Advances in Applied Mathematics 42 (2009) 504–518

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Density of periodic points, invariant measures and almost equicontinuous points of cellular automata

Pierre Tisseur

Universidade Federal do ABC, Centro de Matemática, Computação e Cognição, Santo André, São Paulo, Brazil

ARTICLE INFO

Article history:
Received 27 April 2006
Revised 14 November 2007
Accepted 6 August 2008
Available online 20 February 2009

MSC:
37B15
37A05
28D05
37A35
37B10
37B20

Keywords:
Cellular automata
Invariant measures
Periodic points

ABSTRACT

Revisiting the notion of μ-almost equicontinuous cellular automata introduced by R. Gilman, we show that the sequence of image measures of a shift ergodic measure μ by iterations of such automata converges in Cesàro mean to an invariant measure μ_c. Moreover the dynamical system (cellular automaton F, invariant measure μ_c) has still the μ_c-almost equicontinuity property and the set of periodic points is dense in the topological support of the measure μ_c. We also show that the density of periodic points in the topological support of a measure μ occurs for each μ-almost equicontinuous cellular automaton when μ is an invariant and shift ergodic measure. Finally using most of these results we give a non-trivial example of a couple (μ-equicontinuous cellular automaton F, shift and F-invariant measure μ) such that the restriction of F to the topological support of μ has no equicontinuous points.

1. Introduction

A one-dimensional cellular automaton (CA) is a discrete mathematical idealization of a space–time physical system. The space, called configuration space, is the set of doubly infinite sequences of elements of a finite set A. The discrete time is represented by the action of a CA on this space. Using extensive computer simulations, Wolfram in [9] has proposed a first empirical (visual) classification of one-dimensional CA. In [5] Gilman proposes a formal and measurable classification by roughly dividing the set of all CA in two parts, those with *almost equicontinuous points or equicontinuous points*

E-mail address: pierre.tisseur@ufabc.edu.br.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2008.08.001

and those with *almost expansive points* (partition in order and disorder). The Gilman's classification is defined thanks to Bernoulli measures and corresponds to the Wolfram's classification based on simulations that use random entries. The measure does not need to be invariant, so the Gilman's classification can be applied to any CA. In [7], Kůrka introduces a topological classification based on the equicontinuity, sensitiveness and expansiveness properties. If a CA has equicontinuous points, then there exist finite configurations that stop the propagation of the perturbations on the one-dimensional lattice. If a CA has μ-almost equicontinuous points then the probability that a perturbation move to infinity is equal to zero (see [5]). Remark that the class of CA with almost equicontinuous points contains the topological class of CA with equicontinuous points. In this paper, we consider the definitions of Gilman (μ-expansiveness and μ-equicontinuity) in the more general case of probability measure on the configuration space $A^{\mathbb{Z}}$. In this case we call μ-equicontinuous CA, any CA which has a set of measure one of μ-equicontinuous points. Our main goal is to study the μ-equicontinuous CA when μ is an invariant measure and show the existence of such a measure. Here we prove (see Theorem 1) that if μ is a shift ergodic measure and F a CA which has μ-equicontinuous points then the sequence $(\mu \circ F^{-n})$ converges in Cesàro mean to an F-invariant measure μ_c. We also show that this automaton F is still a μ_c-equicontinuous CA. Then, we describe properties of μ-equicontinuous CA when μ is an F-invariant measure. Here we show that (see Proposition 4) under these assumptions, the measure entropy is equal to zero. If the invariant measure we consider is shift ergodic or is given by Theorem 1 (measure μ_c), the set of F-periodic points is dense in the topological support of this measure (see Propositions 5 and 6). This result extends a previous result on the density of periodic points of surjective CA with equicontinuous points acting on a mixing subshift of finite type (see [1]). Remark that in [3] Boyle and Kitchen have shown that closing CA always have a dense set of periodic points. The expansive CA and some other CA with equicontinuous points belong to this large class. In [5] Gilman gives an example of a μ-equicontinuous CA that has no equicontinuous points. The invariant measure μ_c (limit by Cesàro mean of $(\mu \circ F^n)$) that we can construct (using our results) for this particular automaton still has μ_c-equicontinuous points and no equicontinuous points, but the restriction of this CA to the topological support of μ_c always has equicontinuous points. Using most of our results, we describe a particular CA called F_e with a non-trivial dynamic which keep the sensitiveness property (no equicontinuous points) if we restrict its action to the topological support $S(\mu_c)$ of the invariant measure μ_c. The proofs of sensitiveness and μ_c-equicontinuity of the automaton $(S(\mu_c), F_e)$ use non-classical arguments.

