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Minimal $D = 4$ supergravity from the superMaxwell algebra

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Abstract

We show that the first-order $D = 4$, $N = 1$ pure supergravity lagrangian four-form can be obtained geometrically as a quadratic expression in the curvatures of the Maxwell superalgebra. This is achieved by noticing that the relative coefficient between the two terms of the lagrangian that makes the action locally supersymmetric also determines trivial field equations for the gauge fields associated with the extra generators of the Maxwell superalgebra. Along the way, a convenient geometric procedure to check the local supersymmetry of a class of lagrangians is developed.

1. Introduction

Since the advent of supersymmetry, there has been an interest in superalgebras going beyond the standard superPoincaré one. For instance, enlarged $D = 11$ supersymmetry algebras were considered by D'Auria and Fré in [1] and further in [2] in a search for the group structure underlying $D = 11$ supergravity [3], which is hidden due to the presence of the three-form that needs being trivialized as a product of one-forms to be associated with Maurer–Cartan (MC) forms. The resulting superalgebras go beyond the $D = 11$ superPoincaré algebra and contain additional fermionic generators and tensorial charges. Larger supersymmetry algebras (and correspondingly enlarged superspaces), also appear associated with super-p-branes, where

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the tensorial charges are realized as topological charges [4] (see [5] for the M5-brane and further [6]). The 560-dimensional $D = 11$ superlagebra includes 528 vector and tensorial charges and is usually referred to as the M-theory superalgebra [7]; fermionic extensions of the superPoincaré algebra, similar to those of D'Auria and Fré but without tensorial charges, were introduced by Green in [8] by adding an extra spinorial generator. The Green algebra was used by Siegel [9] to produce a superstring action with a manifestly supersymmetric Wess–Zumino term, a procedure further generalized by Bergshoeff and Sezgin to super-p-branes by introducing larger Green-type superalgebras [10] (see also [11]). These algebras can be viewed as the result of successive extensions of the supertranslations algebra [6], and the associated enlarged superspace group manifolds may be used to construct strictly invariant (rather than invariant up to total derivative) Wess–Zumino terms for general p-branes, as discussed in detail in [6].

In a separate context, Hatsuda and Sakaguchi showed that there is a suitable flat limit for the *AdS* superstring that leads to bilinear WZ terms and to enlarged Poincaré superalgebras [12]. They interpreted these [13] as what now are termed (super)algebra expansions, which were studied in detail in [14] (see also [15] for an expansion-related procedure). Expansions are obtained from the original algebras by means of a series expansion (hence their name [14]) of their MC one-forms. As a result, the expansion procedure leads to algebras of higher dimension than the original one; nevertheless, the dimension preserving İnönü–Wigner contraction (and its Weimar-Woods generalization [16]) are obtained as particular cases [14]. Expansions were shown to lead to the full (*i.e.* including the $D = 11$ Lorentz algebra) M-theory superalgebra, which is a particular expansion of $osp(1|32)$ [14]. The (p, q)-superPoincaré algebras [17] governing the $D = 3$ extended supergravities have also been shown [18] to be related to particular expansions of $osp(p + q|2)$.

Recently, a $D = 4$ Maxwell superalgebra has been introduced in [19] as the minimal superalgebra that contains the Maxwell algebra as its bosonic subalgebra (see [20] for Maxwell algebras). This Maxwell superalgebra can be viewed as an extension of the $D = 4$ Green algebra by the tensorial charges algebra, and it was used in [19] to construct a superparticle model. But the Maxwell algebra is also an expansion of the $D = 4$ *adS* algebra $o(3, 2)$ [21]. The minimal Maxwell superalgebra $s\mathcal{M}$ to be considered here also follows from an expansion of the $D = 4$ *adS* superalgebra $osp(N|4)$ (further N-extended superMaxwell algebras are also described in [21] using the expansion method).

The geometric approach to supergravity has a long history (see *e.g.* [22] and references therein). In this paper we wish to show that the minimal $D = 4$ Maxwell superalgebra $s\mathcal{M}$ may be used to derive the action of the basic (or minimal) $N = 1$, $D = 4$ supergravity from the curvature forms of the Cartan structure equations associated with $s\mathcal{M}$. This extends to the supersymmetric case the $D = 4$ gravity results obtained from the bosonic Maxwell algebra [23] (the $D = 1 + 2$ gravity case has been considered very recently in [24]). To this aim, we first briefly review the Maxwell algebra and its relation to ordinary gravity. In Section 3, a family of lagrangian forms depending on a parameter k will be constructed geometrically in terms of curvatures associated with the $D = 4$ Maxwell superalgebra $s\mathcal{M}$. To show that a value of k provides the lagrangian of $D = 4$ minimal supergravity we present first in Section 4, for a generic algebra, a procedure to analyze the local invariance of a class of $D = 4$ lagrangians that includes the Chern–Simons lagrangians as a particular case. Section 5 applies the method to the $s\mathcal{M}$ superalgebra and to supergravity. The final section contains some comments.

