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A cubical model for a fibration☆

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Abstract

In the paper the notion of *truncating twisting function* from a simplicial set to a cubical set and the corresponding notion of twisted Cartesian product of these sets are introduced. The latter becomes a cubical set. Using this construction together with the theory of twisted tensor products for homotopy G-algebras a strictly associative multiplicative model for a fibration is obtained.

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1. Introduction

In this paper we construct a *cubical set* which models the total space of a fibration. The normalized cubical chain complex of this cubical model *coincides* (as a chain complex) with the twisted tensor product of the singular simplicial complex of the base and the singular cubical complex of the fiber with respect to a certain specific twisting cochain which we call "truncating". Hence the twisted tensor product may be endowed with all structures which exist on the chain complex of a cubical set including the Serre diagonal, Steenrod chain (co)operations and other (co)chain operations. In this paper we concentrate only on the *strictly coassociative* Serre diagonal (the cubical analog of the Alexander–Whitney (AW) diagonal, see [30]). The combinatorial analysis of the Serre diagonal allows us to give

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explicit formulas for a strictly associative multiplication on the twisted tensor product in terms of the $\smile_1$-product and other related cochain operations measuring the deviation of $\smile_1$ from being a derivation with respect to the $\smile$ product. Using the standard triangulation of cubes we also obtain a strictly coassociative diagonal on Brown's twisted tensor product of the singular simplicial complex of the base and the singular simplicial complex of the fiber with respect to some specific twisting cochain.

For a fibration $F \to E \to Y$, Brown [8] introduced a *twisted* differential d_ϕ on the tensor product $C^*(Y) \otimes C^*(F)$ such that the homology of the cochain complex $(C^*(Y) \otimes C^*(F), d_\phi)$ is additively isomorphic to the cohomology $H^*(E)$. There are several papers (see, for example, Lambe and Stasheff [23] for references) where various multiplications are introduced on the twisted tensor product $C^*(Y) \bigotimes_\phi C^*(F) = (C^*(Y) \otimes C^*(F), d_\phi)$ to describe $H^*(E)$ as an algebra as well. But these multiplications are either not associative or the differential d_ϕ is not a derivation except in special cases, for example, for $Y = S^n$ [31].

The difficulty of introducing of such a multiplication rely on the following fact. Consider the standard simplicial model of a fibration: let X be a 1-reduced ($X_0 = X_1 = pt$) simplicial set, G a simplicial group, N a simplicial G-module, $t : X_* \to G_{*-1}$ a twisting function, and $X \times_t N$ the corresponding twisted Cartesian product. Applying chain functor to t we obtain a twisting cochain $t_* = C_*(t) : C_*(X) \to C_{*-1}(G)$ such that there is a *contraction* of $C_*(X \times_t N)$ to $C_*(X) \bigotimes_\phi C_*(N)$ where $\phi = t_*$. The simplicial structure of $X \times_t N$ induces the AW diagonal on $C_*(X \times_t N)$. The standard procedure, which uses the basic perturbation lemma, transports the AW diagonal to the twisted tensor product $C_*(X) \bigotimes_\phi C_*(N)$. But the resulting (co)multiplication is (co)associative only up to higher homotopies [15,23].

The situation changes radically if we replace a simplicial group G by a *monoidal cubical set* and suitably modify the notion of a twisting function. This yields a *cubical* model of a fibration which, as a by-product, induces a strictly associative multiplication on the above tensor product.

Let us give some more details. Let X be a 1-reduced simplicial set, Q a *monoidal cubical set*, and L a cubical Q-module, i.e., Q and L are cubical sets with given associative cubical maps $Q \times Q \to Q$ and $Q \times L \to L$. We introduce the notion of *truncating twisting function* $\tau : X_* \to Q_{*-1}$ from a simplicial set to a monoidal cubical set (the term *truncating* comes from the universal example $\tau_U : \Delta^n \to I^{n-1}$ of such functions obtained by the standard truncation procedure, see Section 4 below). Such a twisting function τ determines the *twisted Cartesian product* $X \times_\tau L$ as *a cubical set*. We remark that the study of twisting functions from cubical sets to permutahedral sets and the appropriate twisted Cartesian product is continued in a forthcoming paper [22].

We construct a functor which assigns to a simplicial set X a monoidal cubical set $\mathbf{\Omega} X$ and present a truncating twisting function $\tau_U : X \to \mathbf{\Omega} X$ which is universal in the following sense: Given an arbitrary truncating function $\tau : X_* \to Q_{*-1,}$ there is a monoidal cubical map $f_\tau : \mathbf{\Omega} X \to Q$ such that $\tau = f_\tau \tau_U$. The twisted Cartesian product $\mathbf{P} X = X \times_\tau \mathbf{\Omega} X$ is a cubical set that depends functorially on X. Note that $\mathbf{\Omega} X$ models the loop space $\Omega|X|$ and $\mathbf{P} X$ models the path fibration on $|X|$.

The normalized cubical chain functor $C_*^\square$ applied to the cubical set $\mathbf{\Omega} X$ produces $C_*^\square(\mathbf{\Omega} X)$, and this chain complex *coincides* with Adams' cobar construction $\Omega C_*(X)$ (equality (i) of (4)); similarly $C_*^\square(\mathbf{P} X)$ *coincides* with the acyclic cobar construction

$\Omega(C_*(X);C_*(X))$ (equality (ii) of (4)); furthermore $\tau_*=C_*(\tau): C_*(X)\to C^{\square}_{*-1}(Q)$ is a twisting cochain and $C^{\square}_*(X\times_\tau L)$ *coincides* with the twisted tensor product $C_*(X)\bigotimes_{\tau_*} C^{\square}_*(L)$ (equality (iii) of (4)).

The obtained cubical structures of the cobar construction $\Omega C_*(X)$ and the twisted tensor product $C_*(X)\bigotimes_{\tau_*} C^{\square}_*(L)$ have the following advantage.

The normalized chain complex of a cubical set admits the *Serre diagonal* (see [30] and below (3)), which turns it into a dg coalgebra. Since the identification $C^{\square}_*(\mathbf{\Omega}X)=\Omega C_*(X)$ the cubical structure of $\mathbf{\Omega}X$ determines a *strictly coassociative comultiplication* on the cobar construction $\Omega C_*(X)$. Similarly, the cubical structure of $X\times_\tau L$ determines a *strictly coassociative comultiplication* on the twisted tensor product $C^{\square}_*(X\times_\tau L)=C_*(X)\bigotimes_{\tau_*} C^{\square}_*(L)$. Dually, we immediately obtain the desired strictly associative multiplication on $C^*(X)\bigotimes_{\tau^*} C^*_{\square}(L)\subset C^*_{\square}(X\times_\tau L)$ (here we have equality when the graded sets have finite type).

Also note that the chain operations dual to Steenrod $\smile_i$ operations are defined for cubical sets in [18,19] and the equality $C^{\square}_*(\mathbf{\Omega}X)=\Omega C_*(X)$ allows to define these operations on the cobar construction $\Omega C_*(X)$; similarly since $C^{\square}_*(X\times_\tau L)=C_*(X)\bigotimes_{\tau_*} C^{\square}_*(L)$ it is possible to introduce Steenrod operations on multiplicative twisted tensor products.

Next, we express the resulting comultiplication on $C_*(X)\bigotimes_{\tau_*} C^{\square}_*(L)$ in terms of certain chain operations of degree k

$$E^{k,1}: C_*(X)\to C_*(X)^{\otimes k}\otimes C_*(X),\ \ k\geqslant 0,$$

which give $C_*(X)$ a *homotopy* G-*coalgebra structure* (dual to a *homotopy* G-*algebra* in the sense of Gerstenhaber and Voronov [12]). This structure is a consequence of the Serre diagonal on $C^{\square}_*(\mathbf{\Omega}X)=\Omega C_*(X)$: The Serre diagonal of $C^{\square}_*(\mathbf{\Omega}X)$ induces the diagonal $\Omega C_*(X)\to\Omega C_*(X)\otimes\Omega C_*(X)$ being a multiplicative map, thus it extends a certain homomorphism $C_*(X)\to\Omega C_*(X)\otimes\Omega C_*(X)$, which itself consists of components $E^{k,t}: C_*(X)\to C_*(X)^{\otimes k}\otimes C_*(X)^{\otimes\ell},\ k,\ell\geqslant 0$, with $E^{k,\ell}=0$ for $\ell\geqslant 2$. The operation $E^{1,1}$ is dual to the Steenrod $\smile_1$-cochain operation; thus when $E^{1,1}=0$ a homotopy G-coalgebra specializes to a cocommutative dg coalgebra (and dually for homotopy G-algebras). We note that Baues constructed a homotopy G-coalgebra structure on the normalized chain complex $C^{\mathrm{N}}_*(X)$ in [2,3].

Towards the end of the paper we develop the theory of multiplicative twisted tensor products for homotopy G-algebras, which provides a general algebraic framework for our multiplicative model of a fibration. First, we review the theory of multiplicative twisted products due to Proutè (see [27]): Suppose C is a dg Hopf algebra, A is a *commutative* dg algebra, $\phi: C\to A$ is a *coprimitive* twisting cochain (referred to as a *multiplicative* cochain below), and M is simultaneously a dg algebra and a comodule over C with multiplicative $M\to C\otimes M$. Then the twisted tensor product $A\bigotimes_\phi M$ is a dga with respect to the standard multiplication on the tensor product $A\otimes M$ of dga's. Now replace Proutè's commutative A by a homotopy G-algebra A. By definition, there is a strictly associative multiplication on BA, which can be viewed as a perturbation of the shuffle product and is compatible with the coproduct. Thus BA is a dg Hopf algebra. We say that a twisting cochain $\phi: C\to A$ is *multiplicative* if the induced map $C\to BA$ is a dg Hopf algebra map. We introduce a twisted associative multiplication μ_ϕ on $A\bigotimes_\phi M$ in terms of ϕ and the homotopy G-algebra structure of A by the same formulas as in the case $A=C^*(X)$, $C=C^*_{\square}(Q)$ and $M=C^*_{\square}(L)$; then $\tau^*: C^*_{\square}(Q)\to C^*(X)$ provides a basic example of a multiplicative twisting cochain. Thus, the theory outlined above unifies the general commutative and homotopy commutative theories; in particular, this unifies the singular and Sullivan–deRham cochain complexes of topological spaces.

We remark that the idea of using of *cubical* cochains of a structure group and fiber is found in recent results due to N. Berikashvili, who constructed a multiplicative model with associative multiplication when the fiber F is the *cubical* version of an Eilenberg–MacLane space (see [5]) and a multiplicative model $C^*(Y)\bigotimes_{\phi}C^*_{\square}(F)$, $\phi: C^*_{\square}(G)\to C^{*+1}(Y)$, where $C^*(Y)$ is the singular *simplicial* cochain complex of the base and $C^*_{\square}(G)$ and $C^*_{\square}(F)$ are the singular *cubical* cochain complexes of the structure group and the fiber (see [6]); however, there is no notion of underlying truncating twisting functions in general setting as a map form a simplicial set to a cubical one leading to the cubical model; also it lacks the analysis of the Serre cubical diagonal generating the cooperations $E^{k,1}$, and, consequently, the general algebraic theory of twisted tensor products of homotopy commutative dg (co)algebras.

Applying our machinery to a fibration $F\to E\to Y$ on a 1-connected space Y and an associated principal G-fibration $G\to P\to Y$ with action $G\times F\to F$ we obtain the following cubical model (Theorem (5.1): Let $X=\mathrm{Sing}^1 Y\subset \mathrm{Sing}\, Y$ be the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point of Y. Let $Q=\mathrm{Sing}^I G$ and $M=\mathrm{Sing}^I F$ be the singular cubical sets. Then Adams' map $\omega_*: \Omega C_*(Y)=C_*(\mathbf{\Omega}X)\to C^{\square}_*(\Omega Y)$ is realized by a monoidal cubical map $\omega: \mathbf{\Omega}X\to \mathrm{Sing}^I\,\Omega Y$. Composing ω with the map of monoidal cubical sets $\mathrm{Sing}^I\,\Omega Y\to Q$ induced by the canonical map $\Omega Y\to G$ of monoids we immediately obtain a truncating twisting function $\tau: X\to Q$. The resulting twisted Cartesian product $X\times_{\tau}M$ provides the required cubical model of E; and there exists a cubical weak equivalence $X\times_{\tau}M\to Sing^I\,E$. Applying the cochain functor we obtain Berikashvili's multiplicative twisted tensor product in [6].

