Stable Clifford extensions of modules

Ziqun Lu

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Received 9 September 2005
Available online 17 April 2006
Communicated by Jan Saxl

Let $G$ be a finite group. Let $(K, O, F)$ be a $p$-modular system, where $O$ is a complete discrete valuation ring, $K$ is the quotient field of $R$ of characteristic 0, and $F$ is the residual field of $O$ of characteristic $p > 0$. We assume that $K$ and $F$ are big enough. Let $R$ be $O$ or $F$. In this paper, a module means a finitely generated right module. For a subgroup $H$ of $G$, and for an $RG$-module $X$ and an $RH$-module $Y$, we write $X_H$ for the restriction of $X$ to $H$ and $Y^G$ for the induction of $Y$ to $G$. When $H \triangleleft G$ and $Y$ is an $RH$-module, we denote $I_G(Y)$ the inertia subgroup of $Y$ in $G$.

1. Clifford extensions of indecomposable modules

Let $H$ be a normal subgroup of $G$, and let $W$ be an indecomposable $FH$-module. Assume that $I_G(W) = G$. Set $E = \text{End}_{RG}(W^G)$ and $\Lambda = \text{End}_{RH}(W)$. We can write $E$ in the form $E = \sum_{x \in X} E_x$ where $X = G/H$ and $E_x$ is the $R$-submodule of $E$ mapping $W = W \otimes 1$ to $W \otimes x$ inside $W^G$, and $E_x \cong \text{Hom}_{RH}(W, Wx)$ (as $R$-module) by [3, Chapter 4, Lemma 6.4]. Clearly $E_{\tilde{x}}E_{\tilde{y}} \subset E_{\tilde{x} \tilde{y}}$, for $\tilde{x}, \tilde{y} \in X$. Also we can use the stability hypothesis to choose an element $\varphi_{\tilde{x}} \in E_{\tilde{x}}$ mapping $W \otimes 1$ isomorphically onto $W \otimes x$; it follows that $\varphi_{\tilde{x}}$ is a unit in $E$. Since $E_{\tilde{x}}$ can be identified with $\Lambda$, we have $E_{\tilde{x}} = \Lambda \varphi_{\tilde{x}} = \varphi_{\tilde{x}} \Lambda$, so $E$ is a free right $\Lambda$-module. The module $E \otimes_{\Lambda} W$ is an $E$-$RG$-bimodule with action $(e \otimes w) \cdot y = e \varphi_{\tilde{x}} \otimes \varphi_{\tilde{y}}^{-1}(w \otimes y)$, where $\tilde{y} = yH$, and $e' \cdot (e \otimes w) = e'e \otimes w$. Then we have the following proposition due to Cline.

Proposition 1. [1] There is an $E$-$RG$-bimodule isomorphism

$$E \otimes_{\Lambda} W \cong W^G$$

given by $f : e \otimes w \mapsto e(w)$, for $e \in E$ and $w \in W$. 

0021-8693/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2006.03.004
By the isomorphism in Proposition 1, the indecomposable direct summands of $W^G$ can be given by the indecomposable direct summands of $E$.

**Proposition 2.** [1, Corollary 3.15] *Keep the notation as above. Let $E = \sum \bigoplus U_i$ be a decomposition into indecomposable right $E$-modules. Then $V^G = \sum \bigoplus U_i W \cong \sum \bigoplus U_i \otimes_A W$ is a decomposition into indecomposable right $RG$-modules. Moreover, we have that dim$(U_i W) = \text{rank}_A(U_i) \dim(W)$, and that $U_i \cong U_j$ as $E$-modules if and only if $U_i W \cong U_j W$ as $RG$-module.*