2. Maxwell algebra and the gravity action

The $D = 4$ Maxwell algebra $\mathcal{M}$ is given by the following commutators:

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}, \\
&[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
&[P_a, P_b] = Z_{ab}, \\
&[M_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}
\end{aligned}
\tag{2.1}
$$

($a = 0, \dots, 3$), where η_{ab} is the (mostly plus) Minkowski metric. Besides the Poincaré generators, the Maxwell algebra contains six additional tensorial charges Z_{ab} that extend centrally the abelian translation algebra and that behave tensorially under the Lorentz algebra $\mathcal{L}$.

It is convenient to describe this algebra through its MC equations satisfied by the one-forms ω^{ab}, e^a, f^{ab} dual to the generators M_{ab}, P_a, Z_{ab},

$$
\omega^{ab}(M_{cd}) = \delta^{ab}_{cd}, \qquad e^a(P_b) = \delta^a_b, \qquad f^{ab}(Z_{cd}) = \delta^{ab}_{cd}. \tag{2.2}
$$

They are given by

$$
\begin{aligned}
&0 = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \\
&0 = de^a + \omega^a{}_b \wedge e^b, \\
&0 = df^{ab} + e^a \wedge e^b + \omega^a{}_c \wedge f^{cb} - \omega^b{}_c \wedge f^{ca}.
\end{aligned}
\tag{2.3}
$$

The ‘soft’ version of these MC equations introduce the gauge curvatures R^{ab}, T^a and F^{ab} in terms of the gauge field forms. Using without risk of confusion the same notation for the MC one-forms and the gauge field ones, the Cartan structure equations $\Omega = d\theta + \theta \wedge \theta = d\theta + \frac{1}{2}[\theta, \theta]$ where $\theta = e^a P_a + \frac{1}{2}\omega^{ab} M_{ab} + \frac{1}{2} f^{ab} Z_{ab}$ determine the various curvatures. Writing $\Omega = \frac{1}{2} R^{ab} M_{ab} + T^a P_a + \frac{1}{2} F^{ab} Z_{ab}$, they are found to be

$$
\begin{aligned}
&R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \\
&T^a = de^a + \omega^a{}_b \wedge e^b, \\
&F^{ab} = df^{ab} + e^a \wedge e^b + \omega^a{}_c \wedge f^{cb} - \omega^b{}_c \wedge f^{ca}.
\end{aligned}
\tag{2.4}
$$

The Lorentz covariant differentials of the curvatures $DR = dR + [\omega, R]$, $DT = dT + [\omega, T] = [R, e]$, $DF = dF + [\omega, F] = [R, f] + [T, e]$ are then

$$
\begin{aligned}
&DR^{ab} = (dR + \omega \wedge R - R \wedge \omega)^{ab} = 0, \\
&DT^a = R^{ab} \wedge e_b, \\
&DF^{ab} = R^a{}_c \wedge f^{cb} - R^b{}_c \wedge f^{ca} + T^a \wedge e^b - e^a \wedge T^b.
\end{aligned}
\tag{2.5}
$$

The following Lorentz invariant lagrangian four-form constructed from the Maxwell curvatures (with length dimensions of an action in $D = 4$ (L^2) in geometrized $\kappa = 1 = c$) units, was considered in [23]

$$
\mathcal{L} \sim \epsilon_{abcd} R^{ab} \wedge F^{cd}; \tag{2.6}
$$

other possibilities were also discussed there. Since the extra field f_{ab} appears in an exterior differential this lagrangian leads, up to boundary terms that will be disregarded here, to the standard Einstein–Hilbert action for gravity,

$$
\int_M \mathcal{L}_{EH} \sim \int_M \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d. \tag{2.7}
$$

Thus, since the gauging of the Maxwell group provides a geometric framework to derive the gravity lagrangian, it is natural to ask [23] whether a minimal supersymmetrization of the Maxwell algebra may lead to the pure gravity lagrangian. Our aim is to show that this is the case.