At the end of the paper we use the theory of multiplicative twisted tensor products for homotopy G-algebras outlined above to obtain the multiplicative twisted tensor product $C^*(Y)\bigotimes_{\phi}C^*_{\mathrm{N}}(F)$, where C^*_{N} denotes the normalized singular *simplicial* cochains. The twisting cochain ϕ here is the composition $\phi: C^*_{\mathrm{N}}(G)\xrightarrow{\varphi} C^*_{\square}(G)\xrightarrow{\tau^*} C^*(Y)$, where φ is a map of dg Hopf algebras defined by the standard triangulation of cubes (see Proof 7.2). In other words, we use a special twisting cochain to introduce an associative multiplication on Brown's model.

As an example we present fibrations with the base being a suspension (in this case the homotopy G-algebra structure consists just of $E_{1,1}=\smile_1$ and all other operations $E_{k,1}$ are trivial) and for which the formula for the multiplication in the twisted tensor product has a very simple form. Moreover in this case we present small multiplicative model being the twisted tensor product of cohomologies of base and fiber with the multiplicative structure purely defined by the $\smile$ and $\smile_1$ operations.

Finally, we mention that the geometric realization $|\mathbf{\Omega}\mathrm{Sing}^1 Y|$ of $\mathbf{\Omega}\mathrm{Sing}^1 Y$ is homeomorphic to the cellular model for a loop space observed by Carlsson and Milgram [9]. In [2,3], Baues defined a geometric coassociative and homotopy cocommutative diagonal on the cobar construction $\Omega C_*^{\mathrm{N}}(Y)$ of the normalized chains $C_*^{\mathrm{N}}(Y)$ by means of a certain cellular model for the loop space (homotopically equivalent to $|\mathbf{\Omega}\mathrm{Sing}^1 Y|$) whose cellular chains coincide with $\Omega C_*^{\mathrm{N}}(Y)$; consequently, one obtains a homotopy G-coalgebra structure on $C_*^{\mathrm{N}}(Y)$. Another modification of Adams' cobar construction is considered by Felix et al. [10].

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2. Notation and preliminaries

Let R be a commutative ring with unit 1. A *differential graded algebra* (dga) is a graded R-module $C = \{C^i\},\ i \in \mathbb{Z}$, with an associative multiplication $\mu\colon C^i \otimes C^j \to C^{i+j}$ and a homomorphism (*a differential*) $d\colon C^i \to C^{i+1}$ with $d^2 = 0$ and satisfying the Leibniz rule $d\mu = \mu(d \otimes Id + Id \otimes d)$. We assume that a dga has a unit $\eta\colon R \to C$ such that $\mu(\eta \otimes Id) = \mu(Id \otimes \eta) = Id$. A non-negatively graded dga C is *connected* if $C^0 = R$. A connected dga C is *n-reduced* if $C^i = 0$, $1 \leqslant i \leqslant n$. A dga is *commutative* if $\mu = \mu T$, where $T(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$. In general, we use Koszul's sign commutation rule: Whenever two symbols u and v are interchanged, affix the sign $(-1)^{|u||v|}$.

A *differential graded coalgebra* (dgc) is a graded R-module $C = \{C_i\},\ i \in \mathbb{Z}$, with an coassociative comultiplication $\varDelta\colon C \to C \otimes C$ and a homomorphism (*a differential*) $d\colon C_i \to C_{i-1}$ with $d^2 = 0$ and satisfying $\Delta d = (d \otimes Id + Id \otimes d)\varDelta$. A dgc C is assumed to have a counit $\varepsilon\colon C \to R,\quad (\varepsilon \otimes Id)\varDelta = (Id \otimes \varepsilon)\varDelta = Id$. A non-negatively graded dgc C is *connected* if $C_0 = R$. A connected dgc C is *n-reduced* if $C_i = 0$, $1 \leqslant i \leqslant n$. A dgc is *cocommutative* if $\varDelta = \Delta T$.

A (connected) *differential graded Hopf algebra* (dgha) $(C, \mu, \varDelta)$ is simultaneously a connected dga (C, μ) and a connected dgc $(C, \varDelta)$ such that $\varDelta\colon C \to C \otimes C$ is an algebra map; note that a graded connected Hopf algebra has a canonical antipode [26], so that the antipode is not an issue.

A dga M is a (left) *comodule* over a dgha C if $\nu\colon M \to C \otimes M$ is a dga map. Let (M', ν') and (M, ν) be comodules over C' and C, respectively, and let $\varphi\colon C' \to C$ be a dgc morphism. A map $\psi\colon M' \to M$ is a *morphism* of comodules if $\nu\psi = (\varphi \otimes \psi)\nu'$.

2.1. Cobar and bar constructions

For an R-module M, let $T(M)$ be the tensor algebra of M, i.e., $T(M) = \bigoplus_{i=0}^{\infty} M^{\otimes i}$. An element $a_1 \otimes \ldots \otimes a_n \in M^{\otimes n}$ is denoted by $[a_1, \ldots, a_n]$. We denote by $s^{-1}M$ the desuspension of M, i.e., $(s^{-1}M)_i = M_{i+1}$.

Let $(C, d_C, \varDelta)$ be a 1-reduced dgc. Denote $\bar{C} = s^{-1}(C_{>0})$. Let $\varDelta = Id \otimes 1 + 1 \otimes Id + \varDelta'$. The (reduced) cobar construction ΩC on C is the tensor algebra $T(\bar{C})$, with differential $d = d_1 + d_2$ defined for $\bar{c} \in \bar{C}_{>0}$ by

$$d_1[\bar{c}] = -[\overline{d_C(c)}]$$

and

$$d_2[\bar{c}] = \sum(-1)^{|c'|}[\bar{c'}|\bar{c''}] \quad \text{for } \Delta'(c) = \sum c' \otimes c''$$

extended as a derivation. The acyclic cobar construction $\Omega(C;C)$ is the twisted tensor product $C \otimes \Omega C$ in which the tensor differential is twisted by the universal twisting cochain $C \to \Omega C$ being an inclusion of degree -1 (see below).

Let (A, d_A, μ) be a 1-reduced dga. The (reduced) bar construction BA on A is the tensor coalgebra $T(\bar{A})$, $\bar{A}=s^{-1}(A_{>0})$, with differential $d=d_1+d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n] \in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n] = -\sum_{i=1}^{n}(-1)^{\varepsilon_i}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n]$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n] = -\sum_{i=2}^{n}(-1)^{\varepsilon_i}[\bar{a}_1|\cdots|\overline{a_{i-1}a_i}|\cdots|\bar{a}_n],$$

where $\varepsilon_i = \sum_{j<i}|\bar{a_j}|$. The acyclic bar construction $B(A;A)$ is the twisted tensor product $A \otimes BA$ in which the tensor differential is twisted by the universal twisting cochain $BA \to A$ being a projection of degree 1.

2.2. Twisting cochains

Let $(C, d, \Delta\colon C \to C\otimes C)$ be a dgc, $(A, d, \mu\colon A\otimes A \to A)$ be a dga, and $(M, d, \nu\colon M \to C \otimes M)$ be a dg comodule over C. A twisting cochain [8] is a homomorphism $\phi\colon C \to A$ of degree 1 satisfying Brown's condition

$$d\phi + \phi d = -\phi \smile \phi, \tag{1}$$

where $\phi \smile \phi' = \mu_A(\phi \otimes \phi')\Delta_C$. There are universal twisting cochains $C \to \Omega C$ and $BA \to A$ being the obvious inclusion and projection, respectively. Let $T(C, A)$ be the set of all twisting cochains $\phi\colon C \to A$. Three essential consequences of Brown's condition (1) are

(i) *The multiplicative extension* $f_\phi\colon \Omega C \to A$ *is a dga map, so there is a bijection* $T(C,A) \leftrightarrow Hom_{dga}(\Omega C, A)$;
(ii) *The comultiplicative extension* $g_\phi\colon C \to BA$ *is a dgc map, so there is a bijection* $T(C,A) \leftrightarrow Hom_{dgc}(C, BA)$;
(iii) *The homomorphism* $d_\phi = d \otimes Id + Id \otimes d + \phi \cap -\colon A \otimes M \to A \otimes M$, *where* $\phi \cap (m \otimes a) = (\mu \otimes Id)(Id \otimes \phi \otimes Id)(Id \otimes \nu)(a \otimes m)$, *is a differential, i.e.,* $d_\phi d_\phi = 0$.

The dg C-comodule $(A \otimes M, d_\phi)$ is called a twisted tensor product and is denoted by $A \bigotimes_\phi M$. The twisted tensor product is functorial in the following sense: Let $\eta\colon A' \to A$ be a dga morphism, $\varphi\colon C' \to C$ be a dgc morphism, $\psi\colon M' \to M$ be a morphism of comodules and $\phi' : C' \to A'$ be a twisting cochain such that $\eta\phi' = \phi\varphi$. Then $\eta \otimes \psi : A' \bigotimes_{\phi'} M' \to A \bigotimes_{\phi} M$ is a chain map.

2.3. Adams' cobar construction

Let X be a 1-reduced simplicial set, i.e., $X = \{X_0 = X_1 = \{*\}, X_2, X_3, \cdots\}$, and let $\tilde{C}_*(X)$ be its chain complex in the ordinary sense. Define the chain complex $C_*(X)$ as the quotient

$$C_*(X) = \tilde{C}_*(X)/\tilde{C}_{>0}(*).$$

Clearly $C_*(X)$ is a 1-reduced dgc with respect to the AW diagonal.

Now let $\operatorname{Sing} Y$ be the singular simplicial set of a based topological space Y and $X = \operatorname{Sing}^1 Y \subset \operatorname{Sing} Y$ be the (Eilenberg) 1-subcomplex generated by those singular simplices which send the 1-skeleton of the standard simplex Δ^n, $n \geqslant 0$, to the base point $y \in Y$. Define the dgc $C_*(Y)$ as $C_*(X)$. Then Adams' cobar construction $\Omega C_*(Y)$ of a space Y is the cobar construction of the dgc $C_*(Y)$.

2.4. Cubical sets

A cubical set is a graded set $Q = \{Q_n\}_{n \geqslant 0}$ with face operators $d_i^\varepsilon : Q_n \to Q_{n-1}$, $\varepsilon = 0, 1$, $i = 1, 2, \ldots, n$, and degeneracy operators $\eta_i : Q_n \to Q_{n+1}$, $i = 1, 2, \ldots, n+1$, satisfying the following standard cubical identities [17]:

$$\begin{aligned}
&d_j^\varepsilon d_i^{\varepsilon'} = d_i^{\varepsilon'} d_{j+1}^\varepsilon, \quad i \leqslant j\\
&d_i^\varepsilon \eta_j = \begin{cases} \eta_{j-1} d_i^\varepsilon & i < j\\ 1 & i = j\\ \eta_j d_{i-1}^\varepsilon & i > j \end{cases}\\
&\eta_i \eta_j = \eta_{j+1}\eta_i, \quad i \leqslant j.
\end{aligned} \tag{2}$$

For an example, let Y be a space and let $\operatorname{Sing}^I Y = \{\operatorname{Sing}_n^I Y\}_{n \geqslant 0}$, where $\operatorname{Sing}_n^I Y$ is the set of all continuous maps $I^n \to Y$. Then $\operatorname{Sing}^I Y$ is a cubical set [24].

