \tilde{x}(z)\,\mathrm{d}z \quad \text{for } k \in \mathbb{N}_0, \tag{4.9}$$

where Γ denotes the positively oriented circle $\{z \in \mathbb{C}\colon\ |z - h_*| = A\}$.

Proof. For any $j \in \mathbb{N}_0$, we have from (4.3) that

$$(z - h_*)^j \tilde{x}(z) = \sum_{k=0}^{\infty} \frac{x(kh)}{h^k} (z - h_*)^{j-k-1}.$$

Integrating both sides over the circle Γ and noting that we can integrate under the sum sign by the uniform convergence of the series, we obtain

$$\int_\Gamma (z - h_*)^j \tilde{x}(z)\,\mathrm{d}z = \sum_{k=0}^{\infty} \frac{x(kh)}{h^k} \int_\Gamma (z - h_*)^{j-1-k}\,\mathrm{d}z.$$

Since

$$\int_\Gamma (z - h_*)^{j-1-k}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

the latter implies

$$\int_\Gamma (z - h_*)^j \tilde{x}(z)\,\mathrm{d}z = 2\pi i \frac{x(jh)}{h^j}.$$

This concludes the proof. □

Theorem 4.9 means that the inverse of the h-Laplace transform is given by the formula

$$\mathcal{L}^{-1}\{\tilde{x}\}(kh) = \frac{h^k}{2\pi i} \int_\Gamma (z - h_*)^k \tilde{x}(z)\,\mathrm{d}z \quad \text{for } k \in \mathbb{N}_0.$$

5. The *q*-Laplace transform

In this section we consider the time scale

$$\mathbb{T} = q^{\mathbb{N}_0} = \{q^k\colon\ k \in \mathbb{N}_0\} = \{1, q, q^2, q^3, \ldots\},$$

where $q > 1$ is a fixed number. Calculus on this time scale is called *quantum calculus*, see [13]. On this time scale we have

$$\sigma(t) = qt \quad \text{and} \quad \mu(t) = (q - 1)t.$$

For a function $f : q^{\mathbb{N}_0} \to \mathbb{C}$ its Δ-derivative (also called q-derivative or *Jackson derivative*) is

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q - 1)t} \quad \text{for all } t \in q^{\mathbb{N}_0}. \tag{5.1}$$

Therefore for any complex number z, the initial value problem

$$y^{\Delta}(t) = zy(t), \qquad y(s) = 1, \quad t, s \in \mathbb{T}$$

becomes

$$y(qt) = (1 + q'tz)y(t), \qquad y(s) = 1, \quad t, s \in q^{\mathbb{N}_0},$$

where $q' = q - 1$. It follows that if we put

$$t = q^n \quad \text{and} \quad s = q^m \quad \text{with } m, n \in \mathbb{N}_0,$$

then $e_z(t, s)$ has the form

$$e_z\big(q^n, q^m\big) = \prod_{k=m}^{n-1} \big(1 + q'q^k z\big) \quad \text{if } n \geqslant m \tag{5.2}$$

and

$$e_z\big(q^n, q^m\big) = \frac{1}{\prod_{k=n}^{m-1}(1 + q'q^k z)} \quad \text{if } n \leqslant m, \tag{5.3}$$

where the products for $n = m$ are understood, as usual, to be 1. Next we will assume that

$$z \neq -\frac{1}{q'q^k} \quad \text{for all } k \in \mathbb{N}_0. \tag{5.4}$$

Since

$$(\ominus z)(t) = -\frac{z}{1 + \mu(t)z} = -\frac{1}{1 + (q-1)tz} = -\frac{1}{1 + q'tz},$$

the initial value problem

$$y^{\Delta}(t) = (\ominus z)(t)y(t), \qquad y(s) = 1, \quad t, s \in \mathbb{T}$$

becomes

$$y(qt) = \frac{1}{1 + q'tz} y(t), \qquad y(s) = 1, \quad t, s \in q^{\mathbb{N}_0}.$$

It follows that $e_{\ominus z}(q^n, q^m)$, where $m, n \in \mathbb{N}_0$ and z satisfies (5.4), has the form

$$e_{\ominus z}\big(q^n, q^m\big) = \frac{1}{\prod_{k=m}^{n-1}(1 + q'q^k z)} \quad \text{if } n \geqslant m \tag{5.5}$$

and

$$e_{\ominus z}\big(q^n, q^m\big) = \prod_{k=n}^{m-1} \big(1 + q'q^k z\big) \quad \text{if } n \leqslant m.$$

Comparing these with (5.2) and (5.3), we see that, as is expected,

$$e_{\ominus z}\big(q^n, q^m\big) = \frac{1}{e_z(q^n, q^m)}.$$

Taking into account the general definition (3.1) of the Laplace transform and (5.5), we get that for any function $x : [t_0, \infty)_{q^{\mathbb{N}_0}} \to \mathbb{C}$ with $t_0 \in q^{\mathbb{N}_0}$, its Laplace transform has the form

$$\begin{aligned} \tilde{x}(z) = \mathcal{L}\{x\}(z) &= \sum_{t \in [t_0, \infty)_{q^{\mathbb{N}_0}}} \mu(t) e_{\ominus z}(qt, t_0) x(t) \\ &= (q-1) \sum_{n=n_0}^{\infty} q^n e_{\ominus z}\big(q^{n+1}, q^{n_0}\big) x\big(q^n\big) = q' \sum_{n=n_0}^{\infty} \frac{q^n x(q^n)}{\prod_{k=n_0}^{n}(1 + q'q^k z)}. \end{aligned}$$

Thus, taking $t_0 = 1$, i.e., $n_0 = 0$, we can introduce the following definition.

