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On the Stability of a Multiplicative Functional Equation

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In this paper, we will introduce a new multiplicative functional Eq. (1) and prove that the given equation is equivalent to the well known "original" one, $f(xy) = f(x)f(y)$. Moreover, we will investigate the stability problem of Eq. (1) in the sense of R. Ger.

Key Words: multiplicative functional equation; stability.

1. INTRODUCTION

The study of stability problems for functional equations originated from a famous talk given by S. M. Ulam in 1940. In the talk, Ulam discussed a number of important unsolved mathematical problems (see [12]). One of them was a question concerning the stability of group homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year 1941, D. H. Hyers affirmatively answered the question of Ulam for the case where G_1 and G_2 are Banach spaces (see [7]). The terminology Hyers–Ulam stability originates from this historical background (Refs. [3, 8–10]).

In 1978, Th. M. Rassias succeeded in generalizing Hyers's theorem by considering a stability problem with unbounded Cauchy differences (see [11]). Since then, the stability problems of several functional equations have been extensively investigated.

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J. Baker, J. Lawrence, and F. Zorzitto found a new type of stability by investigating the stability problem of the exponential equation $f(x+y) = f(x)f(y)$ (see [2]). More precisely, they proved that if a complex-valued function f defined on a normed space satisfies the inequality

$$|f(x+y) - f(x)f(y)| \leq \delta$$

for some $\delta \geq 0$ and for all x, y, then either f is bounded or f is exponential. In general, such type of stability is called the superstability.

In [4], R. Ger pointed out that the superstability phenomenon of the exponential equation is caused by the fact that the natural group structure in the range space is disregarded. He posed the stability problem in the form

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \varepsilon$$

and proved the stability of the exponential equation (cf. [5]). We promise in this paper that such type of stability is called the stability in the sense of R. Ger.

Recently, K. J. Heuvers introduced in his paper [6] a new type of logarithmic functional equation

$$f(x+y) - f(x) - f(y) = f(x^{-1} + y^{-1})$$

and proved that this equation is equivalent to the "original" logarithmic equation $f(xy) = f(x) + f(y)$ in the class of functions $f\colon (0, \infty) \to \mathbf{R}$.

If we slightly modify the above equation of Heuvers, we may obtain a new functional equation

$$f(x+y) = f(x)f(y)f(x^{-1} + y^{-1}) \tag{1}$$

which we may call a multiplicative functional equation because the function $f(x) = x^a$ is a solution of this equation.

By using the theorem of Heuvers, we can easily prove that if both the domain and range of relevant functions are positive real numbers, then Eq. (1) is equivalent to the "original" multiplicative equation $f(xy) = f(x)f(y)$.

By modifying an idea of Heuvers [6], we will prove in this paper that Eq. (1) and the equation $f(xy) = f(x)f(y)$ are equivalent to each other in the class of functions $f\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$. Moreover, we will investigate a stability problem of Eq. (1) in the sense of R. Ger. By applying this result, we will also prove the Hyers–Ulam stability of the equation of Heuvers.

Every solution of the "original" multiplicative (logarithmic) functional equation is called a multiplicative (logarithmic) function. For more information on multiplicative functions or logarithmic functions, one can refer to [1].

Throughout this paper, we will denote by **N** and **R** the set of positive integers and of real numbers, respectively.

2. PRELIMINARIES

First, we will introduce a lemma which is essential to prove Theorem 4 and Theorem 6 which are main theorems of this paper. The proof of the following lemma is elementary.

LEMMA 1. *It holds*

$$\Big\{(u,v) \mid u = x^{-1} + y^{-1};\ v = 1 - x(x+y)^{-1}y^{-1};$$
$$x, y \in \mathbf{R} \setminus \{0\} \text{ with } x + y \neq 0\Big\}$$
$$\supset \big\{(u,v) \mid u \in \mathbf{R} \setminus \{0\},\ v \in \mathbf{R} \text{ with } u + v \neq 1 \text{ and } u(1-v) > 0\big\}.$$

Proof. Let us consider the following system of equations

$$\begin{cases} x^{-1} + y^{-1} = u, \\ x(x+y)^{-1}y^{-1} = 1 - v \end{cases} \tag{2}$$

with variables x and y, where $u \in \mathbf{R} \setminus \{0\}$ and $v \in \mathbf{R}$ with $u + v \neq 1$ and $u(1 - v) > 0$. It suffices to prove that the system has at least one solution (x, y) with $x, y, x + y \in \mathbf{R} \setminus \{0\}$.