3. Maxwell superalgebra and geometric ingredients of minimal supergravity

Pure, simple $D=4$ supergravity just includes the graviton and the gravitino, with two on-shell degrees of freedom each. To express its lagrangian in terms of curvature bilinears we consider the 24-dimensional minimal superMaxwell algebra $s\mathcal{M}$ [19]. It contains the 16-dimensional Maxwell algebra (2.1) as its even subalgebra, and the brackets involving the odd generators are

$$\begin{aligned}
&[M_{ab}, Q_\alpha] = \frac{1}{2}\gamma_{ab}{}^\beta{}_\alpha Q_\beta, \\
&[M_{ab}, \Sigma_\alpha] = \frac{1}{2}\gamma_{ab}{}^\beta{}_\alpha \Sigma_\beta, \\
&\{Q_\alpha, Q_\beta\} = \gamma^a{}_{\alpha\beta} P_a, \\
&[P_a, Q_\alpha] = \frac{1}{2}\gamma_a{}^\alpha{}_\beta \Sigma_\beta, \\
&\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}\gamma^{ab}{}_{\alpha\beta} Z_{ab},
\end{aligned} \tag{3.8}$$

where $Q_\alpha, \alpha = 1, \ldots, 4$, is the supersymmetry generator ($[Q] = L^{-1/2}$) and, as in the Green algebra, the $[P, Q]$ commutator produces an additional spinor generator Σ_α, $[\Sigma] = L^{-3/2}$. All spinors above and below are Majorana spinors.

The dual MC one-forms of $s\mathcal{M}$ are defined by the duality conditions (2.2) plus

$$\psi^\alpha(Q_\beta) = \delta^\alpha_\beta = \xi^\alpha(\Sigma_\beta), \tag{3.9}$$

and their MC equations, $0 = d\theta + \theta \wedge \theta$, where now $\theta = e^a P_a + \frac{1}{2}\omega^{ab} M_{ab} + \frac{1}{2} f^{ab} Z_{ab} + \psi^\alpha Q_\alpha + \xi^\alpha \Sigma_\alpha$, are given by

$$\begin{aligned}
&0 = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \\
&0 = de^a + \omega^a{}_b \wedge e^b - \frac{1}{2}\bar\psi\gamma_a \wedge \psi, \\
&0 = df^{ab} + \omega^a{}_c \wedge f^{cb} - \omega^b{}_c \wedge f^{ca} + \bar\xi\gamma^{ab} \wedge \psi + e^a \wedge e^b, \\
&0 = d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab} \wedge \psi, \\
&0 = d\xi + \frac{1}{4}\omega_{ab}\gamma^{ab} \wedge \xi + \frac{1}{2}e_a\gamma^a \wedge \psi.
\end{aligned} \tag{3.10}$$

We use $(\lambda\lambda')^* = \lambda^*\lambda'^*$ for the conjugation of bilinears of odd scalars, so that both ω_{ab} and e_a are real. The gauge curvatures, again using the same notation for the gauge field one-forms and for the MC ones, are given by the structure equations $\Omega = d\theta + \theta \wedge \theta$, where $\Omega = \frac{1}{2}R^{ab}M_{ab} + T^a P_a + \frac{1}{2}F^{ab}Z_{ab} + \Psi^\alpha Q_\alpha + \xi^\alpha \Sigma_\alpha$. Explicitly,

$$\begin{aligned}
&R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \\
&T^a = de^a + \omega^a{}_b \wedge e^b - \frac{1}{2}\bar\psi\gamma_a \wedge \psi, \\
&F^{ab} = df^{ab} + \omega^a{}_c \wedge f^{cb} - \omega^b{}_c \wedge f^{ca} + \bar\xi\gamma^{ab} \wedge \psi + e^a \wedge e^b,
\end{aligned}$$

$$\Psi^\alpha = d\psi^\alpha + \frac{1}{4}\omega_{ab}\big(\gamma^{ab} \wedge \psi\big)^\alpha,$$
$$\Xi^\alpha = d\xi^\alpha + \frac{1}{4}\omega_{ab}\big(\gamma^{ab} \wedge \xi\big)^\alpha + \frac{1}{2}e_a\big(\gamma^a \wedge \psi\big)^\alpha. \tag{3.11}$$

These curvatures have dimensions $[R] = L^0$, $[\Psi] = L^{1/2}$, $[T] = L$, $[\Xi] = L^{3/2}$, $[F] = L^2$.