Definition 5.1. If $x : q^{\mathbb{N}_0} \to \mathbb{C}$ is a function, then its *q-Laplace transform* is defined by

$$\tilde{x}(z) = \mathcal{L}\{x\}(z) = q' \sum_{n=0}^{\infty} \frac{q^n x(q^n)}{\prod_{k=0}^{n}(1 + q'q^k z)} \tag{5.6}$$

for those values of $z \neq -1/q'q^k$, $k \in \mathbb{N}_0$, for which this series converges, where $q' = q - 1$.

Let us set

$$P_n(z) = \prod_{k=0}^{n}\bigl(1 + q'q^k z\bigr), \quad n \in \mathbb{N}_0, \tag{5.7}$$

which is a polynomial in z of degree $n+1$. It is easily verified that the equations

$$P_n(z) - P_{n-1}(z) = zq'q^n P_{n-1}(z), \quad n \in \mathbb{N}_0 \tag{5.8}$$

and

$$\frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} = z\frac{q'q^n}{P_n(z)}, \quad n \in \mathbb{N}_0 \tag{5.9}$$

hold, where $P_{-1}(z) = 1$. The numbers

$$\alpha_k = -\frac{1}{q'q^k}, \quad k \in \mathbb{N}_0,$$

where $q' = q - 1$, belong to the real axis interval $[-(q-1)^{-1}, 0)$ and tend to zero as $k \to \infty$. For any $\delta > 0$ and $k \in \mathbb{N}_0$, we set

$$D_\delta^k = \bigl\{z \in \mathbb{C}\colon\ |z - \alpha_k| < \delta\bigr\}$$

and

$$\Omega_\delta = \mathbb{C} \setminus \bigcup_{k=0}^{\infty} D_\delta^k = \bigl\{z \in \mathbb{C}\colon\ |z - \alpha_k| \geqslant \delta,\ \forall k \in \mathbb{N}_0\bigr\}$$

so that Ω_δ is a closed domain of the complex plane $\mathbb{C}$, whose points are in distance not less than δ from the set $\{\alpha_k\colon\ k \in \mathbb{N}_0\}$.

Lemma 5.2. *For any $z \in \Omega_\delta$,*

$$\bigl|P_n(z)\bigr| \geqslant (q'\delta)^{n+1} q^{\frac{n(n+1)}{2}}, \quad n \in \mathbb{N}_0 \cup \{-1\}. \tag{5.10}$$

Therefore, for an arbitrary number $R > 0$, there exists a positive integer $n_0 = n_0(R, \delta, q)$ such that

$$\bigl|P_n(z)\bigr| \geqslant R^{n+1} \quad \text{for all } n \geqslant n_0 \text{ and } z \in \Omega_\delta. \tag{5.11}$$

In particular,

$$\lim_{n\to\infty} P_n(z) = \infty \quad \text{for all } z \in \Omega_\delta. \tag{5.12}$$

Proof. For any $z \in \Omega_\delta$, we have

$$\begin{aligned}\bigl|P_n(z)\bigr| &= \left|\prod_{k=0}^{n}\bigl(1 + q'q^k z\bigr)\right| = \left|\prod_{k=0}^{n} q'q^k(z - \alpha_k)\right| \\ &\geqslant \prod_{k=0}^{n} q'q^k\delta = (q'\delta)^{n+1}\prod_{k=0}^{n} q^k = (q'\delta)^{n+1} q^{\frac{n(n+1)}{2}}.\end{aligned}$$

Hence (5.10) holds. Further, we can rewrite (5.10) in the form

$$\bigl|P_n(z)\bigr| \geqslant \bigl(q'\delta q^{\frac{n}{2}}\bigr)^{n+1}.$$

On the other hand, since $q > 1$, we can choose for any given number $R > 0$ a positive integer $n_0 = n_0(R, \delta, q)$ such that

$$q'\delta q^{\frac{n}{2}} \geqslant R \quad \text{for all } n \geqslant n_0.$$

Therefore (5.11) follows. □

Example 5.3. First we find the q-Laplace transform of $x(t) \equiv 1$. We have, using (5.6), (5.7), (5.9), and (5.12),

$$\begin{aligned}\mathcal{L}\{1\}(z) &= q' \sum_{n=0}^{\infty} \frac{q^n}{P_n(z)} = \frac{1}{z} \sum_{n=0}^{\infty} \left[\frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} \right] \\ &= \frac{1}{z} \lim_{m\to\infty} \left[1 - \frac{1}{P_m(z)} \right] = \frac{1}{z}.\end{aligned}$$

Now we find the q-Laplace transform of the function $e_\alpha(t) = e_\alpha(t, 1)$, for which we have by (5.2) and (5.7),

$$e_\alpha(q^n) = \prod_{k=0}^{n-1} (1 + q' q^k \alpha) = P_{n-1}(\alpha) \quad \text{for } n \in \mathbb{N}_0.$$