Combining both equations in the system, we get a quadratic equation

$$(u^2 - u + uv)x^2 - 2ux + 1 = 0.$$

Applying the quadratic formula we find the solutions of the above equation:

$$x = \frac{u \pm \sqrt{u(1-v)}}{u(u+v-1)} \neq 0; \qquad y = \pm \frac{1}{\sqrt{u(1-v)}} \neq 0.$$

Furthermore, we see by the first equation of the system (2) that $x + y \neq 0$ because of $u \in \mathbf{R} \setminus \{0\}$. ∎

We will verify in the following lemma that if a function $f\colon \mathbf{R} \to \mathbf{R}$ is a solution of Eq. (1) and $f(x) = 0$ for some $x \in \mathbf{R} \setminus \{0\}$, then f is a null function.

LEMMA 2. *If a function $f\colon \mathbf{R} \to \mathbf{R}$ satisfies Eq.* (1) *for all $x, y \in \mathbf{R} \setminus \{0\}$ and if there is an $x_0 \neq 0$ with $f(x_0) = 0$, then $f(x) = 0$ for any $x \in \mathbf{R}$.*

Proof. Put $x = x_0$ in (1) to obtain

$$f(x_0 + y) = f(x_0)f(y)f\left(x_0^{-1} + y^{-1}\right) = 0$$

for each $y \in \mathbf{R} \setminus \{0\}$. ∎

Two sets of solutions of Eq. (1) with $f(1) = 1$ resp. $f(1) = -1$ are equivalent to each other. In particular, we introduce the following lemma whose proof is trivial.

LEMMA 3. *If a function $f\colon \mathbf{R} \to \mathbf{R}$ is a solution of the functional equation* (1) *for all $x, y \in \mathbf{R} \setminus \{0\}$ and if $f(1) = -1$, then the function $g\colon \mathbf{R} \to \mathbf{R}$ defined by $g(x) = -f(x)$ is also a solution of Eq.* (1) *for any $x, y \in \mathbf{R} \setminus \{0\}$ with $g(1) = 1$.*

3. SOLUTION OF EQ. (1)

In the following theorem, we will prove that the new multiplicative Eq. (1) is equivalent to the "original" one, $f(xy) = f(x)f(y)$, in the class of functions $f\colon \mathbf{R} \to \mathbf{R}$.

THEOREM 4. *A function $f\colon \mathbf{R} \to \mathbf{R}$ satisfies Eq.* (1) *for every $x, y \in \mathbf{R} \setminus \{0\}$ if and only if there exist a constant $\sigma \in \{-1, 1\}$ and a multiplicative function $m\colon \mathbf{R} \to \mathbf{R}$, i.e.,*

$$m(xy) = m(x)m(y) \qquad \text{for all } x, y \in \mathbf{R} \setminus \{0\},$$

such that $f(x) = \sigma m(x)$ for all $x \in \mathbf{R} \setminus \{0\}$.

Proof. If there exists an $x_0 \neq 0$ with $f(x_0) = 0$, then Lemma 2 implies that $f(x) = 0$ for every $x \in \mathbf{R}$. In this case, we may choose $\sigma = 1$ and a multiplicative function $m \equiv 0$ such that $f(x) = \sigma m(x)$ for all $x \in \mathbf{R} \setminus \{0\}$.

Assume now that $f(x) \neq 0$ for all $x \in \mathbf{R} \setminus \{0\}$. Put $y = x^{-1}$ in (1) to obtain

$$f(x^{-1}) = f(x)^{-1} \tag{3}$$

for each $x \in \mathbf{R} \setminus \{0\}$. With $x = 1$, (3) implies $f(1) = 1$ or $f(1) = -1$.

In view of Lemma 3, we may without loss of generality assume that

$$f(1) = 1. \tag{4}$$

It follows from (3) that $f((x+y)^{-1})f(x+y) = f((y+z)^{-1})f(y+z) = 1$ for $x+y,\ y+z \in \mathbf{R}\setminus\{0\}$. Hence, by using (1) we have

$$\begin{aligned} f\big((x+y)^{-1} + z^{-1}\big)f\big(x^{-1}+y^{-1}\big) &= f\big((x+y)^{-1}\big)f(z^{-1})f(x+y+z)f(x^{-1})f\big(y^{-1}\big)f(x+y) \\ &= f\big((y+z)^{-1}\big)f(x^{-1})f(x+y+z)f\big(y^{-1}\big)f(z^{-1})f(y+z) \\ &= f\big((y+z)^{-1} + x^{-1}\big)f\big(y^{-1}+z^{-1}\big) \end{aligned} \tag{5}$$

for all $x, y, z \in \mathbf{R}\setminus\{0\}$ with $x+y \neq 0$ and $y+z \neq 0$.