Given a cubical set Q and an R-module A, let $(\bar{C}_*^\Box(Q; A), d)$ denote its chain complex with coefficients in A. The normalized chain complex $(C_*^\Box(Q; A), d)$ of Q is defined as the quotient $C_*^\Box(Q; A) = \bar{C}_*^\Box(Q; A)/D_*(Q)$, where $D_*(Q)$ is the subcomplex of $(\bar{C}_*^\Box(Q; A), d)$ generated by the degenerate elements of Q. For a space Y, we denote $C_*^\Box(\operatorname{Sing}^I Y; \mathbb{Z})$ by $C_*^\Box(Y)$. Both $\bar{C}_*^\Box(Q)$ and $C_*^\Box(Q)$ are dg coalgebras with respect to the *Serre diagonal* determined by the Cartesian product decomposition $I^n = I \times \cdots \times I$ of the n-cube [30]: For an element $x \in Q_n$ the *Serre diagonal* is given by

$$\Delta(x) = \Sigma(-1)^\varepsilon d_{j_1}^0 \cdots d_{j_p}^0(x) \otimes d_{i_1}^1 \cdots d_{i_q}^1(x), \tag{3}$$

where the summation is over all shuffles $\{i_1 < \cdots < i_q, j_1 < \cdots < j_p\}$ of the set $\{1, \ldots, n\}$ and $(-1)^\varepsilon$ is the shuffle sign.

Let Q and Q' be cubical sets. The *(tensor) product* of Q and Q' is defined to be

$$Q \times Q' = \{(Q \times Q')_n = \bigcup_{p+q=n} Q_p \times Q'_q\}/\sim,$$

where $(\eta_{p+1}(a), b) \sim (a, \eta_1(b)),\ (a, b) \in Q_p \times Q'_q$. This product is endowed with the obvious face and degeneracy operators [17]. Define a *monoidal cubical set* to be a cubical set Q with an associative cubical multiplication $\mu\colon Q \times Q \to Q$ for which a distinguished element $e \in Q_0$ is a unit. (Warning: since the Q_i's are not assumed to be monoids, Q is not a cubical monoid.) Clearly, the (normalized) chain complex $C_*^{\square}(Q; R)$ on a monoidal cubical set Q and the dual cochain complex $C^*_{\square}(Q; R)$ are dg Hopf algebras. Given a graded monoidal cubical set Q, a *Q-module* is a cubical set L together with associative action $Q \times L \to L$ with the unit of Q acting as identity. In this case, $C^*_{\square}(L; R)$ is a dga comodule over the dg Hopf algebra $(C^*_{\square}(Q; R), d)$.

3. The cubical loop and path functors

3.1. The cubical loop functor

In this section we construct a functor that assigns to a simplicial set $X = \{X_n, \partial_i, s_i\}$ a cubical monoidal set $\mathbf{\Omega} X$, which plays the role of the *loop space of* X. First we construct a cubical monoid *MX without degeneracies*, then enlarge it to $\mathbf{\Omega} X$ with degeneracy operators.

Let $\bar{X} = s^{-1}(X_{>0})$ and define MX to be the free graded monoid (without unit) generated by $\bar{X}$. We denote elements of MX by $\bar{x}_1 \cdots \bar{x}_k$ for $x_j \in X_{m_j+1},\ m_j \geqslant 0, 1 \leqslant j \leqslant k$. The total degree of an element $\bar{x}_1 \cdots \bar{x}_k$ is the sum $m_{(k)} = m_1 + \cdots + m_k,\ m_j = |\bar{x}_j|$, and we write $\bar{x}_1 \cdots \bar{x}_k \in (MX)_{m_{(k)}}$. The product of two elements $\bar{x}_1 \cdots \bar{x}_k$ and $\bar{y}_1 \cdots \bar{y}_\ell$ is defined by concatenation $\bar{x}_1 \cdots \bar{x}_k \bar{y}_1 \cdots \bar{y}_\ell$and is subject only to the associativity relation; there are no other relations whatsoever among these expressions. The graded set MX canonically admits the structure of a cubical set without degeneracies in the following fashion: Let

$$\nu_i : X_n \to X_i \times X_{n-i}, \quad \nu_i(x) = \partial_{i+1} \cdots \partial_n(x) \times \partial_0 \cdots \partial_{i-1}(x), \quad 0 \leqslant i \leqslant n$$

denote the components of the AW diagonal. A superscript n on a simplex $x^n \in X_n$ denotes its dimension. Then for an n-simplex $x^n \in X_n,\ n > 0$, let

$$\nu_i(x^n) = ((x')^i,\ (x'')^{n-i}) \in X_i \times X_{n-i}.$$

First define the face operators $d_i^0, d_i^1 : (MX)_{n-1} \to (MX)_{n-2}$ on a (monoidal) generator $\overline{x^n} \in (\bar{X})_{n-1} = \overline{X_n}$ by

$$\begin{aligned} d_i^0(\overline{x^n}) &= \overline{(x')^i} \cdot \overline{(x'')^{n-i}}, \quad && i = 1, \ldots, n-1, \\ d_i^1(\overline{x^n}) &= \overline{\partial_i(x^n)}, && i = 1, \ldots, n-1. \end{aligned}$$

Thereafter, for any element (word) $\bar{x}_1 \cdots \bar{x}_k$ let

$$\begin{aligned} d_i^0(\bar{x}_1 \cdots \bar{x}_k) &= \bar{x}_1 \cdots \overline{(x'_q)^{j_q}} \cdot \overline{(x''_q)^{m_q+1-j_q}} \cdots \bar{x}_k, \\ d_i^1(\bar{x}_1 \cdots \bar{x}_k) &= \bar{x}_1 \cdots \overline{\partial_{j_q}(x_q)} \cdots \bar{x}_k, \end{aligned}$$

where $m_{(q-1)} < i \leqslant m_{(q)},\ \ j_q = i - m_{(q-1)},\ \ 1 \leqslant q \leqslant k,\ \ 1 \leqslant i \leqslant n-1$.

It is straightforward to check that the defining identities of a cubical set hold for $d_i^0,\ d_i^1$. In particular, the simplicial relations between the ∂_i's imply the cubical relations between the d_i^1's; the associativity relations between the ν_i's imply the cubical relations between the d_i^0's, and the commuting relations between the ∂_i's and v_j's imply the cubical relations between the d_i^1's and d_j^0's. We now enlarge *MX* by enlarging its generating set $\bar{X}$ and introduce the desired degeneracy operators.

For an element $x \in X_n$, we consider formal expressions $\eta_{i_k} \cdots \eta_{i_1}\eta_{i_0}(x)$ with $1 \leqslant i_j \leqslant n + j - 1$, $1 \leqslant j \leqslant k$, $k \geqslant 0$, $\eta_{i_0} = Id$. We call such an expression *normal* if $i_1 \leqslant \cdots \leqslant i_k$. Note that any such expression can be reduced to this normal form by applying the defining identities for a cubical set with degeneracy operators η_i. Let X^c be the graded set of formal expressions with normal form

$$X^c_{n+k} = \{\eta_{i_k} \cdots \eta_{i_1}\eta_{i_0}(x) \mid x \in X_n\}_{n \geqslant 0; k \geqslant 0},$$

where

$$i_1 \leqslant \cdots \leqslant i_k,\ 1 \leqslant i_j \leqslant n + j - 1,\ 1 \leqslant j \leqslant k,\ \eta_{i_0} = Id$$

and let $\bar{X}^c = s^{-1}(X^c_{>0})$. Define $\mathbf{\Omega}''X$ to be the free graded monoid (without unit) generated by $\bar{X}^c$. It is clear that $X \subset X^c$ since $\eta_{i_0}(x) = x$. Thus $MX \subset \mathbf{\Omega}''X$.

Let $\mathbf{\Omega}'X$ be the monoid obtained from $\mathbf{\Omega}''X$ by quotienting with respect to the equivalence relation generated by $\overline{\eta_{p+1}(x)} \cdot \bar{y} \sim \bar{x} \cdot \overline{\eta_1(y)}$ for $|x| = p + 1$, $x, y \in X \subset X^c$. We have the inclusion of graded monoids $MX \subset \mathbf{\Omega}'X$. We claim that $\mathbf{\Omega}'X$ admits the structure of a cubical set. Face operators on the subset $MX \subset \mathbf{\Omega}'X$ were already defined. Now define a degeneracy operator $\eta_i : (\mathbf{\Omega}'X)_{n-1} \to (\mathbf{\Omega}'X)_n$ on a (monoidal) generator $\overline{x} \in (\overline{X^c})_{n-1}$ by

$$\eta_i(\overline{x}) = \overline{\eta_i(x)},$$

(assuming $\eta_i(x)$ is normalized). For any element $\bar{x}_1 \cdots \bar{x}_k$ of $\mathbf{\Omega}'X$ extend the degeneracy operators by

$$\begin{aligned}&\eta_i(\bar{x}_1 \cdots \bar{x}_k) = \bar{x}_1 \cdots \eta_{j_q}(\overline{x_q}) \cdots \bar{x}_k,\\ &\eta_n(\bar{x}_1 \cdots \bar{x}_k) = \bar{x}_1 \cdots \bar{x}_{m_{k-1}} \cdot \eta_{m_k+1}(\overline{x_k}),\end{aligned}$$

where $m_{(q-1)} < i \leqslant m_{(q)}$, $j_q = i - m_{(q-1)}$, $1 \leqslant q \leqslant k$, $1 \leqslant i \leqslant n-1$. Inductively extend the face operators on degenerate elements in such a way that the defining identities for a cubical set are satisfied. Then the cubical set $\{\mathbf{\Omega}'X, d_i^0, d_i^1, \eta_i\}$ depends functorially on *X*.

Now suppose that *X* is a based simplicial set with base point $* \in X_0$, and denote $e = \overline{s_0(*)} \in (\bar{X})_0$. Let $\mathbf{\Omega}X$ be the monoid obtained from $\mathbf{\Omega}'X$ via

$$\mathbf{\Omega}X = \mathbf{\Omega}'X/\sim,$$

where $ea \sim ae \sim a$, for $a \in \mathbf{\Omega}'X$, and $\eta_n(\bar{x}) \sim \overline{s_n(x)}$ for $x \in X_n$, $n > 0$. Obviously $(\mathbf{\Omega}X, d_i^0, d_i^1, \eta_i)$ is a (unital) monoidal cubical set. Note that although the underlying monoidal structure of $\mathbf{\Omega}X$ is not free; all relations involve degenerate elements.

Remark 3.1. In the definition of the face operators d_i^0, d_i^1 of $\mathbf{\Omega}X$ for an *n*-simplex of X_n the first and last face operators ∂_0 and ∂_n of *X* are not used directly. If, in particular, *X* is a 1-reduced simplicial set (i.e., $X_0 = X_1 = \{*\}$), we have the following identities:

$$\begin{aligned}&d_1^0(\overline{x^n}) = \overline{(x')^1} \cdot \overline{(x'')^{n-1}} = e \cdot \overline{(x'')^{n-1}} = \overline{(x'')^{n-1}} = \overline{\partial_0(x^n)},\\ &d_{n-1}^0(\overline{x^n}) = \overline{(x')^{n-1}} \cdot \overline{(x'')^1} = \overline{(x')^{n-1}} \cdot e = \overline{(x')^{n-1}} = \overline{\partial_n(x^n)}, \quad x^n \in X_n.\end{aligned}$$

Thus, all face operators ∂_i of X participate in the definition of $\mathbf{\Omega}X$ in this case.

Remark 3.2. The degeneracies of $\mathbf{\Omega}X$ are formal; we do not use degeneracies of X except for the last one s_n. This is justified by the geometrical fact that in the path fibration, a degenerate singular n-simplex in the base lifts to a singular $(n-1)$-cube of the fiber which need not be degenerate (cf. the proof of Theorem 5.1).

It is convenient to verify the cubical relations by the following combinatorics of the standard cube (compare, [4]). Motivated by the combinatorial description of the standard $(n+1)$-simplex Δ^{n+1}, we denote the set $\{0,1,\ldots,n+1\}$ by $[0,1,\ldots,n+1]$ and assign this to the whole I^n.

Proposition 3.1. *Let*

$$\begin{aligned} d_i^0 &\leftrightarrow x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n, \quad i=1,\ldots,n,\\ d_i^0 &\leftrightarrow x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n, \quad i=1,\ldots,n \end{aligned}$$

denote the face operators of the standard cube I^n in Euclidean coordinates. Then the action of the face operators on $[0,1,\ldots,n+1]$ *by*

$$\begin{aligned} &[0,1,\ldots,n+1] \xrightarrow{d_i^0} [0,1,\ldots,i][i,\ldots,n+1], \quad i=1,\ldots,n,\\ &[0,1,\ldots,n+1] \xrightarrow{d_i^1} [0,1,\ldots,\hat{i},\ldots,n+1], \qquad i=1,\ldots,n \end{aligned}$$

agrees with the cubical identities.