It follows that

$$\begin{aligned}\widetilde{e_\alpha}(z) &= \mathcal{L}\{e_\alpha\}(z) = q' \sum_{n=0}^{\infty} \frac{q^n e_\alpha(q^n)}{P_n(z)} = q' \sum_{n=0}^{\infty} \frac{q^n P_{n-1}(\alpha)}{P_n(z)} \\ &= q' \sum_{n=0}^{\infty} \frac{q^n}{1 + q' q^n z} \prod_{k=0}^{n-1} \frac{1 + q' q^k \alpha}{1 + q' q^k z} = q' \sum_{n=0}^{\infty} \frac{q^n}{1 + q' q^n z} \prod_{k=0}^{n-1} \frac{\alpha - \alpha_k}{z - \alpha_k}.\end{aligned} \tag{5.13}$$

Since the numbers α_k, $k \in \mathbb{N}_0$, are contained in the finite interval $[-(q-1)^{-1}, 0)$, there is a sufficiently large number $R_0 > 0$ such that

$$\left| \frac{\alpha - \alpha_k}{z - \alpha_k} \right| \leqslant \frac{1}{2} \quad \text{for all } |z| \geqslant R_0 \text{ and } k \in \mathbb{N}_0. \tag{5.14}$$

Therefore the series (5.13) converges for $|z| \geqslant R_0$. Next, we can write, using (5.9),

$$\begin{aligned}\widetilde{e_\alpha}(z) &= q' \sum_{n=0}^{\infty} \frac{q^n P_{n-1}(\alpha)}{P_n(z)} = \frac{q'}{P_0(z)} + q' \sum_{n=1}^{\infty} \frac{q^n P_{n-1}(\alpha)}{P_n(z)} \\ &= \frac{q'}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{P_{n-1}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] \\ &= \frac{q'}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{(1 + q' q^{n-1} \alpha) P_{n-2}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] \\ &= \frac{q'}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{P_{n-2}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] + \frac{\alpha q'}{z} \sum_{n=1}^{\infty} \frac{q^{n-1} P_{n-2}(\alpha)}{P_{n-1}(z)} \\ &= \frac{q'}{P_0(z)} + \frac{1}{z P_0(z)} - \frac{1}{z} \lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} + \frac{\alpha}{z} \widetilde{e_\alpha}(z) \\ &= \frac{1}{z} + \frac{\alpha}{z} \widetilde{e_\alpha}(z),\end{aligned}$$

where we have used the fact that

$$\lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} = 0$$

because of

$$\frac{P_{m-1}(\alpha)}{P_m(z)} = \frac{1}{1 + q' q^m z} \prod_{k=0}^{m-1} \frac{\alpha - \alpha_k}{z - \alpha_k}$$

and (5.14). Thus, we have obtained the equality

$$\widetilde{e_\alpha}(z) = \frac{1}{z} + \frac{\alpha}{z} \widetilde{e_\alpha}(z).$$

Hence

$$\widetilde{e_\alpha}(z) = \frac{1}{z - \alpha}.$$

Theorem 5.4. *If the function $x : q^{\mathbb{N}_0} \to \mathbb{C}$ satisfies the condition*

$$|x(q^n)| \leqslant CR^n \quad \text{for all } n \in \mathbb{N}_0, \tag{5.15}$$

where C and R are some positive constants, then the series in (5.6) *converges uniformly with respect to z in the region Ω_δ and therefore its sum $\tilde{x}(z)$ is an analytic* (*holomorphic*) *function in Ω_δ.*

Proof. By Lemma 5.2, for the number R given in (5.15), we can choose an $n_0 \in \mathbb{N}$ such that

$$|P_n(z)| \geqslant [q(1+R)]^{n+1} \quad \text{for all } n \geqslant n_0 \text{ and } z \in \Omega_\delta.$$

Then for the general term of the series in (5.6), we have the estimate

$$\left|\frac{q^n x(q^n)}{P_n(z)}\right| \leqslant \frac{C}{q(1+R)}\left(\frac{R}{1+R}\right)^n \quad \text{for all } n \geqslant n_0 \text{ and } z \in \Omega_\delta.$$

Hence the proof is completed. □

A larger class of functions for which the q-Laplace transform exists is the class $\mathcal{F}_\delta$ of functions $x : q^{\mathbb{N}_0} \to \mathbb{C}$ satisfying the condition

$$\sum_{n=0}^{\infty} (q'\delta)^{-n} q^{-\frac{n(n-1)}{2}} |x(q^n)| < \infty. \tag{5.16}$$

Theorem 5.5. *For any $x \in \mathcal{F}_\delta$, the series in* (5.6) *converges uniformly with respect to z in the region Ω_δ, and therefore its sum $\tilde{x}(z)$ is an analytic function in Ω_δ.*

Proof. The proof follows from (5.10). □

Theorem 5.6. *Let x^Δ be the q-derivative of x, defined by* (5.1). *Suppose that $x \in \mathcal{F}_\delta$. Then*

$$\mathcal{L}\{x^\Delta\}(z) = z\tilde{x}(z) - x(1) \tag{5.17}$$

and

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z^2\tilde{x}(z) - zx(1) - x^\Delta(1). \tag{5.18}$$