If we set

$$u = x^{-1} + y^{-1} \qquad \text{and} \qquad v = (x+y)^{-1} + z^{-1}, \tag{6}$$

then

$$\begin{aligned} uv &= \big(x^{-1}+y^{-1}\big)\big((x+y)^{-1}+z^{-1}\big) = (yz)^{-1} + (xy)^{-1} + (xz)^{-1} \\ &= (y+z)^{-1}\big(y^{-1}+z^{-1}\big) + x^{-1}\big(y^{-1}+z^{-1}\big) \\ &= \big((y+z)^{-1}+x^{-1}\big)\big(y^{-1}+z^{-1}\big). \end{aligned}$$

If we additionally set

$$y^{-1} + z^{-1} = 1 \tag{7}$$

in Eq. (5), then (4), (5), (6), and (7) imply that the function f satisfies

$$f(uv) = f(u)f(v) \tag{8}$$

for all $u \in \mathbf{R}\setminus\{0\}$, $v \in \mathbf{R}$ with $u+v \neq 1$ and $u(1-v) > 0$ (see Lemma 1 and the fact that $u = x^{-1}+y^{-1}$ and $v = (x+y)^{-1}+z^{-1} = 1 - x(x+y)^{-1}y^{-1}$ for some $x, y \in \mathbf{R}\setminus\{0\}$ with $x+y \neq 0$).

Let $\alpha \approx -1.324717956\ldots$ be a real solution of the cubic equation $x^3 - x + 1 = 0$; more precisely, let

$$\alpha := \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{23}{108}}}\,.$$

Using (8) and (3) we obtain

$$f(u) = f(u^2 u^{-1}) = f(u^2) f(u^{-1}) = f(u^2) f(u)^{-1}$$

or

$$f(u^2) = f(u)^2 \tag{9}$$

for any $u < 0$ ($u \neq \alpha$) or $u > 1$ ($u \neq \alpha$ means $u^2 + u^{-1} \neq 1$). Using (3) and (9) we have

$$f(u^{-2}) = f(u^2)^{-1} = f(u)^{-2} = f(u^{-1})^2$$

for $u < 0$ ($u \neq \alpha$) or $u > 1$. Hence, (4) and (9) yield that f satisfies (9) for all $u \in \mathbf{R} \setminus \{0\}$. (The validity of (9) for $u = \alpha^{-1} \neq \alpha$ or $u \geq 1$ implies that $f(u^{-2}) = f(u^{-1})^2$ for $u^{-1} = \alpha$ or $0 < u^{-1} \leq 1$.)

Let $s > t \geq 1$ be given. It then follows from (8), (9), and (3) that

$$f(st) = f(s^2 t s^{-1}) = f(s^2) f(ts^{-1}) = f(s)^2 f(t) f(s^{-1}) = f(s) f(t) \tag{10}$$

for all $s \geq 1$ and $t \geq 1$. (We may replace each of s and t by the other when $t > s \geq 1$ and use (9) and (4) to prove (10) for $s = t$.) From (3) and (10) we get

$$f(s^{-1}t^{-1}) = f(st)^{-1} = f(s)^{-1} f(t)^{-1} = f(s^{-1}) f(t^{-1}),$$

for all $s \geq 1$ and $t \geq 1$, or

$$f(st) = f(s) f(t)$$

for all $0 < s,\ t \leq 1$. For the case when $s \geq 1$ and $0 < t < 1$ ($t \geq 1$ and $0 < s < 1$), we may use (8) to obtain $f(st) = f(s)f(t)$. Altogether, we may conclude by considering (4) that f satisfies (8) for all pairs (u, v) of

$$\{(u,v) \mid u > 0;\ v > 0\}$$
$$\cup \{(u,v) \mid u \in \mathbf{R} \setminus \{0\};\ v \in \mathbf{R};\ u + v \neq 1;\ u(1 - v) > 0\}. \tag{11}$$

By (9) we have $f(u)^2 = f(u^2) = f((-u)^2) = f(-u)^2$ for all $u \in \mathbf{R} \setminus \{0\}$. Hence, we get

$$f(u) = -f(-u) \qquad \text{or} \qquad f(u) = f(-u) \tag{12}$$

for each $u \in \mathbf{R} \setminus \{0\}$.