The Lorentz covariant exterior differential of the curvatures is given by $DR = dR + [\omega, R] = 0$, $DT = dT + [\omega, T] = [R, e] + [\Psi, \psi]$, $DF = dF + [\omega, F] = [R, f] + [T, e] + [\Psi, \xi] + [\Xi, \psi]$, plus $D\Psi = d\Psi + [\omega, \psi]$ and $D\Xi = d\Xi + [\omega, \Xi] + [T, \psi] + [\Psi, e]$. Explicitly,

$$DR^{ab} = (dR + \omega \wedge R - R \wedge \omega)^{ab} = 0,$$
$$DT^a = R^{ab} \wedge e_b + \bar{\psi}\gamma_a \wedge \Psi,$$
$$DF^{ab} = R^a{}_c \wedge f^{cb} - R^b{}_c \wedge f^{ca} + T^a \wedge e^b - e^a \wedge T^b + \bar{\Xi}\gamma^{ab} \wedge \psi - \bar{\xi}\gamma^{ab} \wedge \Psi,$$
$$D\Psi^\alpha = \frac{1}{4}\big(R_{ab}\gamma^{ab} \wedge \psi\big)^\alpha,$$
$$D\Xi^\alpha = \frac{1}{4}\big(R_{ab}\gamma^{ab} \wedge \xi\big)^\alpha + \frac{1}{2}T_a\big(\gamma^a \wedge \psi\big)^\alpha - \frac{1}{2}e_a\big(\gamma^a \wedge \Psi\big)^\alpha. \tag{3.12}$$

To show that $D = 4$ minimal supergravity can be written in terms of the above curvatures, consider lagrangian four-forms B, $[B] = L^2$ given by linear combinations of the type

$$B = \epsilon_{abcd} R^{ab} \wedge F^{cd} + k\bar{\Xi}\gamma_5 \wedge \Psi, \tag{3.13}$$

where k is a constant to be determined and $\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma_5^2 = -1$, $\gamma_5\gamma^{abcd} = \epsilon^{abcd}$ with $\gamma^{abcd} = 1/4!\gamma^{[a}\gamma^b\gamma^c\gamma^{d]}$. It will turn out that there is a value of k for which the field equations for f^{ab} and ξ^α are trivial $(0 = 0)$. For this value of k, the resulting action becomes the well-known action of minimal $D = 4$ supergravity, given by Eq. (2.7) plus the Rarita–Schwinger (R-S) action for the lagrangian

$$\mathcal{L}_{RS} \sim \bar{\psi} \wedge \gamma_5\gamma_a e^a \wedge D\psi = \frac{1}{3!}\epsilon_{abcd}\bar{\psi} \wedge \gamma^{abc} D\psi \wedge e^d. \tag{3.14}$$

To see that this is the case, let us first discuss for a generic superalgebra the problem of local invariance of a class of lagrangians depending on its gauge fields and their curvatures. This includes as a particular case those depending only on the curvatures (as (3.13)).

4. Geometry of local invariance and the field equations

Let us introduce here a geometric procedure to discuss the local invariance of a class of lagrangians. Let H be a form that is a combination of exterior products of the gauge forms A^i associated with a generic Lie superalgebra and of their curvatures, as defined by the Cartan structure equations $F^i = dA^i + \frac{1}{2}C_{jk}{}^i A^j \wedge A_k$. Thus, in general, $H = H(A, F)$. Let us now introduce two inner derivations i_{F^i} and i_{A^i} of degree -2 and -1, respectively, defined by $i_{F^i} A^j = 0$, $i_{F^i} F^j = \delta_i^j$ and $i_{A^i} F^j = 0$, $i_{A^i} A^j = \delta_i^j$. Since $dF^i = C^i{}_{jk} F^j \wedge A^k$, the exterior differential d may be expressed as

$$d = C^i{}_{jk} F^j \wedge A^k i_{F^i} - \frac{1}{2}C^i{}_{jk} A^j \wedge A^k i_{A^i} + F^i i_{A^i}. \tag{4.15}$$

Then, the commutator $[d, i_{F^i}]$ is given by

$$d i_{F^i} - i_{F^i} d = -C^j{}_{ik} A^k i_{F^j} - i_{A^i}. \tag{4.16}$$

Now, let B be a lagrangian form $B = B(A, F)$ which is a potential form of H, $H = dB$. Then, it follows that the A^i field equation is given simply by $i_{F^i} H = 0$. Indeed, let us first compute the variation of $\int_M B$, where M stands for Minkowski space:

$$\begin{aligned}
\delta \int_M B &= \int_M \left\{\delta A^i \wedge i_{A^i} B + \delta F^i \wedge i_{F^i} B\right\} \\
&= \int_M \left\{\delta A^i \wedge i_{A^i} B + \left(d\delta A^i + C^i{}_{jk} \delta A^j \wedge A^k\right) \wedge i_{F^i} B\right\} \\
&= \int_M \delta A^i \wedge \left\{i_{A^i} B + d i_{F^i} B - C^l{}_{ji} A^j \wedge i_{F^l} B\right\} \\
&= \int_M \delta A^i \wedge i_{F^i} H, \qquad (4.17)
\end{aligned}$$