Proof. Using the definition (5.6) of the q-Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{x^\Delta\}(z) &= q' \sum_{n=0}^{\infty} \frac{q^n x^\Delta(q^n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{x(q^{n+1}) - x(q^n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(q^{n+1})}{P_n(z)} - \sum_{n=0}^{\infty} \frac{x(q^n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{x(q^{n+1})}{P_{n+1}(z)}\left(1 + q'q^{n+1}z\right) - \sum_{n=0}^{\infty} \frac{x(q^n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(q^{n+1})}{P_{n+1}(z)} - \sum_{n=0}^{\infty} \frac{x(q^n)}{P_n(z)} + q'z \sum_{n=0}^{\infty} \frac{q^{n+1}x(q^{n+1})}{P_{n+1}(z)} \\
&= -\frac{x(q^0)}{P_0(z)} + z\left[\tilde{x}(z) - \frac{q'x(q^0)}{P_0(z)}\right] \\
&= -\frac{(1+q'z)x(q^0)}{P_0(z)} + z\tilde{x}(z) = -x(1) + z\tilde{x}(z)
\end{aligned}$$

so that (5.17) holds. The formula (5.18) is obtained by applying (5.17) to the second derivative $x^{\Delta\Delta}$. □

Theorem 5.7 *(Initial Value and Final Value Theorem). We have the following*:

(a) *If $x \in \mathcal{F}_\delta$ for some $\delta > 0$, then*

$$x(1) = \lim_{z \to \infty} \{z\tilde{x}(z)\}. \tag{5.19}$$

(b) *If $x \in \mathcal{F}_\delta$ for all $\delta > 0$, then*

$$\lim_{n\to\infty} x(q^n) = \lim_{z\to 0}\{z\tilde{x}(z)\}. \tag{5.20}$$

Proof. Assume $x \in \mathcal{F}_\delta$ for some $\delta > 0$. It follows from (5.6) that

$$\tilde{x}(z) = \frac{q'x(1)}{1+q'z} + \frac{q'qx(q)}{(1+q'z)(1+q'qz)} + \frac{q'q^2x(q)}{(1+q'z)(1+q'qz)(1+q'q^2z)} + \cdots$$

and

$$(1+q'z)\tilde{x}(z) = q'x(1) + \frac{q'qx(q)}{1+q'qz} + \frac{q'q^2x(q)}{(1+q'qz)(1+q'q^2z)} + \cdots.$$

Hence

$$\lim_{z\to\infty}\tilde{x}(z) = 0 \quad \text{and} \quad \lim_{z\to\infty}\{(1+q'z)\tilde{x}(z)\} = q'x(1),$$

which yields (5.19). Note that we have taken a term-by-term limit due to the uniform convergence of the series in the region Ω_δ.

Next, assume $x \in \mathcal{F}_\delta$ for all $\delta > 0$. In the proof of Theorem 5.6 we have obtained the formula

$$\sum_{n=0}^{\infty}\frac{x(q^{n+1})-x(q^n)}{P_n(z)} = z\tilde{x}(z) - x(1).$$

Further we can argue as in the proof of Theorem 4.4(b) taking into account that

$$\lim_{z\to 0} P_n(z) = 1 \quad \text{for any } n \in \mathbb{N}_0$$

and that for any $\delta > 0$ there is a positive integer $n_0 = n_0(\delta)$ such that $|P_n(z)| \geqslant 2^{n+1}$ for all $n \geqslant n_0$ and all $z \in \Omega_\delta$. □

For the case $\mathbb{T} = q^{\mathbb{N}_0}$, the shifting problem (3.2) with $t_0 = 1$ takes the form

$$s[\hat{f}(qt,qs) - \hat{f}(t,qs)] + t[\hat{f}(t,qs) - \hat{f}(t,s)] = 0, \quad t,s \in q^{\mathbb{N}_0},\ t \geqslant s,$$
$$\hat{f}(t,1) = f(t), \quad t \in q^{\mathbb{N}_0}, \tag{5.21}$$

where $f : q^{\mathbb{N}_0} \to \mathbb{C}$ is a given function. The unique solution of this problem is obtained as follows (see [4, Section 6]). Let us use the notation from [13], in particular

$$[\alpha] = [\alpha]_q = \frac{q^\alpha - 1}{q-1}, \quad \alpha \in \mathbb{R},$$
$$[n]! = \prod_{k=1}^{n}[k], \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}, \quad m,n \in \mathbb{N}_0,$$
$$(t-s)_q^n = \prod_{k=0}^{n-1}(t - q^k s), \quad t,s \in \mathbb{T},\ n \in \mathbb{N}_0,$$

with $[0]! = 1$ and $(t-s)_q^0 = 1$. Then if $t,s \in q^{\mathbb{N}_0}$ and $t \geqslant s$, so that $t = q^k s$ for some $k \in \mathbb{N}_0$, we have by [4]

$$\hat{f}(t,s) = \hat{f}(q^k s, s) = \sum_{\nu=0}^{k}\begin{bmatrix} k \\ \nu \end{bmatrix} s^\nu (1-s)_q^{k-\nu} f(q^\nu)$$
$$= \sum_{\nu=0}^{k} f(q^\nu)\begin{bmatrix} k \\ \nu \end{bmatrix} s^\nu \prod_{j=0}^{k-\nu-1}(1 - q^j s).$$