If we assume that $f(u) = f(-u)$ for all $-1 \leq u < 0$ and that there exists a $u_0 < -1$ with $f(u_0) = -f(-u_0)$, then it follows from (3) that

$$f(u_0^{-1}) = f(u_0)^{-1} = -f(-u_0)^{-1} = -f(-u_0^{-1})$$

and $-1 < u_0^{-1} < 0$ which are contrary to our assumption. (Due to Lemma 2, we can assume that $f(-u_0^{-1}) \neq 0$.)

Now, suppose there exists a u_0 $(-1 \leq u_0 < 0)$ with $f(u_0) = -f(-u_0)$. It then follows from (11) that

$$\begin{aligned} f(u_0 v) &= f((-u_0)(-v)) = f(-u_0)f(-v) \\ &= -f(u_0)f(-v) = -f(-u_0 v) \end{aligned}$$

for all $v < -1$ with $v \neq u_0 - 1$, i.e.,

$$f(u) = -f(-u) \tag{13}$$

for all $u := u_0 v > 1$ $(u \neq u_0^2 - u_0)$. Using (3) and (13) we get

$$f(u) = f(u^{-1})^{-1} = -f(-u^{-1})^{-1} = -f(-u)$$

for any $0 < u < 1$ $(u \neq (u_0^2 - u_0)^{-1})$. From (1), (3), (8), and (11) we get

$$\begin{aligned} f(x-1) &= f(x)f(-1)f(x^{-1}-1) = f(x)f(-1)f(1-x)f(x^{-1}) \\ &= f(-1)f(1-x) = -f(-1)f(x-1) \end{aligned}$$

for $0 < x < 1$ and this means that $f(-1) = -1$. Therefore,

$$f(1) = -f(-1).$$

All together, we see that if there exists a u_0 $(-1 \leq u_0 < 0)$ with $f(u_0) = -f(-u_0)$, then f satisfies (13) for all $u > 0$ with possible exceptions at $u_0^2 - u_0$ and $(u_0^2 - u_0)^{-1}$.

Taking (12) into consideration, assume

$$f\left(u_0^2 - u_0\right) = f\left(u_0 - u_0^2\right).$$

By (8) and (11), we have

$$\begin{aligned} f\left(\left(u_0 - u_0^2\right)v\right) &= f\left(\left(u_0^2 - u_0\right)(-v)\right) = f\left(u_0^2 - u_0\right)f(-v) \\ &= f\left(u_0 - u_0^2\right)f(-v) = f\left(-\left(u_0 - u_0^2\right)v\right) \end{aligned}$$

for all $v < -1$ with $v \neq u_0 - u_0^2 - 1$, i.e.,

$$f(u) = f(-u)$$

for each $u := (u_0 - u_0^2)v > u_0^2 - u_0$ with $u \neq (u_0 - u_0^2)(u_0 - u_0^2 - 1)$, which is contrary to the fact that (13) holds true for all $u > 0$ with possible exceptions at two points. (In view of Lemma 2, we may exclude the case when $f(u) = 0$ for some $u \neq 0$.)

Similarly, if we assume

$$f\left(\left(u_0^2 - u_0\right)^{-1}\right) = f\left(\left(u_0 - u_0^2\right)^{-1}\right),$$

then this assumption also leads to a contradiction. Hence, we can conclude that if there exists a u_0 ($-1 \le u_0 < 0$) with $f(u_0) = -f(-u_0)$, then

$$f(u) = -f(-u)$$

for all $u > 0$. If we replace $-u$ by u, we will see that this equation is true also for all $u < 0$.

Therefore, f satisfies either

$$f(u) = -f(-u) \qquad \text{for all } u < 0$$

or

$$f(u) = f(-u) \qquad \text{for all } u < 0.$$

This fact together with (8) and (11) yields

$$f(uv) = f(u)f(v)$$

for all $u, v \in \mathbf{R} \setminus \{0\}$.