where we have integrated by parts the second term in the second equality above and used Eq. (4.16) for $d i_{F^i} B$ in the third one. Thus, the A^i field equation is

$$i_{F^i} H \equiv E_i = 0. \qquad (4.18)$$

We note that the differential $d(i_{F^i} H)$ of the *l.h.s.* of the A^i field equation three-form is, since $dH \equiv 0$,

$$d(i_{F^i} H) = -C^j{}_{ik} A^k i_{F^j} H - i_{A^i} H, \qquad (4.19)$$

a condition that will be relevant for the local invariance below.

Let us now assume that the three-form $i_{A^i} H$ on M adopts the expression $i_{A^i} H = X_i^j \wedge i_{F^j} H$ for some one-forms X^i_j. This means that the $i_{A^i} H$ vanish when the $i_{F^j} H$ do *i.e.*, that they vanish on shell (or are zero). Then, the following lemma holds:

Lemma. *Let $i_{A^j} H$ vanish on shell or be zero. Then, $i_{A^j} H$ has the form $i_{A^j} H = X^i_j \wedge i_{F^i} H$. Let us assume that $X^i_j \wedge i_{F^i} H \neq 0$. Then, the action is invariant under a local symmetry δA^i of the form*

$$\delta A^i = \delta_{gauge} A^i + \delta' A^i = d\alpha^i - C^i{}_{jk} \alpha^j A^k + \delta' A^i \equiv D\alpha^i + \delta' A^i, \qquad (4.20)$$

where

$$\delta' A^i = -X^i_j \alpha^j \qquad (4.21)$$

and the sum is extended to the indices j that make $i_{A^j} H \neq 0$.

Proof. The extra piece $\delta' A^i$ is needed for δA^i in (4.20) to be a symmetry when $i_{A^j} H \neq 0$ (this will be the case for the lagrangian B in Eq. (3.13), because $H = dB$, being a differential of curvatures, will not be given in terms of curvatures only). Indeed, as an arbitrary variation of the action has the form $\delta \int_M B = \int_M \delta A^i \wedge i_{F^i} H$, the specific δA^i in (4.20) leads to

$$\delta \int_M B = \int_M \left(d\alpha^i - C^i{}_{jk} \alpha^j A^k + \delta' A^i\right) \wedge i_{F^i} H$$

$$
\begin{aligned}
&= -\int_M \alpha^i \big(d(i_{F^i} H) + C^j{}_{ik} A^k \wedge i_{F^j} H\big) + \int_M \delta' A^i \wedge i_{F^i} H \\
&= \int_M \big(\alpha^i i_{A^i} H + \delta' A^i \wedge i_{F^i} H\big),
\end{aligned}
\tag{4.22}
$$

where (4.19) has been used in the last equality. We see that the last line of (4.22) vanishes for $\delta' A^i$ given by (4.21). □

In our case, it is at this point where the factor k in (3.13) is fixed so that B becomes the supergravity lagrangian. Note that no on-sell condition has been used; only the expression of the three-forms $E_i \equiv i_{F^i} H$ that determine the field equations ($E_i = 0$) enter in the four-form $\delta A^i \wedge i_{F^i} H$.

The above procedure is reminiscent of the construction of bosonic Chern–Simons (C-S) lagrangians in odd dimensions, where H is a (symmetric, gauge invariant and closed) even polynomial in the curvatures and B in $H = dB$ is the C-S form. In this C-S case, $i_{A^i} H$ is identically zero ($H \neq H(A)$) and $\delta \int_M B = 0$ without any $\delta' A^i$ term so that $\delta A^i = \delta_{gauge} A^i$ is a genuine gauge transformation and, as we know, $\delta_{gauge} B$ a total derivative. To derive this in the above context, let B be a C-S lagrangian. If the two total differentials that were discarded in Eqs. (4.17) and (4.22) are restored and (4.16) is used, then we obtain

$$
\delta_{gauge} B = d\big(\alpha^i i_{A^i} B\big).
\tag{4.23}
$$

Let us now check that this formula reproduces the familiar expression for the gauge variation of a C-S lagrangian form (see *e.g.* [25]). Let $A = A^i T_i$, $F = F^i T_i$ and $\alpha = \alpha^i T_i$, where $[T_i, T_j] = C^k{}_{ij} T_k$. Then, the C-S forms may be constructed as potentials of the closed (Chern) $2l$-forms

$$
H_l = \mathrm{Tr}(F \wedge \overset{l}{\ldots} \wedge F), \qquad dB_l = H_l
\tag{4.24}
$$