Hence, if we put $t = q^n$ and $s = q^m$ for $m,n \in \mathbb{N}_0$ with $n \geqslant m$, then we get

$$\hat{f}(q^n,q^m) = \sum_{\nu=0}^{n-m}\begin{bmatrix} n-m \\ \nu \end{bmatrix} q^{m\nu}(1-q^m)_q^{n-m-\nu} f(q^\nu)$$
$$= \sum_{\nu=0}^{n-m} f(q^\nu)\begin{bmatrix} n-m \\ \nu \end{bmatrix} q^{m\nu}\prod_{j=0}^{n-m-\nu-1}(1 - q^{m+j}) \tag{5.22}$$

for any $n \geqslant m \geqslant 0$. Therefore, according to the general definition (3.3) of the convolution, we can introduce the following definition.

Definition 5.8. For given functions $f, g : q^{\mathbb{N}_0} \to \mathbb{C}$, their *convolution* $f * g$ is defined by

$$\begin{aligned}(f * g)(q^n) &= (q-1)\sum_{k=0}^{n-1} q^k \hat{f}(q^n, q^{k+1}) g(q^k) \\ &= (q-1)\sum_{k=0}^{n-1} q^k \left\{ \sum_{\nu=0}^{n-k-1} f(q^\nu) \begin{bmatrix} n-k-1 \\ \nu \end{bmatrix} q^{(k+1)\nu} \prod_{j=0}^{n-k-\nu-2} (1-q^{k+j+1}) \right\} g(q^k)\end{aligned}$$

with $(f * g)(q^0) = 0$, where $n \in \mathbb{N}_0$.

Theorem 5.9 *(Convolution Theorem). Assume that $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$, and $\mathcal{L}\{f * g\}(z)$ exist for a given $z \in \mathbb{C}$. Then at the point z,*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z). \tag{5.23}$$

Proof. For brevity let us set

$$e_{nm}(z) = e_z(q^n, q^m) \quad \text{and} \quad \hat{f}_{nm} = \hat{f}(q^n, q^m).$$

Then (5.2) gives

$$e_{nn}(z) = 1 \quad \text{for all } n \in \mathbb{N}_0, \tag{5.24}$$

$$e_{n+1,m}(z) = (1 + q'q^n z) e_{nm}(z) \quad \text{for } n, m \in \mathbb{N}_0,\ n \geqslant m, \tag{5.25}$$

$$e_{n,m+1}(z) = \frac{e_{nm}(z)}{1 + q'q^m z} \quad \text{for } n, m \in \mathbb{N}_0,\ n \geqslant m+1, \tag{5.26}$$

and Eqs. (5.21) can be written as, taking $t = q^n$ and $s = q^m$ with $n \geqslant m$,

$$\begin{aligned}&q^m(\hat{f}_{n+1,m+1} - \hat{f}_{n,m+1}) + q^n(\hat{f}_{n,m+1} - \hat{f}_{nm}) = 0, \quad n \geqslant m \geqslant 0, \\ &\hat{f}_{n0} = f(q^n), \quad n \in \mathbb{N}_0.\end{aligned} \tag{5.27}$$

Using definition (5.6) of the q-Laplace transform and Definition 5.8 for the convolution, we have

$$\begin{aligned}\mathcal{L}\{f * g\}(z) &= (q-1)\sum_{n=1}^{\infty} \frac{q^n (f * g)(q^n)}{e_{n+1,0}(z)} \\ &= (q-1)^2 \sum_{n=1}^{\infty} \frac{q^n}{e_{n+1,0}(z)} \sum_{k=0}^{n-1} q^k \hat{f}_{n,k+1} g(q^k) \\ &= (q-1)^2 \sum_{k=0}^{\infty} q^k g(q^k) \sum_{n=k+1}^{\infty} \frac{q^n \hat{f}_{n,k+1}}{e_{n+1,0}(z)}.\end{aligned}$$

Substituting here

$$e_{n+1,0}(z) = e_{n+1,k+1}(z) e_{k+1,0}(z),$$

we get

$$\begin{aligned}\mathcal{L}\{f * g\}(z) &= (q-1)^2 \sum_{k=0}^{\infty} \frac{q^k g(q^k)}{e_{k+1,0}(z)} \sum_{n=k+1}^{\infty} \frac{q^n \hat{f}_{n,k+1}}{e_{n+1,k+1}(z)} \\ &= \mathcal{L}\{g\}(z) \cdot (q-1) \sum_{n=k+1}^{\infty} \frac{q^n \hat{f}_{n,k+1}}{e_{n+1,k+1}(z)}.\end{aligned} \tag{5.28}$$

Let us set

$$\psi_m = \sum_{n=m}^{\infty} \frac{q^n \hat{f}_{nm}}{e_{n+1,m}(z)}, \quad m \in \mathbb{N}_0. \tag{5.29}$$