2. Definitions and preliminary results

2.1. Symbolics systems and cellular automata

Let A be a finite set or alphabet. Denote by A^* the set of all concatenations of letters in A. These concatenations are called words. The length of a word $u \in A^*$ is denoted by $|u|$. The set of bi-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ is denoted by $A^{\mathbb{Z}}$. A point $x \in A^{\mathbb{Z}}$ is called a configuration. For integers i, j with $i \leqslant j$ we denote by $x(i, j)$ the word $x_i \dots x_j$ and by $x(i,\infty)$ the infinite sequence $(v_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ one has $v_n = x_{i+n}$. For each integer t and each word $u = u_1 \dots u_{|u|}$, we call cylinder the set $[u]_t = \{x \in A^{\mathbb{Z}}\colon\ x_t = u_1 \dots;\ \ x_{t+|u|} = u_{|u|}\}$. We endow $A^{\mathbb{Z}}$ with the product topology of the discrete topologies on the sets A. For this topology $A^{\mathbb{Z}}$ is a compact metric space. A metric compatible with this topology can be defined by the distance $d(x, y) = 2^{-i}$ where $i = \min\{|j|$ such that $x_j \neq y_j\}$. The shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by: $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is called the full shift. A subshift X is a closed shift-invariant subset X of $A^{\mathbb{Z}}$. Consider a probability measure μ on the Borel sigma-algebra $\mathcal{B}$ of $A^{\mathbb{Z}}$. If μ is σ-invariant then the topological support of μ (which is the smallest closed subset of measure 1) is a subshift denoted by $S(\mu)$. If $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ are two partitions of a compact space X, denote by $\alpha \vee \beta$ the partition $\{A_i \cap A_j;\ 1 \leqslant i \leqslant n;\ 1 \leqslant j \leqslant m\}$. The metric entropy $h_\mu(T)$ of a transformation T is an isomorphism invariant between two μ-preserving transformations. Put $H_\mu(\alpha) = \sum_{A\in\alpha} \mu(A)\log\mu(A)$. The entropy of the partition α is defined as $h_\mu(\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha)$ and the entropy of (X, T, μ) as $\sup_\alpha h_\mu(\alpha)$. A cellular automaton is a continuous self-map F on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis–Hedlund–Lyndon theorem states that for every F there exist an integer r and a block map f from A^{2r+1} to A such that: $F(x)_i = f(x_{i-r}, \ldots, x_i, \ldots, x_{i+r})$. The integer r is called the radius of the CA. If X is a subshift of $A^{\mathbb{Z}}$ and one has $F(X) \subset X$, the restriction of F to X determines a dynamical system (X, F); it is called a CA on X. For example, given any shift invariant measure we can consider the restriction of F to $S(\mu)$. A closed subset of $Y \subset A^{\mathbb{Z}}$ (not necessarily shift-invariant) such that $F(Y) \subset Y$ is said F-invariant. The omega limit set of any set $S \subset A^{\mathbb{Z}}$ under the action of F is denoted by $w(S, F) = \lim_{n\to\infty} \bigcap_{j=0}^{n} \bigcup_{i=j}^{\infty} \{F^i(S)\}$.

2.2. Almost equicontinuous points of cellular automata

In [5] Gilman shows that considering any Bernoulli measure μ, it is possible to divide the set of all CA in the three following classes: The class of CA where there exist equicontinuous points, the class of CA with μ-almost equicontinuous points but without equicontinuous point and the class of μ-almost expansive CA. In this section we recall the topological and classical definitions for the expansive and equicontinuous classes of CA acting on $A^{\mathbb{Z}}$ and we extend the Gilman's measurable definitions to any probability measure μ.

For any integer $n \geqslant 0$ and point $x \in A^{\mathbb{Z}}$, we denote by $B_n(x)$ the set of points y such that for all $i \in \mathbb{N}$ one has $d(F^i(x), F^i(y)) \leqslant 2^{-(n+1)}$ and by $C_n(x)$ the set of points y such that $y_j = x_j$ with $-n \leqslant j \leqslant n$.