Let $W$ be a $G$-invariant indecomposable $RH$-module with vertex $Q$. Let $W'$ be the Green correspondent of $W$ with respect to $(H, Q, NH(Q))$. Then $G = HN_G(Q)$ (see Theorem 3), and $W'$ is $N_G(Q)$-invariant. Thus $W^G \cong E \otimes_A W$ as right $RG$-modules by Proposition 1. Set $E' = \text{End}_{RG(Q)}((W')^{NG(Q)})$, and $A' = \text{End}_{RG(Q)}(W')$. Then $E' \otimes_{A'} W' \cong (W')^{NG(Q)}$ as right $RN_G(Q)$-modules. It is well known that $E' \otimes_{A'} W'$ is an algebra. It can be easily derived from [3, Chapter 4, Lemma 6.4] that $\dim(E' \otimes_{A'} W') = \dim(E \otimes_A W)$. Moreover, we have that $E/J(\Lambda)E$ (respectively $E'/J(\Lambda')E'$) is a twisted group algebra.

We now come to the main result. We are grateful to the referee for pointing out that this already appears without proof in Cline’s paper [2].

**Theorem 3.** $G = HN_G(Q)$. Under the canonical isomorphism $G/H \cong N_G(Q)/NH(Q)$ (x $H \mapsto xNH(Q)$, $x \in N_G(Q)$), we have $E/J(\Lambda)E \cong E'/J(\Lambda')E'$ as twisted group algebras.

**Proof.** Since $W$ is $G$-invariant, $Q^x$ is also a vertex of $W$ for any $x \in G$. But $Q^x$ and $Q$ are conjugate in $H$, so $x \in HN_G(Q)$. Thus $G = HN_G(Q)$. For convenience, we set $U = N_G(Q)$, $V = NH(Q)$, and fix a set of right coset representatives $H \setminus G = \{ x_1, \ldots, x_n \}$ of $H$ in $G$ with $x_i \in N_G(Q) = U$ ($i = 1, 2, \ldots, n$). Then $\{ x_1, \ldots, x_n \}$ is a set of right coset representatives of $V$ in $U$. We have $E = \sum x_i \oplus E_{\bar{x}_i}$ (respectively $E' = \sum x_i \oplus E'_{\bar{x}_i}$), where $E_{\bar{x}_i} \cong \text{Hom}_{RG}(W, W)$ (respectively $E'_{\bar{x}_i} \cong \text{Hom}_{RV}(W', W')$), and $E_{\bar{x}_i} \cong \text{Hom}_{RH}(W, Wx_{\bar{i}})$ (respectively $E'_{\bar{x}_i} \cong \text{Hom}_{RV}(W', W'x_{\bar{i}})$). Thus $E_{\bar{x}_i} \cong \text{Hom}_{RV}(W, Wx_{\bar{i}})$ to be the composition of $f$ with $\bar{h}$, where $\bar{h}$ is defined by

$$\bar{h} : W_{x_{\bar{i}}} \longrightarrow W_{x_{\bar{i}}},$$

$$w_{x_{\bar{i}}} \longmapsto h(w)x_{\bar{i}}.$$

Under the above multiplication, $\sum x_i \oplus \text{Hom}_{RH}(W, Wx_{\bar{i}})$ is an algebra. It can be easily derived from [3, Chapter 4, Lemma 6.4] that $\sum x_i \oplus \text{Hom}_{RH}(W, Wx_{\bar{i}})$ is isomorphic to $E$ as algebras. Thus we identify $E$ with $\sum x_i \oplus \text{Hom}_{RH}(W, Wx_{\bar{i}})$, and by the same way we identify $E'$ with $\sum x_i \oplus \text{Hom}_{RV}(W', W'x_{\bar{i}})$.