The proof of the reverse assertion is clear. ∎

If a function $f: \mathbf{R} \to \mathbf{R}$ is a solution of Eq. (1) for all $x, y \in \mathbf{R} \setminus \{0\}$ and additionally satisfies $f(0) \neq 0$, then we see by putting $y = -x$ in (1) and considering (12) that $f(x) \in \{-1, 1\}$ for all $x \in \mathbf{R} \setminus \{0\}$.

Therefore, we have

COROLLARY 5. *An unbounded function $f: \mathbf{R} \to \mathbf{R}$ is a solution of the functional Eq.* (1) *for all $x, y \in \mathbf{R} \setminus \{0\}$ if and only if there exist a constant $\sigma \in \{-1, 1\}$ and an unbounded multiplicative function $m: \mathbf{R} \to \mathbf{R}$, i.e.*,

$$m(xy) = m(x)m(y) \qquad \text{for all } x, y \in \mathbf{R},$$

such that $f(x) = \sigma m(x)$ for all $x \in \mathbf{R}$.

4. STABILITY OF EQ. (1)

In the following theorem, we will prove the stability of the functional equation (1) in the sense of R. Ger.

THEOREM 6. *If a function $f: \mathbf{R} \to (0, \infty)$ satisfies the functional inequality*

$$\left| \frac{f(x+y)}{f(x)f(y)f\left(x^{-1} + y^{-1}\right)} - 1 \right| \le \varepsilon \tag{14}$$

for some $0 \leq \varepsilon < 1$ *and for all* $x, y \in \mathbf{R} \setminus \{0\}$, *then there exists a unique multiplicative function* $m\colon \mathbf{R} \setminus \{0\} \to (0, \infty)$ *such that*

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^5 (1-\varepsilon)^{1/2} \leq \frac{m(x)}{f(x)} \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^5 (1+\varepsilon)^{1/2}$$

for any $x \in \mathbf{R} \setminus \{0\}$. *If* f *is additionally assumed to be unbounded, then the domain of* m *can be extended to the whole real space* $\mathbf{R}$ *with* $m(0) = 0$.

Proof. From (14) it follows that

$$\frac{1}{1+\varepsilon} \leq \frac{f(x)f(y)f(x^{-1}+y^{-1})}{f(x+y)} \leq \frac{1}{1-\varepsilon} \tag{15}$$

for any $x, y \in \mathbf{R} \setminus \{0\}$. By putting $y = x^{-1}$ in (15) we have

$$\frac{1}{1+\varepsilon} \leq f(x)f(x^{-1}) \leq \frac{1}{1-\varepsilon} \tag{16}$$

for each $x \in \mathbf{R} \setminus \{0\}$. With $x = 1$, (16) yields

$$(1+\varepsilon)^{-1/2} \leq f(1) \leq (1-\varepsilon)^{-1/2}. \tag{17}$$

From (14) we get

$$(1-\varepsilon)^2 \leq \frac{f\big((x+y)^{-1}+z^{-1}\big)f(x^{-1}+y^{-1})}{f\big((x+y)^{-1}\big)f(z^{-1})f(x+y+z)f(x^{-1})f(y^{-1})f(x+y)} \leq (1+\varepsilon)^2 \tag{18}$$

and

$$(1-\varepsilon)^2 \leq \frac{f\big((y+z)^{-1}+x^{-1}\big)f(y^{-1}+z^{-1})}{f\big((y+z)^{-1}\big)f(x^{-1})f(x+y+z)f(y^{-1})f(z^{-1})f(y+z)} \leq (1+\varepsilon)^2 \tag{19}$$

for all $x, y, z \in \mathbf{R} \setminus \{0\}$ with $x + y \neq 0$ resp. $y + z \neq 0$.

If we divide the inequalities in (18) by those in (19) and consider (16), then

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 \leq \frac{f\big((x+y)^{-1}+z^{-1}\big)f(x^{-1}+y^{-1})}{f\big((y+z)^{-1}+x^{-1}\big)f(y^{-1}+z^{-1})} \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^3 \tag{20}$$

for all $x, y, z \in \mathbf{R} \setminus \{0\}$ with $x + y \neq 0$ and $y + z \neq 0$.