Definitions 1 *(Equicontinuity).*

- A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point if for all positive integers n there exists another positive integer m such that $B_n(x) \supset C_m(x)$.
- A CA is called almost equicontinuous when there exist equicontinuous points.
- A CA is equicontinuous when all points in $A^{\mathbb{Z}}$ are equicontinuous.
- A point x is μ-almost equicontinuous if for all $m \in \mathbb{N}$ one has

$$\lim_{n\to\infty} \frac{\mu(C_n(x) \cap B_m(x))}{\mu(C_n(x))} = 1.$$

- A CA is μ-almost equicontinuous if there exists a set of full measure of μ-almost equicontinuous points.

The last definition extends the notion of μ-equicontinuity to all measurable CA $(A^{\mathbb{Z}}, F, \mu)$. If x is a μ-equicontinuous point which belongs to the topological support $S(\mu)$ of some measure μ then x is also a μ-equicontinuous point. Remark that for CA, to have no equicontinuous points is equivalent to sensitiveness (see [7]).

Definitions 2 *(Expansiveness).*

- A Cellular automaton is positively expansive if there exists a positive integer n such that for all $x \in A^{\mathbb{Z}}$ one has $B_n(x) = \{x\}$.
- A cellular automaton F is almost expansive if there exists a positive integer n such that for all $x \in A^{\mathbb{Z}}$, $\mu(B_n(x)) = 0$.

In [5] Gilman proves the following proposition that allows to establish a classification based on μ-equicontinuity. Since in [5], the proof of the original Proposition 1 requires only the ergodicity of any Bernoulli measure, it can be extended to any shift ergodic measure.

Proposition 1. *(See [5].) Let μ be a shift-ergodic measure and F a cellular automaton with radius r. The following properties are equivalent*:

(i) *F has a μ-equicontinuous point.*
(ii) *There exists a point $x \in A^{\mathbb{Z}}$ such that $\mu(B_m(x)) > 0$ for all $m \geqslant 0$.*
(iii) *There exists a point $x \in A^{\mathbb{Z}}$ such that $\mu(B_r(x)) > 0$.*
(iv) *The set of μ-equicontinuous points has measure 1 for F.*

When μ is a shift ergodic measure and F has no μ-equicontinuous points then F is an almost expansive CA. A point x is an equicontinuous point when the interior of $B_n(x)$ is non-empty for all $n \in \mathbb{N}$. In [5], Gilman introduces a measure-theoretic analogue of the interior of a set. For any measurable set E, define $\rho_E(x) = \lambda$ if $\lim_{n\to\infty} \frac{\mu(C_n(x)\cap E)}{\mu(C_n(x))} = \lambda$ and call E^{μ} the set $\{x \in E\colon\ \rho_E(x) = 1\}$.

Lemma 1 *(Lebesgue). If μ is a Borel probability measure and E any measurable set, we have $\mu(E^{\mu}) = \mu(E)$.*

Remark that x is a μ-equicontinuous point if for all $n \in \mathbb{N}$, $x \in B_n^{\mu}(x)$. The next topological result is due to Gilman (see [6]). We give here a detailed proof of this key result.

Proposition 2. *(See [6].) If there exist a point x and an integer $m \neq 0$ such that $B_n(x) \cap \sigma^{-m} B_n(x) \neq \emptyset$ with $n \geqslant r$ (the radius of the automaton F) then the common sequence $(F^i(y)(-n,n))_{i\in\mathbb{N}}$ of all points $y \in B_n(x)$ is ultimately periodic.*

Proof. First remark that for each shift periodic point $\overline{w}$ of period P, the cardinal of the set $\{F^i(\overline{w}) \mid i \in \mathbb{N}\}$ is finite and less or equal to $(\#A)^P$ (for all integer $k \geqslant 0$ one has $\sigma^P \circ F^k(\overline{w}) = F^k \circ \sigma^P(\overline{w}) = F^k(\overline{w})$). This implies that the sequence $(F^i(\overline{w}))_{i\in\mathbb{N}}$ is ultimately periodic. Since all the elements of $B_n(x)$ share the same ultimately periodic sequence $(F^i(x)(-n,n))_{i\in\mathbb{N}}$, we only need to show that $B_n(x)$ contains a shift periodic element $\overline{w}$ to finish the proof. Now suppose without loosing generalities that $m > 0$, pick a point $y_1 \in B_n(x) \cap \sigma^{-m}(B_n(x))$ and put $w := y_1(-n, -n+m-1)$.