Proof. It is straightforward. □

In general, any q-dimensional face a of I^n is expressed as

$$\begin{gathered} a=[0,i_1,\ldots,i_{k_1}][i_{k_1},\ldots,i_{k_2}][i_{k_2},\ldots,i_{k_3}]\ldots[i_{k_{p-1}},\ldots,i_{k_p},n+1],\\ 0<i_1<\cdots<i_{k-p}<n+1,\ \ q=k_p-p+1 \end{gathered}$$

in the above combinatorics; while a cubical degeneracy operator

$$\eta_i \leftrightarrow x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$$

is thought of as adding a formal element $*$ to the set $[0,1,\ldots,n+1]$ at the $(i+1)^{st}$ place

$$\eta_i[0,1,\ldots,n+1]=[0,1,\ldots,i-1,*,i,\ldots,n+1]$$

with the convention that $[0,1,\ldots,i-1,*][*,i,\ldots,n+1]=[0,1,\ldots,n+1]$ guarantees the equality $d_i^0\eta_i=Id=d_i^1\eta_i$.

3.2. *The cubical path functor*

Here we assign to a simplicial set X a cubical set $\mathbf{P}X$ which plays the role of the *path space of* X. In some sense, $\mathbf{P}X$ will be a *twisted Cartesian product* of a simplicial set X and the monoidal cubical set $\mathbf{\Omega}X$.

First we define the cubical set $\mathbf{P}'X$ as follows. Ignoring underlying structure for the moment, consider the Cartesian product

$$X^c \times \mathbf{\Omega}'X = \left\{ (X^c \times \mathbf{\Omega}'X)_n = \bigcup_{p+q=n} X^c_p \times (\mathbf{\Omega}'X)_q \right\}$$

of the graded sets X^c and $\mathbf{\Omega}'X$. Let

$$X^c \tilde{\times} \mathbf{\Omega}'X = X^c \times \mathbf{\Omega}'X/\sim,$$

where $(\eta_{p+1}(x), y) \sim (x, \eta_1(y))$, $(x, y) \in X^c_p \times (\mathbf{\Omega}'X)_q$. Introduce operators d^0_i, d^1_i and η_i on $X^c \tilde{\times} \mathbf{\Omega}'X$ as follows. For an element $(x, y) \in X_p \times (\mathbf{\Omega}'X)_q \subset X^c_p \times (\mathbf{\Omega}'X)_q$, $p+q=n$, let

$$d^0_i(x, y) = \begin{cases} ((x')^{i-1}, \overline{(x'')^{p+1-i}} \cdot y), & 1 \leqslant i \leqslant p, \\ (x, d^0_{i-p}(y)), & p < i \leqslant n, \end{cases}$$

$$d^1_i(x, y) = \begin{cases} (\partial_{i-1}(x), y), & 1 \leqslant i \leqslant p, \\ (x, d^1_{i-p}(y)), & p < i \leqslant n, \end{cases}$$

$$\eta_i(x, y) = (\eta_i(x), y), \qquad 1 \leqslant i \leqslant p,$$
$$\eta_i(x, y) = (x, \eta_{i-p}(y)), \quad p < i \leqslant n+1.$$

It is easy to check that these face operators satisfy the canonical cubical identities. The data uniquely extends to the structure of a cubical set on the whole $X^c \tilde{\times} \mathbf{\Omega}'X$. The resulting cubical set is denoted by $\mathbf{P}'X$; the cubical set $\mathbf{P}X$ is obtained by replacing $\mathbf{\Omega}'X$ by $\mathbf{\Omega}X$ in the definition of $\mathbf{P}'X$. There is the canonical inclusion of graded sets $\mathbf{\Omega}X \to \mathbf{P}X$ defined by $y \mapsto (*, y)$, $* \in X_0$, and the canonical projection $\xi : \mathbf{P}X \to X$ defined by $(x, y) \mapsto x$.

The cubical relations in $\mathbf{P}'X$ can be verified by means of the following combinatorics of the standard cube (compare with Proposition 3.1). The top dimensional cell of I^{n+1} is identified with the set $0, 1, \ldots, n+1]$ while any proper q-face a of I^{n+1} is expressed as

$$a = j_1, \ldots, j_{s_1}][j_{s_1}, \ldots, j_{s_2}][j_{s_2}, \ldots, j_{s_3}] \ldots [j_{s_{t-1}}, \ldots, j_{s_t}, n+1],$$
$$0 \leqslant j_1 < \cdots < j_{s_t} < n+1, \ \ q = s_t - t + 1.$$

The dimension of the first block $j_1, \ldots, j_{s_1}]$ is $\dim([j_1, \ldots, j_{s_1}]) + 1$.

Proposition 3.2. *Let the face operators* d^ε, $\varepsilon=0, 1$, *act on a face* a *of* I^{n+1} *as in Proposition* 3.1, *but for its first block as*

$$j_1, \ldots, j_{s_1}] \xrightarrow{d^0_i} j_1, \ldots, j_i][j_i, \ldots, j_{s_1}], \quad 1 \leqslant i < s_1,$$
$$j_1, \ldots, j_{s_1}] \xrightarrow{d^1_i} j_1, \ldots, \widehat{j_i}, \ldots, j_{s_1}], \qquad 1 \leqslant i < s_1.$$

Then the relations among d^ε*'s again agree with the cubical identities.*

Proof. It is straightforward. $\square$

The canonical cellular map $\psi : I^{n+1} \to \Delta^{n+1}$ [30] is combinatorially defined by

$$j_1, \ldots, j_{s_1}][j_{s_1}, \ldots, j_{s_2}][j_{s_2}, \ldots, j_{s_3}] \ldots [j_{s_{t-1}}, \ldots, j_{s_t}] \to j_1, \ldots, j_{s_1}$$

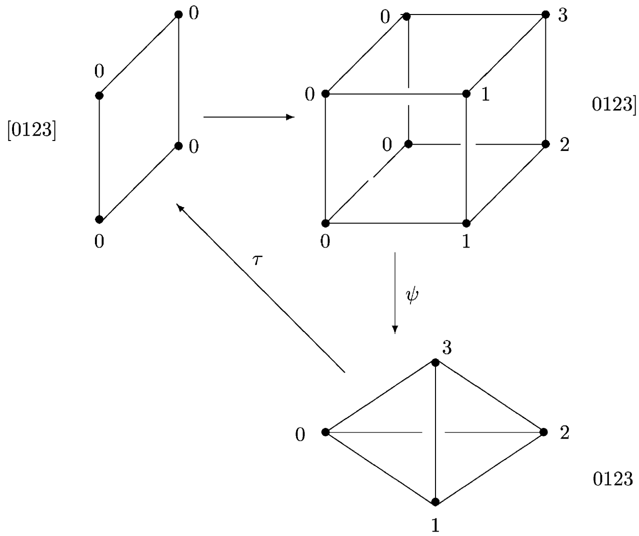

Fig. 1. The universal truncating twisting function τ.

(see Fig. 1). In particular the face $0][0, 1, \dots, n+1]$ of I^{n+1}, i.e., d_1^0, goes to the minimal vertex (the base point) $0 \in \Delta^{n+1}$.

The map ψ can be thought of as a combinatorial model of the projection $\mathbf{P}X \xrightarrow{\xi} X$.

4. Truncating twisting functions and twisted Cartesian products

There is the classical notion of a *twisting function* $\tau\colon X \to G$ from a simplicial set to a simplicial group. Such τ defines a *twisted Cartesian product* for a simplicial G-module M as a simplicial set $X \times_\tau M$. In this section we introduce the notion of a twisting function between graded sets in which the domain and the target have face and degeneracy operators of different types; moreover, the group structure on each homogeneous component of the target is replaced by a graded monoidal structure reflecting the standard Cartesian product of cubes. Namely, we define a *truncating twisting function* $\tau\colon X \to Q$ from a simplicial set X to a monoidal cubical set Q. For a cubical Q-module with action $Q \times L \to L$, such τ defines a *twisted Cartesian product* $X \times_\tau L$ as a *cubical set*.

These notions are motivated by the cubical set $\mathbf{P}X$, which can be viewed as a twisted Cartesian product determined by the canonical inclusion $\tau\colon X \to \mathbf{\Omega}X,\ x \mapsto \bar{x}$ of degree -1, referred to as the *universal truncating twisting function.*

Definition 4.1. Let X be a 1-reduced simplicial set and Q be a monoidal cubical set. A sequence of functions $\tau = \{\tau_n : X_n \to Q_{n-1}\}_{n \geqslant 1}$ of degree -1 is called a truncating twisting function if it satisfies:

$$\begin{array}{lll} \tau(x) = e, & & x \in X_1, \\ d_i^0\tau(x) = \tau\partial_{i+1}\cdots\partial_n(x)\cdot\tau\partial_0\cdots\partial_{i-1}(x), & i=1,\dots,n-1, & x\in X_n,\ n\geqslant 1 \\ d_i^1\tau(x) = \tau\partial_i(x), & i=1,\dots,n-1, & x\in X_n,\ n\geqslant 1 \\ \eta_n\tau(x) = \tau s_n(x), & & x\in X_n,\ n\geqslant 1. \end{array}$$

Remark 4.1. Note that by definition, a truncating twisting function commutes only with the last degeneracy operators (compare [30]), since this is so for the universal truncating function.

The next proposition is an analog of the property (ii) of a twisting cochain from 2.2.

Proposition 4.1. *Let X be a 1-reduced simplicial set and Q be a monoidal cubical set. A sequence of functions $\tau=\{\tau_n : X_n \to Q_{n-1}\}_{n\geqslant 1}$ of degree -1 is a truncating twisting function if and only if the monoidal map $f : \mathbf{\Omega}X \to Q$ defined by $f(\bar{x}_1\dots\bar{x}_k)=\tau(x_1)\dots\tau(x_k)$ is a map of cubical sets.*

Proof. Since f is completely determined by its restriction to monoidal generators, use the argument of verification of cubical identities for a given single generator $\bar{\sigma}$ in ΩX being equivalent to that of identities of the universal truncating function $\tau_{\mathrm{U}} : \sigma \to \bar{\sigma}$. □

The following construction is an analog of the property (iii) of a twisting cochain from 2.2. Given a truncating twisting function $\tau : X \to Q$ and a cubical set L, which is a Q-module via $Q\times L \to L$, define the corresponding twisted Cartesian product $X\times_\tau L$ by replacing $\mathbf{\Omega}X$ with L in the definition of $\mathbf{P}X$. This gives the following:

Definition 4.2. Let X be a 1-reduced simplicial set, Q be a monoidal cubical set, and L be a Q-module via $Q\times L \to L$. Let $\tau=\{\tau_n : X_n \to Q_{n-1}\}_{n\geqslant 1}$ be a truncating twisting function. The twisted Cartesian product $X\times_\tau L$ is the graded set

$$X\times_\tau L = X^c \times L/\sim,$$

where $(\eta_{p+1}(x), y) \sim (x, \eta_1(y)),\ (x,y)\in X_p^c\times L_q$, and is endowed with the face d_i^0, d_i^1 and degeneracy η_i operators defined for $(x,y)\in X_p\times L_q \subset X_p^c\times L_q$ by

$$d_i^0(x,y) = \begin{cases} (\partial_1\cdots\partial_p(x), \tau(x)\cdot y), & i=1, \\ (\partial_i\cdots\partial_p(x), \tau\partial_0\cdots\partial_{i-2}(x)\cdot y), & 1<i\leqslant p, \\ (x, d_{i-p}^0(y)), & p<i\leqslant n, \end{cases}$$

$$d_i^1(x,y) = \begin{cases} (\partial_{i-1}(x), y), & 1\leqslant i\leqslant p, \\ (x, d_{i-p}^1(y)), & p<i\leqslant n, \end{cases}$$

$$\begin{array}{ll} \eta_i(x,y) = (\eta_i(x), y), & 1\leqslant i\leqslant p, \\ \eta_i(x,y) = (x, \eta_{i-p}(y)), & p<i\leqslant n+1. \end{array}$$

For any $(x,y)\in X\times_\tau L$ the operators uniquely extend to form the cubical set $(X\times_\tau L, d_i^0, d_i^1, \eta_i)$.