If we define u and v by (6), then

$$uv = \left((y+z)^{-1} + x^{-1}\right)\left(y^{-1} + z^{-1}\right)$$

as we see in the proof of Theorem 4. If an additional condition (7) is assumed to be hold, then (17), (20), (6), and (7) imply

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 (1+\varepsilon)^{-1/2} \le \frac{f(u)f(v)}{f(uv)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^3 (1-\varepsilon)^{-1/2} \tag{21}$$

for all $u \in \mathbf{R} \setminus \{0\}$, $v \in \mathbf{R}$ with $u + v \neq 1$ and $u(1-v) > 0$ (see Lemma 1 and the fact that $u = x^{-1} + y^{-1}$ and $v = (x+y)^{-1} + z^{-1} = 1 - x(x+y)^{-1}y^{-1}$ for some $x, y \in \mathbf{R} \setminus \{0\}$ with $x + y \neq 0$).

Let us define $\alpha \approx -1.324717956\ldots$ as in the proof of Theorem 4. From the relation

$$\frac{f(u^2)}{f(u)^2} = \frac{f(u^2)f(u^{-1})}{f(u)} \frac{1}{f(u)f(u^{-1})}$$

and from (21) and (16) it follows that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{7/2} (1-\varepsilon)^{1/2} \le \frac{f(u^2)}{f(u)^2} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{7/2} (1+\varepsilon)^{1/2} \tag{22}$$

for any $u < 0$ $(u \neq \alpha)$ or $u > 1$ $(u \neq \alpha$ implies $u^2 + u^{-1} \neq 1)$. On account of (16) and (22), we obtain

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{5} (1-\varepsilon)^{1/2} \le \frac{f(u^{-2})}{f(u^{-1})^2} = \frac{f(u^2)f(u^{-2})}{f(u)^2 f(u^{-1})^2} \frac{f(u)^2}{f(u^2)}$$
$$\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{5} (1+\varepsilon)^{1/2}$$

for $u < 0$ $(u \neq \alpha)$ or $u > 1$. This fact together with (22) and (17) yields

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{5} (1-\varepsilon)^{1/2} \le \frac{f(u^2)}{f(u)^2} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{5} (1+\varepsilon)^{1/2} \tag{23}$$

for all $u \in \mathbf{R} \setminus \{0\}$, since $\alpha^{-1} \neq \alpha$.

Let $s > t \ge 1$. With

$$\frac{f(s)f(t)}{f(st)} = \frac{f(s^2)f(ts^{-1})}{f(s^2 \cdot ts^{-1})} \frac{f(t)f(s^{-1})}{f(ts^{-1})} \frac{f(s)^2}{f(s^2)} \frac{1}{f(s)f(s^{-1})},$$

(21), (23), (16), and (17) yield

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{12}(1+\varepsilon)^{-1/2} \le \frac{f(s)f(t)}{f(st)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{12}(1-\varepsilon)^{-1/2} \tag{24}$$

for all $s \ge 1$ and $t \ge 1$. (We can replace each of s and t by the other when $t > s \ge 1$ and we apply (23) to the proof of (24) for the case $s = t$.) By

$$\frac{f(s^{-1})f(t^{-1})}{f\big((st)^{-1}\big)} = f(s)f(s^{-1})f(t)f(t^{-1})\frac{f(st)}{f(s)f(t)}\frac{1}{f(st)f\big((st)^{-1}\big)},$$

and using (16) and (24) we obtain

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{27/2}(1+\varepsilon)^{-1/2} \le \frac{f(s^{-1})f(t^{-1})}{f\big((st)^{-1}\big)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{27/2}(1-\varepsilon)^{-1/2} \tag{25}$$

for all $s \ge 1$ and $t \ge 1$. Hence, by (21), (24), and (25) we conclude that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{27/2}(1+\varepsilon)^{-1/2} \le \frac{f(u)f(v)}{f(uv)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{27/2}(1-\varepsilon)^{-1/2} \tag{26}$$

for each pair (u, v) of

$$\{(u,v) \mid u > 0;\ v > 0\} \cup \{(u,v) \mid u \in \mathbf{R}\setminus\{0\};\ v \in \mathbf{R};\ u + v \ne 1;\ u(1-v) > 0\}. \tag{27}$$

(We can use (21) to verify inequalities in (26) either for $s \ge 1$ and $0 < t < 1$ or for $t \ge 1$ and $0 < s < 1$.)