We claim that for all points x, integers $n \geqslant r$ and $y \in B_n(x)$, all points z, z' that verify $z(-\infty,-n) = y(-\infty,-n)$, $z(-n+1,+\infty) = x(-n+1,+\infty)$, $z'(-\infty,-n) = x(-\infty,-n)$ and $z'(-n+1,+\infty) = y(-n+1,+\infty)$ belong to $B_n(x)$.

We prove the claim only for z using a recurrence proof. To simplify, we indifferently denote by f, the local rule of F which is a block map from A^{2r+1} to A and also all the finite extensions of f which are block maps from A^{2r+1+k} to A^{k+1} with $k \in \mathbb{N}$. Since $y \in B_n(x)$ we have $z(-n,n) = y(-n,n) = x(-n,n)$. Suppose that for $i > 0$ one has $F^t(z)(-n,n) = F^t(x)(-n,n)$ for $0 \leqslant t \leqslant i$. In this case we have $F^t(z)(-\infty,-n) = F^t(y)(-\infty,-n)$ for $0 \leqslant t \leqslant i$ and $F^{i+1}(z)(-n,0) = f(F^i(z)(-n-r,r)) = f[F^i(z)(-r-n,-n-1)F^i(x)(-n,r)] = f[F^i(y)(-r-n,-n-1)F^i(x)(-n,r)] = F^{i+1}(x)(-n,0)$. Since $F^{i+1}(z)(1,n) = f(F^i(x)(-r+1,n+r)) = F^{i+1}(x)(1,n)$ it follows that $F^{i+1}(z)(-n,n) = F^{i+1}(x)(-n,n)$. We can conclude saying that $(F^k(z)(-n,n))_{k\in\mathbb{N}} = (F^k(x)(-n,n))_{k\in\mathbb{N}}$ which implies that $z \in B_n(x)$.

Now we apply the claim to a point $\sigma^m(y_1) \in \sigma^m B_n(x) \cap B_n(x)$. The points y_1 and $\sigma^m(y_1)$ belong to $B_n(x)$, so the point y_2 such that $y_2(-\infty,-n) = \sigma^m(y_1)(-\infty,-n)$ and $y_2(-n+1,+\infty) = y_1(-n+1,+\infty)$ belongs to $B_n(x)$. We can see that $y_2(-n-m,-n+m-1) = ww$ and $y_2 \in \sigma^m B_n(x) \cap B_n(x) \cap \sigma^{-m} B_n(x)$. Next we construct y_3 by applying the claim to $\sigma^m(y_2)$ and $\sigma^{-m}(y_2)$ (remark that $\sigma^m(y_2) \in B_n(\sigma^{-m}(y_2)) = B_n(x)$). The point y_3 is such that $y_3(-\infty,-n) = \sigma^m(y_2)(-\infty,-n)$ and $y_3(-n+1,+\infty) = \sigma^{-m}(y_2)(-n+1,+\infty)$. Since we have $y_3(-n-2m, 2m-1) = wwww$, we can repeat the same process to $\sigma^{2m}(y_3)$ and $\sigma^{-2m}(y_3)$ and construct a point y_4 such that $y_4(-n-4m, 4m-1) = w^8$. Finally, the sequence $(y_n)_{n\in\mathbb{N}}$ of points of $B_n(x)$ that we construct by this algorithm converges to the shift periodic point $\overline{w} = {}^{\infty}w^{\infty} = \ldots wwww \ldots$. Since $B_n(x)$ is closed and compact, $\overline{w}$ is in $B_n(x)$. $\square$

In [6] Gilman states the following result using the ergodic properties of any Bernoulli measure μ.

Proposition 3. *(See [6].) Let μ be a shift ergodic measure. If a cellular automaton F has a μ-equicontinuous point, then for all $\epsilon > 0$ there exists an F-invariant closed set Y such that $\mu(Y) > 1 - \epsilon$ and the restriction of F to Y is equicontinuous.*