The geometrical interpretation of $\tau : X \to \mathbf{\Omega}X$ is the following: The standard n-simplex (the base) is converted into the $(n-1)$-cube (the fiber) by the canonical truncation procedure;

this truncation yields the n-cube (the total space) as well, and the latter is thought of as the "twisted Cartesian product" of the simplex and the cube (see Fig. 1); so that projection ψ is a "healing" map. This justifies the name "truncating twisting function".

Example 4.1. Let $M=\{e_k\}_{k\geqslant 0}$ be the free graded monoid on a single generator $e_1\in M_1$ with trivial cubical set structure and $\tau: X\to M$ the sequence of constant maps $\tau_n: X_n\to M_{n-1},\ n\geqslant 1$. Then the twisted Cartesian product $X\times_\tau M$ can be thought of as a cubical resolution of the 1-reduced simplicial set X.

The normalized cubical chain functor $C_*^{\square}$ applied to the cubical sets $\mathbf{\Omega}X,\ \mathbf{P}X,\ X\times_\tau L$ produce dg modules $C_*^{\square}(\mathbf{\Omega}X),\ C_*^{\square}(\mathbf{P}X),\ C_*^{\square}(X\times_\tau L)$. It is straightforward to check that

(i) $C_*^{\square}(\mathbf{\Omega}X)=\Omega C_*(X)$;

(ii) $C_*^{\square}(\mathbf{P}X)=\Omega(C_*(X);C_*(X))$;

(iii) $C_*^{\square}(X\times_\tau L)=C_*(X)\bigotimes_{\tau_*}C_*^{\square}(L)$. (4)

5. The cubical model of the path fibration

Let Y be a topological space. In [1], Adams constructed a morphism

$$\omega_*: \Omega C_*(Y)\to C_*^{\square}(\Omega Y) \tag{5}$$

of dg algebras that is a weak equivalence for simply connected Y. There are explicit combinatorial interpretations of Adams' cobar construction, the above map ω_*, and the acyclic cobar construction $\Omega(C_*(Y);C_*(Y))$ in terms of cubical sets. Indeed, we have the following theorem (compare, [25,9,2,3,10]).

Theorem 5.1. *Let $\Omega Y\to PY\xrightarrow{\pi} Y$ be the Moore path fibration.*

(i) *There are natural morphisms ω, p, ψ such that*

$$\begin{tikzcd}
\mathrm{Sing}^I\,\Omega Y \arrow[r] & \mathrm{Sing}^I\,PY \arrow[r, "\pi_*"] & \mathrm{Sing}^I\,Y \\
\mathbf{\Omega}\mathrm{Sing}^1\,Y \arrow[r] \arrow[u, "\omega"] & \mathbf{P}\mathrm{Sing}^1\,Y \arrow[r, "\xi"] \arrow[u, "p"] & \mathrm{Sing}^1\,Y \arrow[u, "\psi"]
\end{tikzcd} \tag{6}$$

$\psi: \mathrm{Sing}^1\,Y\to \mathrm{Sing}^I\,Y$ is a map of graded sets induced by $\psi: I^n\to\Delta^n$, while p is a morphism of cubical sets, and ω a morphism of monoidal cubical sets; moreover, the cubical maps are homotopy equivalences whenever Y is simply connected.

(ii) *The chain complex $C_*^{\square}(\mathbf{\Omega}\mathrm{Sing}^1\,Y)$ coincides with the cobar construction $\Omega C_*(Y)$, see* 2.3. *Moreover, for a simply connected space, Y, the Adams weak equivalence* (5)

$$\omega_*: \Omega C_*(Y)=C_*^{\square}(\Omega\mathrm{Sing}^1\,Y)\to C_*^{\square}(\Omega Y)=C_*(\mathrm{Sing}^I\,\Omega Y)$$

is induced by the morphism of monoidal cubical sets ω (and consequently it preserves all structures which one has in the chain complex of a cubical set).

(iii) *The chain complex* $C_*^{\square}(\mathbf{P}\,\mathrm{Sing}^1 Y)$ *coincides with the acyclic cobar construction* $\Omega(C_*(Y); C_*(Y))$.

Proof. (i). Morphisms p and ω are constructed simultaneously by induction on the dimension of singular simplices in $\mathrm{Sing}^1 Y$. For $i=0,1$ and $(\sigma, e)\in \mathbf{P}\,\mathrm{Sing}^1 Y$, $\sigma\in(\mathrm{Sing}^1 Y)_i$, define $p(\sigma,e)$ as the constant map $I^i \to PY$ to the base point y, where e denotes the unit of the monoid $\mathbf{\Omega}\,\mathrm{Sing}^1 Y$ (and of the monoid $\mathrm{Sing}^I \Omega Y$ as well). Put $\omega(e)=e$. Denote by $P\,\mathrm{Sing}^1 Y_{(i,j)}$ the subset in $\mathbf{P}\,\mathrm{Sing}^1 Y$ consisting of the elements (σ,σ') with $|\sigma|\leqslant i$, and $\sigma'\in \Omega\,\mathrm{Sing}^1 Y_{(j)}$, a submonoid in $\mathbf{\Omega}\,\mathrm{Sing}^1 Y$ having (monoidal) generators $\bar\sigma$ with $|\bar\sigma|\leqslant j$.

Suppose by induction that we have constructed p and ω on $\mathbf{P}\,\mathrm{Sing}^1 Y_{(n-1,n-2)}$ and $\mathbf{\Omega}\,\mathrm{Sing}^1 Y_{(n-2)}$ respectively such that

$$p(\sigma,\sigma') = p(\sigma,e)\cdot\omega(\sigma') \quad \text{and} \quad \omega(\bar\sigma)=p(d_1^0(\sigma,e)),$$

where the $\cdot$ product is determined by the action $PY\times\Omega Y\to\Omega Y$. Let $\bar I^n\subset I^n$ be the union of the $(n-1)$-faces $d_i^{\varepsilon}(I^n)$ of I^n except the $d_1^0(I^n)=(0,x_2,\ldots,x_n)$ and then for a singular simplex $\sigma\colon \Delta^n\to Y$ define the map

$$\bar p\colon \bar I^n \to PY$$

by

$$\bar p|_{d_i^{\varepsilon}(I^n)} = p(d_i^{\varepsilon}(\sigma,e)),\ \ \varepsilon=0,1, \quad \text{and} \quad i\neq 1 \ \text{ for } \ \varepsilon=0.$$

Then the following diagram commutes:

$$\begin{tikzcd}
\bar I^n \arrow[r, "\bar p"] \arrow[d, "i"'] & P_\sigma Y \arrow[r, "g_\sigma"] \arrow[d, "\pi_\sigma"'] & PY \arrow[d, "\pi"'] \\
I^n \arrow[r, "\psi"] & \Delta^n \arrow[r, "\sigma"] & Y.
\end{tikzcd}$$

Clearly, i is a strong deformation retraction and we define $p(\sigma,e)\colon I^n\to P_\sigma Y$ as a lift of ψ. Define $\omega(\bar\sigma)=p(d_1^0(\sigma,e))$. The proof of p and ω being homotopy equivalences (after the geometric realizations) immediately follows, for example, from the observation that ξ induces a long exact homotopy sequence. The last statement is a consequence of the following two facts: (1) $|P\,\mathrm{Sing}^1 X|$ is contractible, and (2) the projection ξ induces an isomorphism $\pi_*(|\mathbf{P}\,\mathrm{Sing}^1 Y|, |\mathbf{\Omega}\,\mathrm{Sing}^1 Y|)\xrightarrow{\xi_*}\pi_*(|\mathrm{Sing}^1 Y|)$.

(ii)–(iii). This is straightforward. $\square$

Thus, by passing to chain complexes in diagram (6) we obtain the following comultiplicative model of the path fibration π formed by dgc's.

Corollary 5.1. *For the path fibration* $\Omega Y\to PY\xrightarrow{\pi} Y$ *there is a comultiplicative model formed by coassociative dgc's which is natural in* Y

$$
\begin{tikzcd}
C_*^{\square}(\Omega Y) \arrow[rr] & & C_*^{\square}(PY) \arrow[r, "\pi_*"] & C_*^{\square}(Y) \\
\Omega C_*(Y) \arrow[u, "\omega_*"] \arrow[rr] & & \Omega(C_*(Y); C_*(Y)) \arrow[u, "p_*"] \arrow[r, "\xi_*"] & C_*(Y) \arrow[u, "\psi_*"]
\end{tikzcd}
\tag{7}
$$

6. Cubical models for fibrations

Here we prove the main result in this paper. Let G be a topological group, F be a G-space $G \times F \to F$, $G \to P \xrightarrow{\pi} Y$ be a principal G-bundle and $F \to E \xrightarrow{\zeta} Y$ be the associated fibration with the fiber F. Let $X = \operatorname{Sing}^1 Y$, $Q = \operatorname{Sing}^I G$ and $L = \operatorname{Sing}^I F$. The group operation $G \times G \to G$ induces the structure of a monoidal cubical set on Q and the action $G \times F \to F$ induces a Q-module structure $Q \times L \to L$ on L.

Theorem 6.1. *The principal G-fibration $G \to P \xrightarrow{\pi} Y$ determines a truncating twisting function $\tau\colon \operatorname{Sing}^1 Y \to \operatorname{Sing}^I G$ such that the twisted Cartesian product $\operatorname{Sing}^1 Y \times_\tau \operatorname{Sing}^I F$ models the total space E of the associated fibration $F \to E \xrightarrow{\zeta} Y$, that is there exists a cubical map*

$$\operatorname{Sing}^1 Y \times_\tau \operatorname{Sing}^I F \to \operatorname{Sing}^I E$$

inducing homology isomorphism.

Proof. Let $\omega\colon \mathbf{\Omega}X \to \operatorname{Sing}^I \Omega Y$ be the map of monoidal cubical sets from Theorem 5.1. By Proposition 4.1 ω corresponds to a truncating twisting function $\tau'\colon X = \operatorname{Sing}^1 Y \xrightarrow{\tau_U} \mathbf{\Omega}X = \mathbf{\Omega}\operatorname{Sing}^1 Y \xrightarrow{\omega} \operatorname{Sing}^I \Omega Y$. Composing τ' with the map of monoidal cubical sets $\operatorname{Sing}^I \Omega Y \to \operatorname{Sing}^I G = Q$ induced by the canonical map $\Omega Y \to G$ of monoids we obtain a truncating twisting function $\tau\colon X \to Q$. The resulting twisted Cartesian product $\operatorname{Sing}^1 Y \times_\tau \operatorname{Sing}^I F$ is a cubical model of E. Indeed, we have the canonical equality

$$X \times_\tau L = (X \times_\tau Q) \times L/\sim,$$

where $(xg, y) \sim (x, gy)$. Next the argument of the proof of Theorem 5.1 gives a cubical map $f'\colon X \times_{\tau_U} \mathbf{\Omega}X \to \operatorname{Sing}^I P$ preserving the actions of $\mathbf{\Omega}X$ and Q. Hence, this map extends to a cubical map $f\colon X \times_\tau Q \to \operatorname{Sing}^I P$ by $f(x, g) = f'(x, e)g$. The map

$$f \times Id\colon (X \times_\tau Q) \times L \to \operatorname{Sing}^I P \times L \to \operatorname{Sing}^I (P \times F)$$

induces the map

$$\operatorname{Sing}^1 Y \times_\tau \operatorname{Sing}^I F \to \operatorname{Sing}^I E$$

as desired. $\square$

For convenience, assume that X, Q and L are as in the Definition 4.2. On the chain level a truncating twisting function τ induces the twisting cochains $\tau_*\colon C_*(X) \to C_{*-1}(Q)$ and

$\tau^*: C^*(Q) \to C^{*+1}(X)$ in the standard sense ([8,7,14]). Recall the equality of dg modules ((iii) of 4)

$$C_*^{\square}(X \times_\tau L) = C_*(X) \bigotimes_{\tau_*} C_*^{\square}(L) \tag{8}$$

and, consequently, the obvious injection

$$C^*_{\square}(X \times_\tau L) \supset C^*(X) \bigotimes_{\tau^*} C^*_{\square}(L) \tag{9}$$

of dg modules (which is an equality if the graded sets are of finite type).