We will show that ψ_m is independent of m. Then

$$(q-1)\sum_{n=k+1}^{\infty}\frac{q^n\hat{f}_{n,k+1}}{e_{n+1,k+1}(z)}=(q-1)\sum_{n=0}^{\infty}\frac{q^n\hat{f}_{n0}}{e_{n+1,0}(z)}$$
$$=(q-1)\sum_{n=0}^{\infty}\frac{q^nf(q^n)}{e_{n+1,0}(z)}=\mathcal{L}\{f\}(z),$$

and (5.28) gives (5.23). So, it remains to show that the quantity ψ_m defined by (5.29) does not depend on $m\in\mathbb{N}_0$. We have, putting $e_{nm}(z)=e_{nm}$ and using (5.27) and (5.24), (5.25), and (5.26),

$$\begin{aligned}\psi_{m+1}&=\sum_{n=m+1}^{\infty}\frac{q^n\hat{f}_{n,m+1}}{e_{n+1,m+1}}=\sum_{n=m+1}^{\infty}\frac{q^n\hat{f}_{nm}+q^m\hat{f}_{n,m+1}-q^m\hat{f}_{n+1,m+1}}{e_{n+1,m+1}}\\&=\sum_{n=m+1}^{\infty}\frac{q^n\hat{f}_{nm}}{e_{n+1,m+1}}-q^m\sum_{n=m+1}^{\infty}\left[\frac{\hat{f}_{n+1,m+1}}{e_{n+1,m+1}}-\frac{\hat{f}_{n,m+1}}{e_{n,m+1}}+\frac{\hat{f}_{n,m+1}}{e_{n,m+1}}-\frac{\hat{f}_{n,m+1}}{e_{n+1,m+1}}\right]\\&=\sum_{n=m+1}^{\infty}\frac{q^n\hat{f}_{nm}}{e_{n+1,m}}\left(1+q'q^mz\right)+q^m\frac{\hat{f}_{m+1,m+1}}{e_{m+1,m+1}}-q^m\sum_{n=m+1}^{\infty}\frac{\hat{f}_{n,m+1}}{e_{n+1,m+1}}q'q^nz\\&=\left(1+q'q^mz\right)\psi_m-q^m\frac{\hat{f}_{mm}}{e_{m+1,m}}\left(1+q'q^mz\right)+q^m\hat{f}_{m+1,m+1}-q'q^mz\psi_{m+1}\\&=\left(1+q'q^mz\right)\psi_m-q^m\hat{f}_{mm}+q^m\hat{f}_{m+1,m+1}-q'q^mz\psi_{m+1}\\&=\left(1+q'q^mz\right)\psi_m-q'q^mz\psi_{m+1},\end{aligned}$$

where we have used the fact that, as it follows from (5.22), $\hat{f}_{nn}=f(1)$ for all $n\in\mathbb{N}_0$. Consequently

$$\left(1+q'q^mz\right)\psi_{m+1}=\left(1+q'q^mz\right)\psi_m,$$

and hence $\psi_{m+1}=\psi_m$ as $1+q'q^mz\neq 0$ under the condition (5.4). □

Finally we consider the inversion of the q-Laplace transform.

Theorem 5.10 *(Uniqueness Theorem). Let $x:q^{\mathbb{N}_0}\to\mathbb{C}$ be a function in the space $\mathcal{F}_\delta$, i.e., x satisfies* (5.16)*. Further, let $\tilde{x}(z)$ be the q-Laplace transform of x defined by* (5.6) *for $z\in\Omega_\delta$. If $\tilde{x}(z)\equiv 0$ for $z\in\Omega_\delta$, then $x(q^n)=0$ for all $n\in\mathbb{N}_0$.*

Proof. By the assumption, we have

$$\frac{x(q^0)}{1+q'z}+\frac{qx(q)}{(1+q'z)(1+q'qz)}+\frac{q^2x(q)}{(1+q'z)(1+q'qz)(1+q'q^2z)}+\cdots\equiv 0 \tag{5.30}$$

for $z\in\Omega_\delta$. Multiplying (5.30) by $1+q'z$ and then passing to the limit as $|z|\to\infty$ (we can take a term-by-term limit due to the uniform convergence proved in Theorem 5.5), we get $x(q^0)=0$. Now we multiply the remaining equation

$$\frac{qx(q)}{(1+q'z)(1+q'qz)}+\frac{q^2x(q)}{(1+q'z)(1+q'qz)(1+q'q^2z)}+\cdots\equiv 0$$

by $(1+q'z)(1+q'qz)$ and pass then to the limit as $|z|\to\infty$ to obtain $x(q^1)=0$. Repeating this procedure, we find that $x(q^0)=x(q^1)=x(q^2)=\cdots=0$. □

Theorem 5.10 implies that the inverse q-Laplace transform exists. The following theorem gives an integral formula for the inverse q-Laplace transform.

Theorem 5.11. *Let $x\in\mathcal{F}_\delta$ and $\tilde{x}(z)$ be its q-Laplace transform defined by* (5.6)*. Then*

$$x\left(q^n\right)=\frac{1}{2\pi i}\int_{\Gamma}\tilde{x}(z)\prod_{k=0}^{n-1}\left(1+q'q^kz\right)\mathrm{d}z\quad\text{for } n\in\mathbb{N}_0,$$

where Γ is any positively oriented closed curve in the region Ω_δ that encloses all the points $\alpha_k=-(q'q^k)^{-1}$ for $k\in\mathbb{N}_0$.