Proof. Let x be a μ-equicontinuous point and m and integer greater than r, the radius of F. Since μ is a shift ergodic measure and $\mu(B_m(x)) > 0$, for each integer $k \geqslant 0$ and μ-almost all points y, there exist positive integers i, j greater than k such that $y \in \sigma^{-i} B_m(x) \cap \sigma^j B_m(x)$. Then from Proposition 2, the sequences $(u_n)_{n\in\mathbb{N}} = (F^n(y)(i-m, i+m))_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}} = (F^n(y)(-j-m, -j+m))_{n\in\mathbb{N}}$ are ultimately periodic with respective preperiods pp_u and pp_v. For all $n \in \mathbb{N}$, denote by w_n the word $F^n(y)(-j+m+1, i-m)$ and remark that for each integer $n \geqslant 1$, one has $w_n = f(u_{n-1} w_{n-1} v_{n-1})$ where f is a block map. It follows that if a word $w = w_t$ with $t \geqslant pp_u, pp_v$ appears infinitely often in $(w_n)_{n\in\mathbb{N}}$, the sequence $(w_{n+t})_{n\in\mathbb{N}}$ is periodic. This implies that $(w_n)_{n\in\mathbb{N}}$ and $(F^n(y)(-k, k))_{n\in\mathbb{N}}$ are ultimately periodic sequences. Let P be a map from $\mathbb{N}$ to $\mathbb{N}^2$ and $Y_{P(k)}$ be the set of points y such that each sequence $(F^n(y)(-k, k))_{n\in\mathbb{N}}$ are periodic of period $p(k)$ and preperiod $pp(k)$ where $(p(k), pp(k)) = P(k)$. As μ is shift ergodic measure, for each real $\epsilon > 0$ there exists a map $P_\epsilon : \mathbb{N} \to \mathbb{N}^2$ such that for all $n \in \mathbb{N}$ we have $\mu(Y_{P_\epsilon(n)}) > 1 - \epsilon \times 2^{-n}$. Since each $Y_{P_\epsilon(k)}$ is closed and F-invariant, the set $Y_\epsilon = \lim_{n\to\infty} \bigcap_{k=1}^n Y_{P_\epsilon(k)}$ is closed and F-invariant too. Clearly $\mu(Y_\epsilon) > 1 - \epsilon$ and each point $y \in Y_\epsilon$ is an equicontinuous point since for each integer $k \geqslant 0$ one has $C_{l\times r}(y) \subset B_k(y)$ with $l = p(k) + pp(k)$ and $(p(k), pp(k)) = P_\epsilon(k)$. □

3. Results on invariant measures

3.1. Measure entropy and density of the set of periodic points

Proposition 4. *The measure entropy $h_\mu(F)$ of a μ-equicontinuous and μ-invariant cellular automaton F is equal to zero.*

Proof. Let α_p be the partition of $A^{\mathbb{Z}}$ by the $2p+1$ central coordinates. Two points x and y belong to the same element of α_p if and only if $x(-p, p) = y(-p, p)$. Let $\alpha_p^n(x)$ be the element of the partition $\alpha_p \cap F^{-1}\alpha_p \dots F^{-n+1}\alpha_p$ which contains x. Clearly for all $n \in \mathbb{N}$, we have $\alpha_p^n(x) \supset B_p(x)$. Since almost all points are μ-equicontinuous points, there exists a set Z with measure 1 such that if $y \in Z$ then $\mu(B_m(y)) > 0$ for all integer $m \geqslant 0$. This implies that for almost all y and all positive integer p, we have $\lim_{n\to\infty} \frac{-\log\mu(\alpha_p^n(y))}{n} \leqslant \lim_{n\to\infty} \frac{-\log\mu(B_p(x))}{n} = 0$. Using the Shannon–Breiman–McMillan theorem which tells that

$$h_\mu(F, \alpha_p) = \int_{A^{\mathbb{Z}}} \lim_{n\to\infty} \frac{-\log\mu(\alpha_p^n(y))}{n} \, d\mu(y) = 0,$$

we can conclude that $h_\mu(F) = \lim_{p\to\infty} h_\mu(F, \alpha_p) = 0$. □

In Proposition 4, the measure μ does not need to be shift invariant. Note that from the last proposition each cellular automaton F which has equicontinuous points in the topological support of a shift ergodic and F-invariant measure μ verifies $h_\mu(F) = 0$. This result about CA with equicontinuous points first appears in [8].

In [2] it is shown that if a measure is shift and F-invariant, the entropy of the CA $(A^{\mathbb{Z}}, F, \mu)$ is equal to zero when some discrete analogues of Lyapunov exponents are null. Remark that these Lyapunov exponents can be equal to zero for almost expansive CA (see [2]) but even if they are always null when there exists a set of full measure of equicontinuous points (see [8]), they are not in general equal to zero for μ-equicontinuous CA (it can be easily seen in examples of Section 3.3).