The cubical structure of $X \times_\tau L$ induces a dgc structure on $C_*^{\square}(X \times_\tau L)$. Transporting this structure (the Serre diagonal (3)) to the right-hand side of (8) we obtain a *comultiplicative* model $C_*(X) \bigotimes_{\tau_*} C_*^{\square}(L)$ of our fibration. Dually, $C^*_{\square}(X \times_\tau L)$ is a dga, so a dga structure (a multiplication) arises on the right-hand side of (9) and we obtain a *multiplicative* model $C^*(X) \bigotimes_{\tau^*} C^*_{\square}(L)$ of our fibration.

Below we describe these structures (the comultiplication on the $C_*(X) \bigotimes_{\tau_*} C_*^{\square}(L)$ and the multiplication on $C^*(X) \bigotimes_{\tau^*} C^*_{\square}(L)$) in terms of certain (co)chain operations that form a *homotopy* G-*(co)algebra* structure on the (co)chain complex of X.

6.1. *The canonical homotopy G-algebra structure on $C^*(X)$*

To describe these structures in more detail, we focus on equality (i) of (4)

$$C_*^{\square}(\mathbf{\Omega} X) = \Omega C_*(X).$$

As before, the cubical structure of $\mathbf{\Omega} X$ induces a comultiplication (Serre diagonal) on $C_*^{\square}(\mathbf{\Omega} X)$, thus this structure also appears on the right-hand side of the above equality, so that the cobar construction $\Omega C_*(X)$ becomes a dg Hopf algebra. Such a comultiplication was defined on the cobar construction $\Omega C_*^{\mathrm{N}}(X)$ of the normalized complex $C_*^{\mathrm{N}}(X)$ by Baues in [2,3].

In the combinatorics of Proposition 3.1, this diagonal is expressed as

$$\begin{aligned}\Delta[0,1,\ldots,n+1] = \Sigma(-1)^{\varepsilon}\ &[0,1,\ldots,j_1][j_1,\ldots,j_2]\\ &[j_2,\ldots,j_3]\ldots[j_p,\ldots,n+1]\\ &\otimes [0,j_1,j_2,\ldots,j_p,n+1].\end{aligned}$$

Note that the summands $[01\ldots n{+}1]\otimes[0,n{+}1]$ and $[01][12][23]\ldots[n,n{+}1]\otimes[01\ldots n{+}1]$ form the primitive part of the diagonal.

Now regarding the blocks of natural numbers above as faces of the standard $(n+1)$-simplex, we obtain Baues' formula for the coproduct $\Delta: \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$: For a generator $\sigma \in C_{n+1}(X) \subset \Omega C_*(X)$ define

$$\begin{aligned}\Delta[\sigma] = \Sigma(-1)^{\varepsilon}\ &[\sigma(0,1,\ldots,j_1)|\sigma(j_1,\ldots,j_2)|\\ &\sigma(j_2,\ldots,j_3)|\ldots|\sigma(j_p,\ldots,n+1)]\otimes\\ &[\sigma(0,j_1,j_2,\ldots,j_p,n+1)],\end{aligned} \tag{10}$$

where $\sigma(i_1,\ldots,i_k)$ denotes the suitable face of σ. Note that since X is assumed to be 1-reduced, the image $[\sigma(k,\bar{k}+1)]$ of each 1-dimensional face $\sigma(k,k+1)$ is the unit in

$\Omega C_*(X)$ and hence can be omitted. Note also that the formula is highly asymmetric, the left-hand factors of $\Delta[\sigma]$ in $\Omega C_*(X) \otimes \Omega C_*(X)$ have length $\geqslant 1$ and the right-hand factors have length 1; this is a consequence of (3) and the structure of d_i^0, d_i^1 from Proposition 3.1.

Actually, this diagonal consists of *components*

$$E^{k,1} = pr \circ \Delta : C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X) \to C_*(X)^{\otimes k} \otimes C_*(X), \; k \geqslant 1,$$

where *pr* is the obvious projection. The basic component $E^{1,1}$ looks like

$$\begin{aligned} E^{1,1}(\sigma) = {} & \Sigma_{s,t}(-1)^{\varepsilon} \, (\sigma(0,1) \otimes \sigma(1,2) \otimes \ldots \otimes \sigma(s-1,s) \\ & \otimes \sigma(s, s+1, \ldots, t) \otimes \sigma(t, t+1) \\ & \otimes \sigma(n, n+1)) \otimes \sigma(0, 1, \ldots, s-1, s, t, t+1, \ldots, n+1) \\ = {} & \Sigma_{s,t}(-1)^{\varepsilon} \, \sigma(s, s+1, \ldots, t) \\ & \otimes \sigma(0, 1, \ldots, s-1, s, t, t+1, \ldots, n+1) \end{aligned}$$

which is a chain operation dual to Steenrod's $\smile_1$-product.

Dualizing the operations $E^{k,1}$, we obtain the sequence of cochain operations

$$\{E_{k,1} : C^*(X)^{\otimes k} \otimes C^*(X) \to C^*(X)\}_{k \geqslant 1}$$

which define a multiplication on the bar construction $BC^*(X) \otimes B^*(X) \to BC^*(X)$. These cochain operations form a *homotopy G-algebra structure* on $C^*(X)$ (see the next section).

6.2. The non simply-connected case

The operations $\{E_{k,1}\}$ above are restrictions of more general cochain operations that arise on $\tilde{C}^*(X)$ for a based space Y, which is not necessarily 1-connected. In this case, for $X = \operatorname{Sing} Y$ we have the operations

$$\{E_{k,1} : \tilde{C}^*(X)^{\otimes k} \otimes \tilde{C}^*(X) \to \tilde{C}^*(X)\}_{\,k \geqslant 0}$$

given by the following explicit formulas: For $a_i \in \tilde{C}^{m_i}(X),\; m_i \geqslant 2,\; 1 \leqslant i \leqslant k$, let

$$E_{k,1}(a_1, \ldots, a_k; a_0) = \sum_{j \geqslant k} \tilde{E}_{j,1}(\varepsilon^1, a_1, \varepsilon^1, \ldots, \varepsilon^1, a_k, \varepsilon^1; a_0),$$

where $\varepsilon^1 \in \tilde{C}^1(X)$ is the generator represented by the constant singular 1-simplex at the base point $\Delta^1 \to y \in Y$ and the operations $\tilde{E}_{k,1}$ are defined for $c_j \in \tilde{C}^{m_j}(X),\; m_j \geqslant 1,\; 1 \leqslant j \leqslant k$, $c_0 \in \tilde{C}^k(X)$, by

$$\tilde{E}_{k,1}(c_1, c_2, \ldots, c_k; c_0) = c \in \tilde{C}^n(X), \quad n = m_1 + \cdots + m_k,$$

$$\begin{aligned} c(\sigma) = {} & (-1)^{\varepsilon} c_1(\partial_{i_1+1} \cdots \partial_n \sigma) c_2(\partial_0 \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \partial_n \sigma) \cdots \\ & c_k(\partial_0 \cdots \partial_{i_{k-1}-1} \sigma) c_0(\hat{\partial}_0 \partial_1 \hat{\partial}_{i_1} \cdots \hat{\partial}_{i_{k-1}} \cdots \partial_{n-1} \hat{\partial}_n \sigma) \end{aligned}$$

$$\varepsilon = \sum_{j=1}^{k} (j-1)(m_j - 1),$$

where $i_q = m_1 + \cdots + m_q$, $1 \leqslant q \leqslant k-1$, $\sigma \in X_n$, and where $\tilde{E}_{k,1}(c_1, c_2, \ldots, c_k; c_0) = 0$ otherwise.

Remark 6.1. Though each $\tilde{E}_{k,1}$, and in particular $\tilde{E}_{1,1}$ has only one component, the formula for $k = 1$ defines $E_{1,1}$ as the Steenrod cochain $\smile_1$-operation without any restriction on Y. This fact evidently indicates a difference between topological and algebraic interpretation of the operations $\{E_{k,1}\}_{k \geqslant 1}$ in terms of 1-reduced algebras (see also Example 7.3).

6.3. Twisted multiplicative model for a fibration

Next we further explore the twisted Cartesian product $X \times_\tau L$. To describe the corresponding coproduct and product on the right-hand sides of (8) and (9), respectively, it is very convenient to express the Serre diagonal (3) using the combinatorics of Proposition 3.2

$$\begin{aligned} 01\ldots n] \xrightarrow{\Delta} \Sigma(-1)^{\varepsilon}\ 0\ldots j_1][j_1\ldots j_2[j_2\ldots j_3]\ldots[j_k\ldots n]\otimes \\ \hat{0}, \ldots, \widehat{j_1-1}, j_1, \widehat{j_1+1}, \ldots, \widehat{j_2-1}, j_2, \ldots, j_k, \\ \widehat{j_{k+1}}, \ldots, \widehat{n-1}, n], \end{aligned} \tag{11}$$

$0 \leqslant j_1 < \cdots < j_k < n$, where the summands $01\ldots n]\otimes n]$ and $0][01][12][23]\ldots[n-1, n]\otimes 01\ldots n]$ form the primitive part of the diagonal.

Furthermore, the action $Q \times L \to L$ induces a comodule structure $\Delta_L : C^*(L) \to C^*(Q) \otimes C^*(L)$, and it is not hard to see that the cubical multiplication of (9) can be expressed by this comodule structure, diagonal (11), the twisting cochain τ^*, and the operations $\{E_{k,1}\}_{k \geqslant 1}$ by the following formula: Let $a_1 \otimes m_1,\ a_2 \otimes m_2 \in C^*(X) \bigotimes_{\tau^*} C^*_{\square}(L)$ and $\Delta^k_L : C^*(L) \to C^*(Q)^{\otimes k} \otimes C^*(L)$ be the iterated Δ_L with $\Delta^0_L = Id : C^*(L) \to C^*(L)$; let $\Delta^k_L(m_1) = \sum c^1 \otimes \ldots \otimes c^k \otimes m_1^{k+1}$. Then

$$\begin{aligned} &\mu_{\tau^*}((a_1 \otimes m_1) \otimes (a_2 \otimes m_2)) \\ &\quad = \sum_{k \geqslant 0} (-1)^{|a_2||m_1^{k+1}|} a_1 E_{k,1}(\tau^*(c^1), \ldots, \tau^*(c^k); a_2) \otimes m_1^{k+1} m_2. \end{aligned} \tag{12}$$

Corollary 6.1. *Under the circumstances of Theorem* 6.1, *the twisted differential d_τ and multiplication μ turn the tensor product $C^*(Y) \otimes C^*_{\square}(F)$ into a dga $(C^*(Y) \otimes C^*_{\square}(F), d_\tau, \mu_{\tau^*})$ weakly equivalent to the dga $C^*_{\square}(E)$.*

Such a multiplicative model is constructed in [6] without explicit formulas for the multiplication.

Corollary 6.2. *There exists on the acyclic bar construction $B(C^*(Y); C^*(Y))$ the following strictly associative multiplication: for $a = a_0 \otimes [\bar{a}_1|\cdots|\bar{a}_n]$, $b = b_0 \otimes [\bar{b}_1|\cdots|\bar{b}_m]$, $a_i, b_j \in$*

$C^*(Y),\ 0\leqslant i\leqslant n,\ 0\leqslant j\leqslant m$, *let*

$$ab=\sum_{k=0}^{n}(-1)^{|b_0|(|\bar a_{k+1}|+\cdots+|\bar a_n|)}a_0E_{k,1}(a_1,\ldots,a_k;b_0)\otimes[\bar a_{k+1}|\cdots|\bar a_n]\circ[\bar b_1|\cdots|\bar b_m]. \tag{13}$$

Proof. Take $Q=L=\mathbf{\Omega}X$. Then the multiplication (12) looks as (13). □

7. Twisted tensor products for homotopy G-algebras

The notion of homotopy G-(co)algebra naturally generalizes that of a (co)commutative (co)algebra. For commutative dga's there exists the theory of *multiplicative* twisted tensor products. Below we generalize this theory for homotopy G-algebras. Namely, we define a twisted tensor product with both twisted differential and *twisted multiplication* inspired by the formulas (12) and (13) established in the previous section.