Proof. For any $j \in \mathbb{N}_0$ we have from (5.6) that

$$\tilde{x}(z)\prod_{k=0}^{j-1}\left(1+q'q^k z\right) = q'\sum_{n=0}^{j-1} q^n x\left(q^n\right)\prod_{\substack{k=0\\k\neq n}}^{j-1}\left(1+q'q^k z\right) + \frac{q'q^j x(q^j)}{1+q'q^j z} + q'\sum_{n=j+1}^{\infty}\frac{q^n x(q^n)}{\prod_{k=j}^{n}(1+q'q^k z)}.$$

Integrating both sides over the curve Γ and noting that we can integrate under the sum signs by the uniform convergence of the series, we obtain

$$\int_\Gamma \tilde{x}(z)\prod_{k=0}^{j-1}\left(1+q'q^k z\right)\mathrm{d}z = q'\sum_{n=0}^{j-1} q^n x\left(q^n\right)\int_\Gamma \prod_{\substack{k=0\\k\neq n}}^{j-1}\left(1+q'q^k z\right)\mathrm{d}z + q'q^j x\left(q^j\right)\int_\Gamma \frac{\mathrm{d}z}{1+q'q^j z}$$
$$+ q'\sum_{n=j+1}^{\infty} q^n x\left(q^n\right)\int_\Gamma \frac{\mathrm{d}z}{\prod_{k=j}^{n}(1+q'q^k z)}.$$

Since

$$\int_\Gamma \prod_{\substack{k=0\\k\neq n}}^{j-1}\left(1+q'q^k z\right)\mathrm{d}z = 0 \quad \text{for } j \geqslant 0,$$
$$\int_\Gamma \frac{\mathrm{d}z}{1+q'q^j z} = \frac{2\pi i}{q'q^j} \quad \text{for } j \geqslant 0,$$
$$\int_\Gamma \frac{\mathrm{d}z}{\prod_{k=j}^{n}(1+q'q^k z)} = 0 \quad \text{for } j \geqslant 0,\ n \geqslant j+1,$$

we obtain

$$\int_\Gamma \tilde{x}(z)\prod_{k=0}^{j-1}\left(1+q'q^k z\right)\mathrm{d}z = 2\pi i x\left(q^j\right).$$

Note that we have used the following well-known fact from the theory of complex variable functions: If $P(z)$ is any polynomial of degree $\geqslant 2$ and if Γ is any closed contour that encloses all the roots of the polynomial $P(z)$, then

$$\int_\Gamma \frac{\mathrm{d}z}{P(z)} = 0.$$

This completes the proof. □

6. Concluding remarks

1. It follows from (4.1) (respectively, (5.6)) that the values $\tilde{x}(mh)$ (respectively, $\tilde{x}(q^m)$) with $m \in \mathbb{N}_0$ are well defined for a large class of functions $x : h\mathbb{N}_0 \to \mathbb{C}$ (respectively, $x : q^{\mathbb{N}_0} \to \mathbb{C}$). Reasoning similarly as in the proof of Theorem 4.8 (respectively, Theorem 5.10), taking in that proof $z = mh$ (respectively, $z = q^m$) and letting then $m \to \infty$, we can see that the sequence $\{\tilde{x}(mh)\}_{m\in\mathbb{N}_0}$ (respectively, $\{\tilde{x}(q^m)\}_{m\in\mathbb{N}_0}$) is determined uniquely from the sequence $\{x(nh)\}_{n\in\mathbb{N}_0}$ (respectively, $\{x(q^n)\}_{n\in\mathbb{N}_0}$). In connection with this situation, it would be interesting to find an explicit inversion formula expressing $x(nh)$ (respectively, $x(q^n)$) for each $n \in \mathbb{N}_0$ in terms of $\{\tilde{x}(mh)\}_{m\in\mathbb{N}_0}$ (respectively, $\{\tilde{x}(q^m)\}_{m\in\mathbb{N}_0}$). For the result concerning a similar problem see Feinsilver [8].

2. We can see from Theorem 4.4(a) (respectively, Theorem 5.7(a)) that no function has its h-Laplace (respectively, q-Laplace) transform equal to the constant function 1. An important theoretical question is about finding conditions that guarantee that a certain function is a Laplace transform of some function. In the usual case $\mathbb{T} = \mathbb{R}$, there are various criteria on which one can predetermine whether a given function (analytic in a suitable region) is a Laplace transform. However, for other examples of the time scale $\mathbb{T}$, including $h\mathbb{N}_0$ and $q^{\mathbb{N}_0}$, this problem awaits its investigation.

3. Finally, we note that most of the results concerning the Laplace transform on $h\mathbb{N}_0$ and $q^{\mathbb{N}_0}$ as given in this paper can be generalized appropriately to an arbitrary isolated time scale $\mathbb{T} = \{t_n\}_{n\in\mathbb{N}_0}$ such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \inf\{t_{n+1} - t_n \colon n \in \mathbb{N}_0\} > 0.$$

This problem will be considered by the authors in a forthcoming paper.