The fact

$$\frac{f(u)^2}{f(-u)^2} = \frac{f(u)^2}{f(u^2)}\frac{f\big((-u)^2\big)}{f(-u)^2}$$

together with (23) implies

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{21/4} \le \frac{f(u)}{f(-u)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{21/4}$$

for every $u \in \mathbf{R}\setminus\{0\}$. This fact, (26), and (27) together with the relations

$$\frac{f(u)f(v)}{f(uv)} = \frac{f(-u)f(-v)}{f(uv)}\frac{f(u)}{f(-u)}\frac{f(v)}{f(-v)} \qquad (\text{for } u, v < 0)$$

and

$$\frac{f(u)f(v)}{f(uv)} = \frac{f(u)f(-v)}{f(-uv)}\,\frac{f(-uv)}{f(uv)}\,\frac{f(v)}{f(-v)} \qquad (\text{for } u > 0,\ v < 0)$$

imply

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{24}(1-\varepsilon)^{1/2} \le \frac{f(uv)}{f(u)f(v)} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{24}(1+\varepsilon)^{1/2} \tag{28}$$

for any $u, v \in \mathbf{R} \setminus \{0\}$.

We now claim that

$$\varepsilon_1^{1+2+\cdots+2^{n-1}} \le \frac{f(u^{2^n})}{f(u)^{2^n}} \le \varepsilon_2^{1+2+\cdots+2^{n-1}} \tag{29}$$

for all $u \in \mathbf{R} \setminus \{0\}$ and $n \in \mathbf{N}$, where we put

$$\varepsilon_1 := \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^5 (1-\varepsilon)^{1/2} \qquad \text{and} \qquad \varepsilon_2 := \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^5 (1+\varepsilon)^{1/2}.$$

Due to (23) our assertion is obvious for $n = 1$. Assume that (29) is true for some $n \ge 1$. Then, the relation

$$\frac{f(u^{2^{n+1}})}{f(u)^{2^{n+1}}} = \frac{f\big((u^{2^n})^2\big)}{f(u^{2^n})^2}\left[\frac{f(u^{2^n})}{f(u)^{2^n}}\right]^2$$

together with (23) and (29) gives

$$\varepsilon_1^{1+2+\cdots+2^n} \le \frac{f(u^{2^{n+1}})}{f(u)^{2^{n+1}}} \le \varepsilon_2^{1+2+\cdots+2^n}$$

which proves the validity of (29) for all $u \in \mathbf{R} \setminus \{0\}$ and $n \in \mathbf{N}$.

Let us define functions $g_n\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$ by

$$g_n(u) := 2^{-n} \ln f(u^{2^n})$$

for each $u \in \mathbf{R} \setminus \{0\}$ and $n \in \mathbf{N}$. Let $m, n \in \mathbf{N}$ be arbitrarily given with $n > m$. It then follows from (29) that

$$|g_n(u) - g_m(u)| = 2^{-m}\left|2^{-(n-m)}\ln\frac{f\Big((u^{2^m})^{2^{n-m}}\Big)}{f(u^{2^m})^{2^{n-m}}}\right|$$

$$\to 0 \qquad \text{as } m \to \infty.$$

Hence, $\{g_n(u)\}$ is a Cauchy sequence for every fixed $u \in \mathbf{R} \setminus \{0\}$. Therefore, we can define functions l: $\mathbf{R} \setminus \{0\} \to \mathbf{R}$ and m: $\mathbf{R} \setminus \{0\} \to (0, \infty)$ by

$$l(u) := \lim_{n \to \infty} g_n(u) \qquad \text{and} \qquad m(u) := e^{l(u)}.$$

Indeed, we know that

$$m(u) = \lim_{n \to \infty} f\left(u^{2^n}\right)^{2^{-n}}$$

for every $u \in \mathbf{R} \setminus \{0\}$.

Replace u and v in (28) by u^{2^n} and v^{2^n}, respectively, and extract the 2^nth root of the resulting inequalities and then take the limit as $n \to \infty$ to obtain

$$m(uv) = m(u)m(v)$$

for all $u, v \in \mathbf{R} \setminus \{0\}$. Hence, we conclude by considering (29) that there is a multiplicative function m: $\mathbf{R} \setminus \{0\} \to (0, \infty)$ with

$$\varepsilon_1 \le \frac{m(u)}{f(u)} \le \varepsilon_2 \tag{30}$$

for any $u \in \mathbf{R} \setminus \{0\}$.