Proposition 5. *If a cellular automaton F has some μ-equicontinuous points where μ is an F-invariant and shift ergodic measure then the set of F-periodic points is dense in the topological support of μ.*

Proof. It is sufficient to show that for each point z in $S(\mu)$ (the topological support of μ) and positive integer p we can construct a σ and F periodic point $\overline{w} = {}^\infty w^\infty$ (bi-infinite repetition of the word w) such that $\overline{w}(-p, p) = z(-p, p)$. Remark that if x is a μ-equicontinuous point then

$\mu(B_r(x)) > 0$ where r is the radius of F. Since z is in $S(\mu)$ and μ is a shift ergodic measure then $\mu(C_p(z)) > 0$ and there exist $(i, j) \in \mathbb{N}^2$ such that $\mu(C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x)) > 0$. To simplify, write $S = C_p(z) \cap \sigma^{-(i+p)} B_r(x) \cap \sigma^{j+p} B_r(x)$. Clearly there exists a point $y \in S$ such that $\mu([y(-r-i-p, j+p-r-1)]_{-r-i-p} \cap S) > 0$. Denote by S' the set $[y(-r-i-p, j+p-r-1)]_{-r-i-p} \cap S$ and remark that using the Poincaré recurrence theorem, we obtain that there exists an integer $m > 0$ such that $\mu(S' \cap F^{-m} S') > 0$. Remark that all points $y' \in S'$ share the same sequence $(F^n(y')(-r-i-p, j+p-r-1))_{n\in\mathbb{N}}$ since it results of the same combination of block maps on the word $y(-i-p+r+1, j+p-r-1)$ and on the two sequences $(F^n(x)(-i-p-r, -i-p+r))_{n\in\mathbb{N}}$ and $(F^n(x)(j+p-r, j+p+r))_{n\in\mathbb{N}}$. It follows that $\forall y' \in S'$ one has $F^m(y')(-r-i-p, j+p-r-1) = y'(-r-i-p, j+p-r-1)$. From Proposition 2 and its proof, since $\sigma^{-i} B_r(x) \cap \sigma^j B_r(x) \neq \emptyset$, the shift periodic point $\overline{w} = {}^\infty w^\infty$ such that $w = \overline{w}(-r-i-p, j+p-r-1) = y(-r-i-p, j+p-r-1)$ belongs to the set S'. Finally, we obtain that $F^m(\overline{w})(-r-i-p, j+p-r-1) = \overline{w}(-r-i-p, j+p-r-1)$ and since the common σ-period of $\overline{w}$ and $F^m(\overline{w})$ is less or equal to $|w| = 2p + 2r + i + j$, we get that $F^m(\overline{w}) = \overline{w}$ which finishes the proof. □

3.2. Invariant measures as limit of Cesàro means

Proposition 3 allows us to prove a Cesàro mean convergence result.

Theorem 1. *Let μ be a shift-ergodic measure. If a cellular automaton F has some μ-almost equicontinuous points then the sequence $(\mu \circ F^{-n})_{n\in\mathbb{N}}$ converges vaguely in Cesàro mean to an invariant measure μ_c.*

Proof. To show that the sequence of measure $(\frac{1}{n}\sum_{i=0}^{n-1} \mu \circ F^{-i})_{n\in\mathbb{N}} = (\mu_n)_{n\in\mathbb{N}}$ converges vaguely in measure, we need to show that for all $x \in S(\mu)$ and $m \in \mathbb{N}$ the sequence $(\mu_n(C_m(x)))_{n\in\mathbb{N}}$ converges. From Proposition 3 there exists a set Y_ϵ of measure greater than $1-\epsilon$ such that for all points $y \in Y_\epsilon$ and positive integer k the sequences $(F^n(y)(-k, k))_{n\in\mathbb{N}}$ are eventually periodic with preperiod $pp_\epsilon(k)$ and period $p_\epsilon(k)$. Hence for all $x \in A^{\mathbb{Z}}$ and integer $k \geqslant m$

$$\mu_n\big(C_m(x) \cap Y_\epsilon\big) = \frac{1}{n}\sum_{i=0}^{pp_\epsilon(k)-1} \mu\big(F^{-i}\big(C_m(x)\big) \cap Y_\epsilon\big) + \frac{1}{n}\sum_{i=pp_\epsilon(k)}^{n-1} \mu\big(F^{-i}\big(C_m(x)\big) \cap Y_\epsilon\big).$$