(we ignore an l-dependent factor). Explicitly,

$$
B_l = l \int_0^1 \mathrm{Tr}(A \wedge F_t \wedge \overset{l-1}{\ldots} \wedge F_t)\delta t
\tag{4.25}
$$

where $F_t = tF + (t^2 - t)A \wedge A = t\,dA + t^2 A \wedge A$. Using (4.25) and (4.23), the following formula for the gauge variation is obtained:

$$
\begin{aligned}
\delta B_l = d\Bigg[l \int_0^1 \mathrm{Tr}\Bigg(&\alpha F_t \wedge \overset{l-1}{\ldots} \wedge F_t \\
&- \big(t^2 - t\big) A \wedge \sum_{k=0}^{l-2} F_t \wedge \overset{k}{\ldots} \wedge F_t \wedge [\alpha, A] \wedge F_t \wedge \overset{l-2-k}{\ldots} \wedge F_t \Bigg)\delta t \Bigg].
\end{aligned}
\tag{4.26}
$$

For instance, for $l = 2$ ($D = 3$),

$$
\begin{aligned}
\delta B_2 &= d\left[2\,\mathrm{Tr}(\alpha\, F) \int_0^1 t\,\delta t + 6\,\mathrm{Tr}(\alpha A \wedge A) \int_0^1 \big(t^2 - t\big)\delta t \right] \\
&= d\big[\mathrm{Tr}\,\alpha\,(F - A \wedge A)\big] = d\big[\mathrm{Tr}(\alpha\, dA)\big].
\end{aligned}
\tag{4.27}
$$

Similarly, for $l = 3$, $D = 5$, Eq. (4.26) gives

$$\delta B_3 = d\left(\int_0^1 \mathrm{Tr}\big[\alpha F_t \wedge F_t - (t^2 - t)\big(A \wedge [\alpha, A] \wedge F_t + A \wedge F_t \wedge [\alpha, A]\big)\big]\delta t\right). \tag{4.28}$$

Inserting now the expression of F_t and evaluating the integrals, one obtains:

$$\delta B_3 = d\,\mathrm{Tr}\left(\alpha\, d\left[A \wedge dA + \frac{1}{2} A \wedge A \wedge A\right]\right). \tag{4.29}$$

Both δB_2 and δB_3 reproduce the well known expressions for the variation of the C-S three- and five-forms under the infinitesimal gauge function α.

5. Pure supergravity from $s\mathcal{M}$

Let us apply the above to the superMaxwell algebra case. First, we compute the differential of B in (3.13). $H = dB$ is given by

$$\begin{aligned} H = {} & 2\epsilon_{abcd} R^{ab} \wedge T^c \wedge e^d + \left(1 - \frac{k}{8}\right)\epsilon_{abcd} R^{ab} \wedge \bar{\Xi}\gamma^{cd} \wedge \psi \\ & - \left(1 - \frac{k}{8}\right)\epsilon_{abcd} R^{ab} \wedge \bar{\xi}\gamma^{cd} \wedge \Psi + \frac{k}{2}\bar{\Psi} \wedge e_a\gamma^a\gamma_5 \wedge \Psi \\ & - \frac{k}{2}\bar{\psi} \wedge T_a\gamma^a\gamma_5 \wedge \Psi. \end{aligned} \tag{5.30}$$

Now, we observe that $H \neq H(F^{ab})$ and that, when $k = 8$, the Ξ^α dependence is also absent from H; in fact, $H \neq H(\xi^\alpha, \Xi^\alpha, f^{ab}, F^{ab})$. Thus, both the ξ^α and the f_{ab} field equations are trivial for $k = 8$. This implies that the fields ξ^α and f^{ab} are not relevant in the action, since they have to appear in the lagrangian as total derivatives. For the same value of k, the ψ dependence of H is reduced to the last term in (5.30) so that

$$i_{\bar{\psi}^\alpha} H = -4\big(\bar{\Psi} \wedge T_a\gamma^a\gamma_5\big)_\alpha. \tag{5.31}$$

Moreover, the ω_{ab} field equation $E_{ab} = 0$, where $E_{ab} \equiv i_{R^{ab}} H$, and the $T^a = 0$ equation imply each other since the vielbein is invertible. Since (5.31) is, through T^a, related to the equation of motion of ω^{ab}, this means that in $i_{\psi^\alpha} H = X^i_\alpha \wedge E_i$ the only non-vanishing X^i_α corresponds to $i = (ab)$ *i.e.*, to X^{ab}_α. Thus (see (4.20)) only a certain $\delta'_\epsilon \omega^{ab}$ is needed for local supersymmetry invariance since for $\delta_\epsilon \psi$ and $\delta_\epsilon e$ no δ'_ϵ piece appears.