The following definition of homotopy G-algebra (hga) differs from the definition in [12] only by grading (see also [13]). Let A be a dga and consider the dg module $(Hom(BA\otimes BA,A),\nabla)$ with differential ∇. The $\smile$-product induces a dga structure (the tensor product $BA\otimes BA$ is a dgc with the standard coalgebra structure).

Definition 7.1. A homotopy G-algebra is a 1-reduced dga A equipped with multilinear maps

$$E_{p,q}: A^{\otimes p}\otimes A^{\otimes q}\to A,\ \ p,q\geqslant 0,\ \ p+q>0,$$

satisfying the following properties:

(i) $E_{p,q}$ is of degree $1-p-q$;
(ii) $E_{p,q}=0$ except $E_{1,0}=id,\ E_{0,1}=id$ and $E_{k,1},\ k\geqslant 1$;
(iii) the homomorphism $E: BA\otimes BA\to A$ defined by

$$E([\bar a_1|\cdots|\bar a_p]\otimes[\bar b_1|\cdots|\bar b_q])=E_{p,q}(a_1,\ldots,a_p;b_1,\ldots,b_q)$$

is a twisting cochain in the dga $(Hom(BA\otimes BA,A),\nabla,\smile)$, i.e., satisfies $\nabla E=E\smile E$;
(iv) the multiplication μ_E is associative, i.e., *BA* is a dg Hopf algebra.

Condition (iii) implies that the comultiplicative extension $\mu_E: BA\otimes BA\to BA$ is a chain map; conditions (iii) and (iv) can be rewritten in terms of the components $E_{p,q}$ (see [12]). In particular the operation $E_{1,1}$ satisfies conditions similar to Steenrod's $\smile_1$ product: Condition (iii) gives

$$dE_{1,1}(a_1;a_0)-E_{1,1}(da_1;a_0)+(-1)^{|a_1|}E_{1,1}(a_1;da_0)=(-1)^{|a_1|}a_1a_0-(-1)^{|a_1|(|a_0|+1)}a_0a_1, \tag{14}$$

so it measures the non-commutativity of the product of A. Hence, a homotopy G-algebra with $E_{1,1}=0$ is a commutative dga. We denote $E_{1,1}(a,b)$ by $a\smile_1 b$. This notation is also justified by the other condition that follows from (iii), namely,

$$c\smile_1(ab)=(c\smile_1 a)b+(-1)^{|a|(|c|-1)}a(c\smile_1 b). \tag{15}$$

Thus map $a\smile_1 -: A\to A$ is a derivation; when $A=C^*(X)$ formula 15 is called the *Hirsch formula*. On the other hand, the map $-\smile_1 c: A\to A$ is a derivation only *up to homotopy* with the operation $E_{2,1}$ serving as a suitable homotopy: This time condition (iii) gives

$$\begin{aligned} &dE_{2,1}(a,b;c)-E_{2,1}(da,b;c)-(-1)^{|a|}E_{2,1}(a,db;c)-(-1)^{|a|+|b|}E_{2,1}(a,b;dc)\\ &\quad=(-1)^{|a|+|b|}(ab)\smile_1 c-(-1)^{|a|+|b||c|}(a\smile_1 c)b-(-1)^{|a|+|b|}a(b\smile_1 c). \end{aligned}\tag{16}$$

The main examples of hga's are: $C^*(X)$ (see [2,3,13] and previous section) and the Hochschild cochain complex of an associative algebra, with the operations $E_{1,1}$ and $E_{2,1}$ defined by Gerstenhaber in [11] and the higher operations given in [20,13,12]. Another example is the cobar construction of a dg Hopf algebra [21]. Note also that certain algebras (including polynomial algebras) that are realized as the cohomology of topological spaces also admit a non-trivial hga structure [29] (see also Example 7.3 below).

The dual notion is that of a homotopy G-coalgebra (hgc). For an hgc $(C,d,\Delta,\{E^{p,q}: C\to C^{\otimes p}\otimes C^{\otimes q}\})$ the cobar construction ΩC is a dg Hopf algebra with a comultiplication induced by $\{E^{p,q}\}$.

Remark 7.1. For a hga A, the operation $E_{2,1}$, besides of (16), measures the lack of associativity of $E_{1,1}=\smile_1$. In particular, condition (iv) yields

$$a\smile_1(b\smile_1 c)-(a\smile_1 b)\smile_1 c=E_{2,1}(a,b;c)+(-1)^{(|a|+1)(|b|+1)}E_{2,1}(b,a;c) \tag{17}$$

which implies that the commutator $[a,b]=a\smile_1 b-(-1)^{(|a|+1)(|b|+1)}b\smile_1 a$ satisfies the Jacobi identity. In view of (14), this commutator induces a Lie bracket of degree -1 on $H(A)$. Furthermore, (15) and (16) imply that $[a,-]: H(A)\to H(A)$ is a derivation, so that $H(A)$ is a Gerstenhaber algebra [11] (this notion is not a particular case of hga). This structure is generally nontrivial in the Hochschild cohomology of an associative algebra, but the existence of a $\smile_2$ product trivializes the induced Gerstenhaber algebra structure on $H(C^*(X))=H^*(X)$.

7.1. *Multiplicative twisted tensor products*

Let C be a dgc, A a dga and M a dg comodule over C. Brown's twisting cochain $\phi: C\to A$ (see 2.2) determines a dga map $f_\phi: \Omega C\to A$ (the multiplicative extension of ϕ), a dgc map $g_\phi: C\to BA$ (the comultiplicative extension of ϕ) and the twisted differential $d_\phi=d\otimes Id+Id\otimes d+\phi\cap\_: A\otimes M\to A\otimes M$. Suppose furthermore, that C is a dg Hopf algebra, M is a dga, and $M\to C\otimes M$ is a dga map. In general d_ϕ is not a derivation with respect to the multiplication on the tensor product $A\otimes M$. But when A is a *commutative* dga (in this case BA is a dg Hopf algebra with respect to the shuffle product μ_{sh}) and $g_\phi: C\to BA$ is a map of dg Hopf algebras, the twisted differential d_ϕ is a derivation with respect to the standard multiplication of the tensor product $A \otimes C$ and the twisted tensor product $A \bigotimes_{\phi} C$ is a dga (see Proutè [27]). We shall generalize this phenomenon for a homotopy G-algebra A, in which case BA is again a dg Hopf algebra with respect to the multiplication μ_E.

Definition 7.2. A twisting cochain $\phi\colon C \to A$ in $Hom(C, A)$ is multiplicative if the comultiplicative extension $C \to BA$ is an algebra map.

It is clear that if $\phi\colon C \to A$ is a multiplicative twisting cochain and if $g\colon B \to C$ is a map of dg Hopf algebras then the composition $\phi g\colon B \to A$ is again a multiplicative twisting cochain. The canonical projection $BA \to A$ provides an example of the universal multiplicative cochain. For a commutative dga A, the multiplication map μ_E equals μ_{sh}, so Proutè's twisting cochain is multiplicative (see, for example, [28]). The argument for the proof of formula (12) immediately yields the following:

Theorem 7.1. *Let $\phi\colon C \to A$ be a multiplicative twisting cochain. Then the tensor product $A \otimes M$ with the twisting differential $d_\phi = d \otimes Id + Id \otimes d + \phi \cap\_$ becomes a dga $(A \otimes M, d_\phi, \mu_\phi)$ with the twisted multiplication μ_ϕ determined by formula* (12).

Remark 7.2. As in 2.2, this construction is functorial in the following sense: Let $\eta\colon A' \to A$ be a strict morphism of hga's (i.e., η is a morphism of dga's strictly compatible with all $E_{p,q}$'s), $\varphi\colon C' \to C$ be a dg Hopf algebra morphism, $\psi\colon M' \to M$ be simultaneously a morphism of comodules and a dga morphism, and $\phi'\colon C' \to A'$ be a multiplicative twisting cochain such that $\eta\phi' = \phi\varphi$. Then

$$\eta \otimes \psi\colon (A' \otimes M', d_{\phi'}, \mu_{\phi'}) \to (A \otimes M, d_\phi, \mu_\phi)$$

is a morphism dga's.

The above theorem includes the twisted tensor product theory for commutative algebras [27].

Corollary 7.1. *For a homotopy G-algebra A, the acyclic bar-construction $B(A; A)$, endowed with the twisted multiplication determined by formula* (13) *acquires a dga structure.*

7.2. Brown's model as a dga

In conclusion, we replace the cubical cochains $C^*_{\square}(F)$ and $C^*_{\square}(G)$ by the normalized *simplicial* cochains $C^*_{\mathrm{N}}(F)$ and $C^*_{\mathrm{N}}(G)$ in Corollary 6.1 to introduce an associative multiplication on Brown's model $C^*(Y) \bigotimes_{\phi} C^*_{\mathrm{N}}(F)$ for a *special* twisting cochain ϕ. Specifically, we have:

Corollary 7.2. *Let $F \to E \xrightarrow{\zeta} Y$ be a fibration as in Corollary* 6.1. *There exists a multiplicative twisting cochain $\phi\colon C^*_{\mathrm{N}}(G) \to C^{*+1}(Y)$ such that the twisted tensor product $(C^*(Y) \otimes C^*_{\mathrm{N}}(F), d_\phi, \mu_\phi)$ with twisted differential d_ϕ and twisted multiplication μ_ϕ is a dga with cohomology algebra isomorphic to $H^*(E)$.*

Proof. Let us first mention that there exists the following standard triangulation of the cub I^n, see for example [10]. Each vertex of I^n is a sequence $(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_i = 0, 1$. The set of all 2^n vertexes is ordered: $(\varepsilon_1, \ldots, \varepsilon_n) \leqslant (\varepsilon'_1, \ldots, \varepsilon'_n)$ if $\varepsilon_i \leqslant \varepsilon'_i$. There are $n!$ increasing sequences of maximal length $n+1$ which start with minimal vertex $(0, \ldots, 0)$ and end with maximal $(1, \ldots, 1)$. They form $n!$ n-simplices which triangulate I^n.

Let $\varphi\colon C^*_{\mathrm{N}}(G) \to C^*_{\square}(G)$ and $\psi\colon C^*_{\mathrm{N}}(F) \to C^*_{\square}(F)$ be the maps induced by triangulation of cubes (see, for example, [10]), and $\phi = \tau^*\varphi\colon C^*_{\mathrm{N}}(G) \to C^*_{\square}(G) \to C^*(Y)$. Then the 4-tuple $\{\eta = Id,\ \varphi,\ \psi,\ \phi\}$ satisfies the conditions of Remark 7.2, thus

$$Id \otimes \psi\colon (C^*(Y) \otimes C^*_{\mathrm{N}}(F), d_\phi, \mu_\phi) \to (C^*(Y) \otimes C^*_{\square}(F), d_{\tau^*}, \mu_{\tau^*})$$

is a morphism of dga's. A standard spectral sequence argument shows that this is a weak equivalence. $\square$

7.3. Examples

Here we assume that the ground ring R is a field, and all spaces are path connected. We present examples based on the fact that for a space being a suspension the corresponding homotopy G-algebra structure is extremely simple: it consists just of $E_{1,1} = \smile_1$ and all other operations $E_{k>1,1}$ are trivial.

1. The classical Bott–Samelson theorem establishes that the inclusion $i : X \to \Omega SX$ induces an algebra isomorphism $i_* : T\tilde{H}_*(X) \overset{\approx}{\to} H_*(\Omega SX)$, where SX denotes a suspension on a space X. The left-hand side $T\tilde{H}_*(X)$ is a Hopf algebra with respect to the comultiplication which extends the one from $H_*(X)$ multiplicatively, and the Bott–Samelson map i_* is a Hopf algebra isomorphism too. There is the dual statement for the cohomology as well (cf. Appendix in [16]).