Recall that $W'$ is the Green correspondent of $W$ with respect to $(H, Q, V)$. Let $x \in U$. Then it is easy to see that $W'x$ is the Green correspondent of $Wx$ with respect to $(H, Q, V)$. Now we will define a map from $E'$ to $E$. For $x \in U$, first define a trace map from $\text{Hom}_{RV}(W', W'x)$ to $\text{Hom}_{RH}(W', W'x)/\text{Tr}_{Q}(W', W'x)$ by

$$E' \longrightarrow E,$$

$$E' \longmapsto \text{Tr}_{Q}(W', W'x).$$
\[ \text{Tr} : \text{Hom}_{RV}(W', W'x) \mapsto \text{Hom}_{RH}(W'^H, (W'x)^H), \]

\[ f \mapsto \text{Tr}(f) : \sum_{h \in V \setminus H} w'_h \otimes h \mapsto \sum_{h \in V \setminus H} f(w'_h) \otimes h. \]

Let \( f \in \text{Hom}_{RV}(W', W'x) \) and \( h \in \text{Hom}_{RV}(W'x, W'y) \). It is easy to check that \( \text{Tr}(h \cdot f) = \text{Tr}(h) \cdot \text{Tr}(f) \). Assume that \( (W')^H = W \oplus W_1 \) and \( (W'x)^H = Wx \oplus W_2 \) as \( RH \)-modules, respectively. Let \( \iota_W : W \to (W')^H \) and \( \pi_{Wx} : (W'x)^H \to Wx \) be the inclusion map and projection map.

Then there is a map

\[ \alpha : \text{Hom}_{RV}(W', W'x) \to \text{Hom}_{RH}(W, Wx), \]

\[ f \mapsto \pi_{Wx} \cdot \text{Tr}(f) \cdot \iota_W. \]

By the above identification, \( \alpha \) is a map from \( E' \) to \( E \). By [3, Chapter 4, Theorem 5.4], \( \alpha \) is a one-to-one map from \( E'/I'E' \) to \( E/I'E \). By also [3, Chapter 4, Theorem 5.4] and its proof, we have that \( \alpha \) is an algebra homomorphism from \( E'/I'E' \) to \( E/I'E \). Since both \( W \) and \( W' \) are not \( \Omega \)-projective, \( I \subseteq J(\Lambda) \) and \( I' \subseteq J(\Lambda') \). Recall that \( \alpha \) induces an algebra isomorphism from \( \Lambda'/I' \) to \( \Lambda/I \). Thus \( \alpha \) induces a surjective homomorphism from \( \Lambda'/I' \) to \( \Lambda/J(\Lambda) \), and an isomorphism from \( \Lambda'/J(\Lambda') \) to \( \Lambda/J(\Lambda) \). So \( \alpha \) sends \( J(\Lambda') \) to \( J(\Lambda) \). As \( \alpha \) is an isomorphism from \( E'/I'E' \) to \( E/I'E \), we have that \( \alpha \) is an isomorphism from \( E'/J(\Lambda')E' \) to \( E/J(\Lambda)E \). As desired. \( \square \)

**Corollary 4.** Keep notation as in Theorem 3. Then \( W \) can be extended to an \( H \)-projective \( RG \)-module if and only if \( W' \) can be extended to a \( V \)-projective \( RU \)-module.

**Proof.** Let \( L \) be an indecomposable direct summand of \( W^G \). Then by Proposition 2, \( L \cong U_i \otimes A W \) as \( RG \)-modules for some indecomposable direct summand \( U_i \) of \( E \), and \( \text{rank}_R(L) = \text{rank}_R(W) \cdot \text{rank}_A(U_i) = \text{rank}_R(U_i / J(\Lambda)U_i) \). Note that \( U_i / J(\Lambda)U_i \) is an indecomposable direct summand of \( E/J(\Lambda)E \). Thus \( W \) can be extended to an \( H \)-projective \( RG \)-module if and only if the twisted group algebra \( E/J(\Lambda)E \) has a principal indecomposable module of dimension 1. By Theorem 3, \( E/J(\Lambda)E \) has a principal indecomposable module of dimension 1 if and only if \( E'/J(\Lambda')E' \), if and only if \( W' \) can be extended to a \( V \)-projective \( RU \)-module. As desired. \( \square \)

**References**