Since for $k = 8$ the extra one-form fields ξ^α and f^{ab} are not relevant in the action, it is sufficient to consider the procedure in Section 4 for the one-form fields $A^i = (\omega_{ab}, \psi^\alpha, e_a)$ and their curvatures $F^i = (R_{ab}, \Psi^\alpha, T_a)$. From Eqs. (3.11) and (3.12) it is easy to see that, when acting on the relevant variables ($H = H(E^a, T^a, \psi^\alpha, \Psi^\alpha, R^{ab})$), the Lorentz covariant exterior differential is given by

$$\begin{aligned} D = {} & \left(\frac{1}{2}\bar{\psi}\gamma^a \wedge \psi + T_a\right) i_{e_a} + \Psi^\alpha i_{\psi^\alpha} \\ & + \big(R_{ab} \wedge e^b - \bar{\Psi}\gamma_a \wedge \psi\big) i_{T_a} + \frac{1}{4}\big(R_{ab}\gamma^{ab}\big)^\alpha{}_\beta \wedge \psi^\beta i_{\Psi^\alpha}. \end{aligned} \tag{5.32}$$

It follows from this expression that (cf. Eq. (4.16))

$$Di_{\Psi^\alpha} - i_{\Psi^\alpha} D = (\bar{\psi}\gamma^a)_\alpha i_{T^a} - i_{\psi^\alpha}. \tag{5.33}$$

So writing respectively $E_\alpha \equiv i_{\Psi^\alpha} H = 0$, $E_a \equiv i_{T^a} H = 0$ for the ψ^α and e^a field equations, the Lorentz covariant exterior differential of the R-S three-form E_α in the R-S equation satisfies

$$DE_\alpha - (\bar{\psi}\gamma^a)_\alpha E_a = -i_{\psi^\alpha} H. \tag{5.34}$$

Then, under the local supersymmetry variations

$$\delta_\epsilon \psi^\alpha = D\epsilon^\alpha, \qquad \delta_\epsilon e^a = \bar{\epsilon}\gamma_a \psi, \tag{5.35}$$

(since $\delta'_\epsilon \psi = 0$, $\delta'_\epsilon e = 0$) plus a certain non-zero variation $\delta'_\epsilon \omega^{ab}$, the action is invariant:

$$\begin{aligned}
\delta_\epsilon \int_M B &= \int_M \left(\delta_\epsilon \psi^\alpha \wedge E_\alpha + \delta_\epsilon e^a \wedge E_a + \delta'_\epsilon \omega^{ab} \wedge E_{ab}\right) \\
&= \int_M \left(D\epsilon^\alpha \wedge E_\alpha + \bar{\epsilon}\gamma^a \psi \wedge E_a + \delta'_\epsilon \omega^{ab} \wedge E_{ab}\right) \\
&= \int_M \left(-\epsilon^\alpha DE_\alpha + \bar{\epsilon}\gamma^a \psi \wedge E_a + \delta'_\epsilon \omega^{ab} \wedge E_{ab}\right).
\end{aligned} \tag{5.36}$$

Using now (5.34) in (5.36) leads to

$$\delta_\epsilon \int_M B = \int_M \left(\delta'_\epsilon \omega^{ab} \wedge E_{ab} + \epsilon^\alpha i_{\psi^\alpha} H\right). \tag{5.37}$$

Now, there exists on M a set of one-forms ${X_\alpha}^{ab}$ such that

$$i_{\psi^\alpha} H = {X_\alpha}^{ab} \wedge i_{R^{ab}} H \equiv X^{ab}_\alpha \wedge E_{ab}, \tag{5.38}$$

where $i_{\psi^\alpha} H = -4(\bar{\Psi} \wedge T_a \gamma^a \gamma_5)_\alpha$ and $i_{R^{ab}} H = 2\epsilon_{abcd} T^c \wedge e^d$. A computation shows that the one form ${X_\alpha}^{ab}$ is given by

$${X_\alpha}^{ab} = -\frac{1}{2}\left(\epsilon^{abcd} e^g + \epsilon^{bcdg} e^a\right) \wedge (\bar{\Psi}_{cd}\gamma_g\gamma_5)_\alpha - (a \leftrightarrow b), \tag{5.39}$$