The first term tends to 0; using periodicity one gets

$$\lim_{n\to\infty} \mu_n\big(C_m(x) \cap Y_\epsilon\big) = \frac{1}{p_\epsilon(k)}\sum_{i=0}^{p_\epsilon(k)-1} \mu\big(F^{-(i+pp_\epsilon(k))}\big(C_m(x)\big) \cap Y_\epsilon\big).$$

Clearly we have $\lim_{\epsilon\to 0} \mu_n(C_m(x) \cap Y_\epsilon) = \mu_n(C_m(x))$. The convergence is uniform with respect to ϵ since for all x and $m \in \mathbb{N}$

$$\big|\mu_n\big(C_m(x) \cap Y_\epsilon\big) - \mu_n\big(C_m(x)\big)\big| \leqslant \frac{n\epsilon}{n} = \epsilon.$$

Consequently, letting ϵ go to 0, we get the result by inverting the limits

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu \circ F^{-i}\big(C_m(x)\big) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \lim_{\epsilon\to 0} \mu \circ F^{-i}\big(C_m(x) \cap Y_\epsilon\big) \\ &= \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu \circ F^{-i}\big(C_m(x) \cap Y_\epsilon\big)\end{aligned}$$

$$= \lim_{\epsilon\to 0} \frac{1}{p_\epsilon(k)} \sum_{i=0}^{p_\epsilon(k)-1} \mu\big(F^{-(i+pp_\epsilon(k))}\big(C_m(x)\big)\cap Y_\epsilon\big)$$

$$= \mu_c\big(C_m(x)\big).$$

We denote by μ_c the Cesàro mean limit of $(\mu\circ F^n)_{n\in\mathbb{N}}$. $\square$

In the following of this section, we suppose that μ_c is the probability measure which came from the Cesàro mean of the sequence $(\mu\circ F^{-n})_{n\in\mathbb{N}}$ when F is a μ-equicontinuous CA and μ is a shift ergodic measure.

Theorem 2. *If μ is a shift ergodic measure and F has a μ-equicontinuous point then F is also a μ_c-equicontinuous CA.*

Proof. We need to show that there exists a set of measure one (for the measure μ_c) of μ_c-equicontinuous points. We will prove this by showing that μ_c-almost all points y belong to $B_m^{\mu_c}(y)$ for all $m\in\mathbb{N}$. Since F is a μ-equicontinuous CA then there exists a point x such that $x\in B_m^{\mu}(x)$ for all $m\in\mathbb{N}$. Moreover, for all positive integers k one has $\mu(B_k(x))>0$.

For all $m\in\mathbb{N}$, define $Y_m=\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(x)\cap\sigma^{j+m}B_r(x))$ where r is the radius of the automaton F. Since μ is a shift ergodic measure, for all $m\in\mathbb{N}$ we get $\mu(Y_m)=1$. Then consider the omega limit sets of Y_m called $\Omega_m := w(Y_m,F)=\lim_{n\to\infty}\bigcap_{j=0}^{n}\bigcup_{i=j}^{\infty}F^i(Y_m)$. Clearly for all $m\in\mathbb{N}$, we have $\mu_c(\Omega_m)=1$. Let $\Lambda(F)=w(A^{\mathbb{Z}},F)$, the omega limit set of the F. Since μ is a shift ergodic measure then there exists an integer k such that $B_r(x)\cap\sigma^k B_r(x)\neq\emptyset$ and using Proposition 2 we obtain that the sequence $(F^n(x)(-r,r))_{n\in\mathbb{N}}$ is ultimately periodic of period p. It follows that there exist p points $z_0,\dots,z_{p-1}$ such that we have $w(B_r(x),F)=\{B_r(z_l)\cap\Lambda(F)\mid 0\leqslant l\leqslant p-1\}$. This implies that $\Omega_m=\bigcup_{z\in[z_0\dots z_{p-1}]}\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(z)\cap\sigma^{j+m}B_r(z))\cap\Lambda(F)$. Define the set $\Omega'_m=\bigcup_{z\in[z_0\dots z_{p-1}]}\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(z)\cap\sigma^{j+m}B_r(z))^{\mu_c}\cap\Lambda(F)$. Since for any measurable set E one has $\mu(E)=\mu(E^{\mu_c})$ (see Lemma 1) and that we need to take off a countable number of sets of measure zero from Ω_m to obtain Ω'_m, we get that $\mu_c(\Omega'_m)=1$ for all $m\in\mathbb{N}$.