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Appendix A. Some time scale essentials

In this section, for the convenience of the reader, we introduce some basic concepts concerning the calculus on time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, the h-numbers $h\mathbb{Z} = \{hk\colon k \in \mathbb{Z}\}$ with fixed $h > 0$, and the q-numbers $q^{\mathbb{N}_0} = \{q^k\colon k \in \mathbb{N}_0\}$ with fixed $q > 1$ are examples of time scales, as are

$$[-1, 0] \cup [1, 2], \qquad [-1, 0] \cup \mathbb{N}, \qquad \text{and the Cantor set.}$$

The calculus of time scales was initiated by Stefan Hilger in [11] in order to create a theory that can unify discrete and continuous analysis. Indeed, below we introduce the delta derivative f^Δ for a function $f : \mathbb{T} \to \mathbb{C}$, and it turns out that

(i) $f^\Delta(t) = f'(t)$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,
(ii) $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$ is the usual forward difference quotient if $\mathbb{T} = h\mathbb{Z}$,
(iii) $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$ is the usual Jackson derivative if $\mathbb{T} = q^{\mathbb{N}_0}$.

In [11], Hilger also introduced dynamic equations (or Δ-differential equations) on time scales in order to unify and extend the theories of ordinary differential equations and difference equations (including q-difference equations as well). For a general introduction of the calculus on time scales we refer the reader to the original paper by Hilger [11] and to the textbooks by Bohner and Peterson [5,7]. Here we give only those notions and notations connected to time scales, which the reader may need in order to read this paper.

Any time scale $\mathbb{T}$ is a complete metric space with the metric (distance) $d(t, s) = |t - s|$ for $t, s \in \mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighbourhoods of points, open sets, closed sets, compact sets and so on. In particular, for a given $\delta > 0$, the δ-neighbourhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t, s) < \delta$. By a neighbourhood of a point $t \in \mathbb{T}$ is meant an arbitrary set in $\mathbb{T}$ containing a δ-neighbourhood of the point t. Also we have for functions $f : \mathbb{T} \to \mathbb{C}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbb{Z} \to \mathbb{C}$ is continuous at each point of $\mathbb{Z}$). A remarkable circumstance is that, unlike for functions on general metric spaces, we can introduce and investigate a concept of derivative for functions on time scales. This proves to be possible due to the special structure of a time scale.

In the definition of the derivative an important rôle played by the so-called forward and backward jump operators.

Definition A.1. Let $\mathbb{T}$ be a time scale. We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\} \quad \text{for } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\} \quad \text{for } t \in \mathbb{T}.$$

In this definition we put in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Next, let $t \in \mathbb{T}$. If $\sigma(t) > t$, then we say that t is *right-scattered*, while if $\rho(t) < t$, then we say that t is *left-scattered*. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, while if $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense or left-dense are called *dense* and points that are right-scattered and left-scattered at the same time are called *isolated*. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t \quad \text{for all } t \in \mathbb{T}.$$

Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$.

Example A.2. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$ and $\mu(t) = h$. These are time scales with constant graininess. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then $\sigma(t) = qt$ and $\mu(t) = (q - 1)t$.

Let $\mathbb{T}^\kappa$ denote Hilger's truncated ("kappen" = lop off) above set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point. Now we consider a function $f : \mathbb{T} \to \mathbb{C}$ and define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^\kappa$.

Definition A.3. Assume $f : \mathbb{T} \to \mathbb{C}$ is a function and $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U (in $\mathbb{T}$) of t such that

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big[\sigma(t) - s\big]\right| \leqslant \varepsilon\left|\sigma(t) - s\right| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t.

If $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$, then $f^\Delta(t)$ is not uniquely defined, since for such a point t small neighbourhoods U of t consist only of t, and besides, we have $\sigma(t) = t$. Therefore the requirement in Definition A.3 holds for an arbitrary number $f^\Delta(t)$. This is why we omit a maximal left-scattered point.

The Δ-integration is defined as the inverse operation to the Δ-differentiation.

Definition A.4. A function $F : \mathbb{T} \to \mathbb{C}$ is called a Δ-*antiderivative* of $f : \mathbb{T} \to \mathbb{C}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then we define the Δ-*integral* from a to b of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

Example A.5. Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\,\mathrm{d}t,$$

where the last integral is the ordinary integral. If $\mathbb{T} = h\mathbb{Z}$ (with $h > 0$) and $a = hm$, $b = hn$ with $m < n$, then

$$\int_a^b f(t)\Delta t = h\sum_{k=m}^{n-1} f(hk).$$

Finally, if $\mathbb{T} = q^{\mathbb{N}_0}$ (with $q > 1$) and $a = q^m$, $b = q^n$ with $m < n$, then

$$\int_a^b f(t)\Delta t = (q-1)\sum_{k=m}^{n-1} q^k f\big(q^k\big).$$

In order to describe a class of functions that are "delta integrable", we introduce the concept of rd-continuity. It turns out that every rd-continuous function has an antiderivative.

Definition A.6. A function $f : \mathbb{T} \to \mathbb{C}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

For a more general treatment of the delta integral on time scales (Riemann and Lebesgue delta integrals on time scales), see [7, Chapter 5].

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