where $\bar{\Psi} = \bar{\Psi}_{cd}\, e^c \wedge e^d$. Then, using (5.38) in (5.37), we find that there is local supersymmetry for

$$\delta'_\epsilon \omega^{ab} = -\epsilon^\alpha {X_\alpha}^{ab} \tag{5.40}$$

with ${X_\alpha}^{ab}$ given by (5.39), which is seen to coincide with the well known local supersymmetry variation of ω. Thus, the lagrangian (3.13) for $k = 8$ is local sypersymmetry invariant. As for the local Lorentz and translation variations, the same general pattern of Section 4 applies. For the Lorentz variations, $i_{\omega^{ab}} H = 0$ since H in (5.30) does not depend on ω_{ab}. Thus $X^i_{ab} = 0$ for all values of i, and there is no δ' (the action is directly Lorentz invariant). For the local translations, however, $i_{e^a} H \neq 0$. In fact, besides the pieces containing T_a, and hence related to E_{ab}, there is the piece $4\bar{\Psi}\gamma_a\gamma_5 \wedge \Psi$, which can be shown to be related to E_α ($E_\alpha = 0$ being the R-S equation) by $X^\alpha_a = \Psi^\alpha_{ab} e^b$. Therefore, X^α_a and X^{bc}_a are different from zero so that, besides the piece ${\delta'}_\epsilon \omega^{ab}$ in the variation, there is also ${\delta'}_{t^a} \psi^\alpha = -\Psi^\alpha_{ab} t^b$, t^a being the local translation.

We may finally show that the lagrangian (3.13) for $k = 8$ is that of pure $D = 4$ supergravity. It may be rewritten in the form

$$B = \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + 4\bar{\psi} \wedge e_a \gamma^a \gamma_5 \wedge \Psi + d\big(\epsilon_{abcd} R^{ab} \wedge f^{cd} + 8\bar{\xi}\gamma_5 \Psi\big), \tag{5.41}$$

which is the $D = 4$ simple supergravity lagrangian [26–28] (Eq. (2.7) plus Eq. (3.14)) but for the total derivative in the second term.

6. Final comments

We have shown that the first-order lagrangian four-form of $D = 4$ minimal ($N = 1$) supergravity can be written out of bilinears of the curvatures of the gauge fields associated with the minimal Maxwell superalgebra of [19], thus generalizing the results for gravity in [23] to the supergravity case. The action is the sum of two terms in the $s\mathcal{M}$ curvatures, and for a certain relative factor the extra gauge field forms not contained in the supergravity supermultiplet enter in the action inside a total derivative. For this relative factor, the sum gives the action of minimal $D = 4$ supergravity as shown by Eq. (5.41).

This provides one more example of how new geometrical aspects of a theory may be exhibited by formulating it on the enlarged superspaces associated to larger algebras, even if the additional fields in the enlarged superspace variables/fields correspondence (see [2,6,29]) do not have a dynamical character. In the present case, the enlarged superspace would be determined by the supergroup coset sM/L and would contain, besides the four-dimensional Minkowski spacetime, 6 bosonic tensorial variables and the $4 + 4$ fermionic ones of the two Majorana spinors.

Going beyond $D = 4$ presents difficulties. An obvious one is the fact that in odd dimensions there is no way of writing a lagrangian D-form out of curvature two-forms. This would seem to indicate that in odd dimensions the appropriate point of view is to look for lagrangians the differentials of which are written solely in terms of the curvatures. This is, of course, the case of actions of the Chern–Simons type. For instance, for $D = 3$, the $(p + q)$ supergravities [17] are C-S theories [18] for an expansion of $osp(p + q|2, \mathbb{R})$. Another difficulty, also present in $D = 4$, $N > 1$, is the existence in some cases of Lagrange multiplier zero-forms in the first-order actions, which cannot be interpreted in terms of one-form fields for Lie superalgebras. In higher dimensions there are also forms of order higher than one, but these could, in principle, be given a group theoretical interpretation, as done for $D = 11$ supergravity (see [1,2]). Another problem of higher even-dimensional supergravities in our scheme is that, in the present $D = 4$ bosonic case, the extra field f^{ab} has trivial equations of motion since there is a single F^{ab} curvature in the lagrangian (see Eq. (3.13)), which would not be the case for $D > 4$.

It would be interesting to look further at the role of the relative weight of the two terms in the basic supergravity lagrangian, as well as the effect of other possible lagrangian terms in (3.13) which, in the case of simple gravity, lead to a generalized cosmological term [23]. Another case worth studying in an analogous approach would be that of $D = 4$, $N = 1$ AdS supergravity.

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