First we recover the above facts in the following way. Let Y be the suspension over a polyhedron X; explicitly, regard Y as the geometric realization of a quotient simplicial set $Y = SX/C_-X$ where $SX = C_+X \cup C_-X$, the union of two cones over X with the standard simplicial set structure. It is immediate to check by (10) that all $E^{k,1}$ for $k \geqslant 2$ are identically zero, and, moreover, so is the AW diagonal $\Delta\colon C_*(Y) \to C_*(Y) \otimes C_*(Y)$ in positive degrees as well (cf. [29]). Consequently, since of (14) and (17) $E^{1,1}: C_*(Y) \to C_*(Y) \otimes C_*(Y)$ becomes coassociative chain map of degree 1 and thus it induces a binary cooperation of degree 1 on the homology denoted by $Sq^{1,1}: H_*(Y) \to H_*(Y) \otimes H_*(Y)$. Notice that both $(C_*(Y), d, \bar{\Delta} = 0, E^{1,1})$ and $(H_*(Y), d = 0, \bar{\Delta}_* = 0, Sq^{1,1})$ are homotopy G-coalgebras, thus $\Omega C_*(Y)$ and $\Omega H_*(Y)$ both are dg Hopf algebras.

The cycle choosing homomorphism $\imath\colon H_*(Y) \to C_*(Y)$ is a dg coalgebra map in this case. Thus there is a dg algebra map $\Omega\imath\colon \Omega H_*(Y) \to \Omega C_*(Y)$ which induces the Bott-Samelson isomorphism of graded algebras

$$T\tilde{H}_*(X) = \Omega H_*(Y) = H(\Omega H_*(Y)) \xrightarrow{(\Omega\imath)_*} H_*(\Omega C_*(Y)) = H_*(\Omega Y). \tag{18}$$

To show that (18) is a Hopf algebra isomorphism, let first consider the diagram

$$\begin{array}{ccc}
C_*(Y) & \xrightarrow{E^{1,1}} & C_*(Y)\otimes C_*(Y) \\
\uparrow{\scriptstyle\imath} & & \uparrow{\scriptstyle\imath\otimes\imath} \\
\tilde H_*(Y) & \xrightarrow{Sq^{1,1}} & \tilde H_*(Y)\otimes \tilde H_*(Y) \\
\approx\uparrow{\scriptstyle s} & & \approx\uparrow{\scriptstyle s\otimes s} \\
\tilde H_{*-1}(X) & \xrightarrow{\Delta_*} & \tilde H_{*-1}(X)\otimes \tilde H_{*-1}(X),
\end{array}$$

where s is the suspension isomorphism; the upper square is commutative up to a chain homotopy, while the bottom square is strict commutative. This implies that $\Omega\imath$ is also a coalgebra map up to a chain homotopy, consequently (18) is a coalgebra map too.

2. Let $\Omega Y \to PY \xrightarrow{\pi} Y$ be the Moore path fibration with the base Y which is the suspension over a polyhedron X. Let $f: Y \to Z$ be a map, $\Omega Y \times \Omega Z \to \Omega Z$ be the induced action via the composition

$$\Omega Y \times \Omega Z \xrightarrow{\Omega f\times \mathrm{Id}} \Omega Z \times \Omega Z \to \Omega Z,$$

and $\Omega Z \to E_f \xrightarrow{\xi} Y$ be the associated fibration; for simplicity assume that Z is the suspension and simply connected CW-complex of finite type, as well. We present two multiplicative models for the fibration ξ using the cubical model $Y\times_\tau \mathbf{\Omega} Z$ with the universal truncating twisting function $\tau=\tau_{\mathrm{U}}: Y \to \mathbf{\Omega}Y$.

Notice that the twisted differential of the cochain complex $(C^*(Y\times_\tau\mathbf{\Omega}Z),d)=(C^*(Y)\otimes C^*(\mathbf{\Omega}Z),d_{\tau^\#})=(C^*(Y)\otimes BC^*(Z),d_{\tau^\#})$ with universal $\tau^\#: BC^*(Y)\to C^*(Y)$ becomes the form

$$\begin{aligned} d_{\tau^\#}(a\otimes[\bar m^1|\dots|\bar m^n]) &= da\otimes[\bar m^1|\dots|\bar m^n]+\sum_{k=1}^{n} a\otimes[\bar m^1|\dots|d\bar m^k|\dots|\bar m^n] \\ &\quad + a\cdot m_1\otimes[\bar m^2|\dots|\bar m^n]. \end{aligned}$$

Since the simplified structure of the homotopy G-algebra $(C^*(Y),d,\mu=0,E_{1,1})$ formula (12) becomes the following form:

$$\begin{aligned} &\mu_{\tau^\#}((a_1\otimes m_1)(a_2\otimes m_2)) \\ &\quad = a_1a_2\otimes m_1m_2 + a_1E_{1,1}(f^\#(m_1^1),a_2)\otimes[\bar m_1^2|\dots|\bar m_1^n]\cdot m_2, \end{aligned} \qquad (19)$$

where $f^\#: C^*(Z)\to C^*(Y),\ a_1,a_2\in C^*(Y),\ m_1=[\bar m_1^1|\dots|\bar m_1^n], m_2\in BC^*(Z),\ n\geqslant 0$. Note that since the product on $C^{>0}(Y)$ is zero, the twisted part of $\mu_{\tau^\#}$ (the second summand) may be non-zero only for $a_1\in C^0(Y)$.

So that we get that $H(C^*(Y)\otimes BC^*(Z),d_{\tau^\#},\mu_{\tau^\#})$ and $H^*(E_f)$ are isomorphic as algebras.

On the other hand, let us consider the following multiplicative twisted tensor product $(H^*(Y)\otimes H^*(\mathbf{\Omega}Z), d_{\tau^*})=(H^*(Y)\otimes BH^*(Z), d_{\tau^*})$ with universal $\tau^*: BH^*(Y) \to H^*(Y)$. The differential here is of the form:

$$d_{\tau^*}(a \otimes [\bar{m}^1|\dots|\bar{m}^n]) = a \cdot m_1 \otimes [\bar{m}^2|\dots|\bar{m}^n].$$

Again since the simplified structure of the homotopy G-algebra $(H^*(Y), d{=}0, \mu^*{=}0, Sq_{1,1})$ formula (12) becomes the following form:

$$\begin{aligned}&\mu_{\tau^*}((a_1 \otimes m_1)(a_2 \otimes m_2))\\ &\quad = a_1a_2 \otimes m_1m_2 + a_1Sq_{1,1}(f^*(m_1^1), a_2) \otimes [\bar{m}_1^2|\dots|\bar{m}_1^n] \cdot m_2,\end{aligned} \tag{20}$$

where $f^*: H^*(Z) \to H^*(Y)$, $a_1, a_2 \in H^*(Y)$, $m_1{=}[\bar{m}_1^1|\dots|\bar{m}_1^n]$, $m_2 \in BH^*(Z)$, $n \geqslant 0$. Note that since the product on $H^{>0}(Y)$ is zero, the twisted part of μ_{τ^*} (the second summand) may be non-zero only for $a_1 \in H^0(Y)$. Also we remark that for an element $a \in H^*(Y)$, one gets $Sq_{1,1}(a, a) = Sq_1(a)$, the Steenrod square.

We claim that $(H^*(Y) \otimes BH^*(Z), d_{\tau^*})$ is a "small" multiplicative model of the fibration ξ, i.e $H(H^*(Y) \otimes BH^*(Z), d_{\tau^*})$ and $H^*(E_f)$ are isomorphic as algebras. Indeed, it is straightforward to calculate (or using the standard spectral sequence argument) that additively

$$\begin{aligned}&H(C^*(Y) \otimes BC^*(Z), d_{\tau^\#})\\ &\quad \approx H(H^*(Y) \otimes BH^*(Z), d_{\tau^*})\\ &\quad \approx H^0(Y) \otimes T_f(H^*(Z)) \oplus H^*(Y)/Imf^* \otimes BH^*(Z),\end{aligned}$$

where $T_f(H^*(Z)) = s^{-1}(Kerf^*) + s^{-1}(Kerf^*) \otimes s^{-1}H^*(Z) + \dots + s^{-1}(Kerf^*) \otimes (s^{-1}H^*(Z))^{\otimes n} + \cdots$, $n \geqslant 1$. Since the explicit formulas (19) and (20) it is easy to calculate that the twisted parts of $\mu_{\tau^\#}$ and μ_{τ^*} annihilate in homology, thus they induce the same multiplication on $H^*(E_f)$. As a byproduct we obtain that the multiplicative structure of the total space E_f does not depend on a map f in a sense that if $f^*{=}g^*$ then $H^*(E_f){=}H^*(E_g)$ as algebras. Note also that this multiplicative structure is purely defined by the $\smile$ and $\smile_1$ operations.

References

[1] J.F. Adams, On the cobar construction, Proc. Nat. Acad. Sci. USA 42 (1956) 409–412.

[2] H.-J. Baues, Geometry of loop spaces and the cobar construction, Mem. Amer. Math. Soc. 25 (1980) 1–170.

[3] H.-J. Baues, The double bar and cobar construction, Compositio Math. 43 (1981) 331–341.

[4] H.-J. Baues, The cobar construction as a Hopf algebra, Invent. Math. 132 (1998) 467–489.

[5] N. Berikashvili, An algebraic model of fibration with the fiber $K(\pi, n)$-space, Georgian Math. J. 3 (1) (1996) 27–48.

[6] N. Berikashvili, D. Makalatia, The multiplicative version of twisted tensor product theorem, Bull. Georgian Acad. Sci. 154 (1996) 327–329.

[7] N. Berikashvili, On the differentials of spectral sequences (Russian), Proc. Tbilisi Mat. Inst. 51 (1976) 1–105.

[8] E. Brown, Twisted tensor products, Ann. of Math. 69 (1959) 223–246.

[9] G. Carlsson, R.J. Milgram, Stable homotopy and iterated loop spaces, in: I.M. James (Ed.), Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, pp. 505–583.

[10] Y. Felix, S. Halperin, J.-C. Thomas, Adams' cobar equivalence, Trans. Amer. Math. Soc. 329 (1992) 531–549.
[11] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.
[12] M. Gerstenhaber, A.A. Voronov, Higher operations on the Hochschild complex, Funct. Anal. Appl. 29 (1995) 1–5.
[13] E. Getzler, J. D. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint, 1995.
[14] V.K.A.M. Gugenheim, On the chain complex of a fibration, Ill. J. Math. 16 (1972) 398–414.
[15] J. Huebschmann, The homotopy type $F\Psi^q$, the complex and sympletic cases, Contemp. Math. 55 (1986) 487–518.
[16] D. Husemoller, Fibre bundles, 3rd Edition, Graduate Texts in Mathematics, Vol. 20, Springer, New York, 1994.
[17] D.M. Kan, Abstract homotopy I, Proc. Nat. Acad. Sci. USA 41 (1955) 1092–1096.
[18] T. Kadeishvili, DG Hopf Algebras with Steenrods *i*th coproducts, Bull. Georgian Acad. Sci. 158 (1998) 203–206.
[19] T. Kadeishvili, Cochain operations defining Steenrod $\smile_i$-products in the bar construction, Georgian Math. J. 10 (2003) 115–125.
[20] T. Kadeishvili, The $A(\infty)$-algebra structure and cohomology of Hochschild and Harrison (Russian), Proc. Tbilisi Math. Inst. 91 (1988) 19–27.
[21] T. Kadeishvili, Measuring the noncommutativty of DG algebras, Journal of Mathematical Sciences 119 (2004) 494–512.
[22] T. Kadeishvili, S. Saneblidze, The twisted Cartesian model for the double path space fibration, preprint, AT/0210224.
[23] L. Lambe, J. Stasheff, Applications of perturbation theory to iterated fibrations, Manuscripta Math. 58 (1987) 363–376.
[24] W.S. Massey, A Basic Course in Algebraic Topology, Springer, New York, 1991.
[25] R.J. Milgram, Iterated loop spaces, Ann. of Math. 84 (1966) 386–403.
[26] J. Milnor, J.C. Moore, On the structure of Hopf Algebras, Ann. of Math. 81 (1965) 211–264.
[27] A. Proute, A_∞-structures, modele minimal de Bauess–Lemaire des fibrations, preprint.
[28] S. Saneblidze, Perturbation and obstruction theories in the fibre spaces, Proc. A. Razmadze Math. Inst. 111 (1994) 1–106.
[29] S. Saneblidze, The Hochschild complex of a space is the complex of the Hochschild set, preprint.
[30] J.-P. Serre, Homologie singuliere des espaces fibres, applications, Ann. Math. 54 (1951) 429–505.
[31] R.H. Szczarba, The homology of twisted cartesian products, Trans. Amer. Math. Soc. 100 (1961) 197–216.