Suppose m': $\mathbf{R} \setminus \{0\} \to (0, \infty)$ is another multiplicative function satisfying (30) instead of m. Since m and m' are multiplicative, we see that

$$m\left(u^{2^n}\right) = m(u)^{2^n} \qquad \text{and} \qquad m'\left(u^{2^n}\right) = m'(u)^{2^n}.$$

Thus, it follows from (30) that

$$\begin{aligned} \frac{m(u)}{m'(u)} &= \left[\frac{m\left(u^{2^n}\right)}{f\left(u^{2^n}\right)}\right]^{2^{-n}} \left[\frac{f\left(u^{2^n}\right)}{m'\left(u^{2^n}\right)}\right]^{2^{-n}} \\ &\to 1 \qquad \text{as } n \to \infty, \end{aligned}$$

which implies the uniqueness of m.

By (30) we see that m is unbounded if and only if f is so. Hence, it is not difficult to show that if f is unbounded, then the domain of m can be extended to the whole real space $\mathbf{R}$ by defining $m(0) = 0$. ∎

COROLLARY 7. *If a function f: $\mathbf{R} \to (-\infty, 0)$ satisfies the functional inequality* (14) *for some $0 \le \varepsilon < 1$ and for all $x, y \in \mathbf{R} \setminus \{0\}$, then there exists a unique multiplicative function m: $\mathbf{R} \setminus \{0\} \to (0, \infty)$ with*

$$-\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^5 (1+\varepsilon)^{1/2} \le \frac{m(x)}{f(x)} \le -\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^5 (1-\varepsilon)^{1/2}$$

for each $x \in \mathbf{R} \setminus \{0\}$. *Moreover, if* f *is unbounded, then the domain of* m *can be extended to the whole real space* $\mathbf{R}$ *with* $m(0) = 0$.

We will now apply Theorem 6 to the proof of the Hyers–Ulam stability of the equation of Heuvers.

THEOREM 8. *If a function* $f\colon \mathbf{R} \to \mathbf{R}$ *satisfies the functional inequality*

$$|f(x+y) - f(x) - f(y) - f(x^{-1} + y^{-1})| \le \delta \tag{31}$$

for some $0 \le \delta < \ln 2$ *and for all* $x, y \in \mathbf{R} \setminus \{0\}$, *then there exists a unique logarithmic function* $l\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$ *such that*

$$|f(x) - l(x)| \le 5\delta - \frac{11}{2}\ln(2 - e^{\delta}) \tag{32}$$

for each $x \in \mathbf{R} \setminus \{0\}$.

Proof. If we define a function $g\colon \mathbf{R} \to (0, \infty)$ by

$$g(x) := e^{f(x)}, \tag{33}$$

then it follows from (31) that

$$\left| \frac{g(x+y)}{g(x)g(y)g(x^{-1} + y^{-1})} - 1 \right| \le e^{\delta} - 1$$

for all $x, y \in \mathbf{R} \setminus \{0\}$. According to Theorem 6, there exists a multiplicative function $m\colon \mathbf{R} \setminus \{0\} \to (0, \infty)$ with

$$(2 - e^{\delta})^{11/2} e^{-5\delta} \le \frac{m(x)}{g(x)} \le e^{11\delta/2}(2 - e^{\delta})^{-5} \tag{34}$$

for $x \in \mathbf{R} \setminus \{0\}$. Define a function $l\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$ by

$$l(x) := \ln m(x). \tag{35}$$

Then, l is a logarithmic function. Since

$$5\delta - \frac{11}{2}\ln(2 - e^{\delta}) \ge \frac{11}{2}\delta - 5\ln(2 - e^{\delta}),$$

we can conclude by (33), (34), and (35) that the inequality (32) holds true for any $x \in \mathbf{R} \setminus \{0\}$.

Let $l^*\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$ be another logarithmic function satisfying (32) for each $x \in \mathbf{R} \setminus \{0\}$. Since l and l^* are logarithmic, we get

$$l(x^{2^n}) = 2^n l(x) \qquad \text{and} \qquad l^*(x^{2^n}) = 2^n l^*(x)$$

for all $x \in \mathbf{R} \setminus \{0\}$ and $n \in \mathbf{N}$. Hence, it follows from (32) that

$$\begin{aligned}|l(x) - l^*(x)| &= 2^{-n}|l(x^{2^n}) - l^*(x^{2^n})| \\ &\leq 2^{-n}|l(x^{2^n}) - f(x^{2^n})| + 2^{-n}|f(x^{2^n}) - l^*(x^{2^n})| \\ &\to 0 \qquad \text{as } n \to \infty,\end{aligned}$$

which implies the uniqueness of l. ∎

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