Define the set $\Omega:=\bigcap_{m\in\mathbb{N}}\Omega'_m$ and remark that $\mu_c(\Omega)=1$. Clearly, for all $y\in\Omega$ and all $m\in\mathbb{N}$, there exist $i,j\geqslant m$ and $z\in w(x,F)$ such that $y\in(\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))^{\mu_c}$. Since for all $y'\in(\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))^{\mu_c}$, the sequence $(F^n(y')(-m,m))_{n\in\mathbb{N}}$ depends only on $y'(-m,m)$ and the common sequences $(F^n(y'')(-r+j,j+r))_{n\in\mathbb{N}}$ and $(F^n(y'')(-r-i,-i+r))_{n\in\mathbb{N}}$ where y'' is any point in $(\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))^{\mu_c}$, for all $k\geqslant m$, we have $C_k(y)\cap(\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))^{\mu_c}\subset B_m(y)$. Since $y\in C_k(y)\cap(\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))^{\mu_c}$, we get that $y\in B_m(y)^{\mu_c}$ which finishes the proof. $\square$

Proposition 6. *If μ_c and F are respectively a measure and a cellular automaton that verify the assumptions of Theorem 2 then the set of F-periodic points is dense in $S(\mu_c)$ (the topological support of μ_c).*

Proof. Let x be a μ-equicontinuous point. From the proof and definitions of Theorem 2, there exists a finite number of points $z_0,\dots,z_{p-1}\in w(\{x\},F)$ such that $\forall m\in\mathbb{N}$, the sets $\Omega_m=\bigcup_{z\in[z_0\dots z_{p-1}]}\bigcup_{i,j\in\mathbb{N}^2}(\sigma^{-i-m}B_r(z)\cap\sigma^{j+m}B_r(z))\cap\Lambda(F)$ have full measure with respect to the invariant measure μ_c. It follows that for each point $y\in S(\mu_c)$ and $k\in\mathbb{N}$, one has $\mu_c(C_k(y)\cap\Omega_m)>0$ ($\forall m\in\mathbb{N}$). Since for each $y\in S(\mu_c)$ and $k\in\mathbb{N}$ one has $\mu_c(C_k(y)\cap\Omega_k)>0$, using the Σ-additivity of μ_c, we can see that there always exist integers $i,j\geqslant k$ and point $z\in w(\{x\},F)$ such that $\mu_c(C_k(y)\cap\sigma^{-i}B_r(z)\cap\sigma^{j}B_r(z))>0$ (I). Finally, using the final part of the proof of Proposition 5 that only use the F-invariance of μ_c and inequality (I), we get that there exists a dense set of F-periodic points in $S(\mu_c)$. $\square$

We remark that the measure μ_c is not necessarily shift ergodic.

3.3. Examples of μ-equicontinuous CA without equicontinuous points

In [5] Gilman gives an example of a μ-equicontinuous CA F_s that has no equicontinuous points. The automaton F_s acts on $\{0,1,2\}^{\mathbb{Z}}$ and is defined thank to the following block map of radius 1:

$$\begin{vmatrix} *00 & *01 & *02 & *10 & *11 & *12 & *20 & *21 & *22 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \end{vmatrix}$$

where $*$ stands for any letter in $\{0,1,2\}$. Considering 0 as a background element, the 2s move straight down, 1s move to the left and 1 and 2 collide annihilate each other. In this case the measure μ is a Bernoulli measure on $\{0,1,2\}^{\mathbb{Z}}$ and the existence of μ-equicontinuous points depends on the parameters $p(0), p(1), p(2)$ of this measure. In [5] it is shown that if $p(2) > p(1)$ then the probability that a 2 is never annihilated is positive and this implies that there exist μ-equicontinuous points. Nevertheless for this interesting example, there is no Bernoulli measure μ which is preserving by this automaton F. In this case a “natural” way to obtain an invariant measure μ' such that F_s is a μ'-equicontinuous CA is to use Cesàro mean of image by F_s of an appropriate Bernoulli measure using Theorem 1. The dynamical system $(\{0,1,2\}^{\mathbb{Z}}, F_s, \mu')$ we obtain is again a μ'-equicontinuous CA which is sensitive (without equicontinuous points) but the CA $(S(\mu'), F_s, \mu')$ where $S(\mu')$ is the topological support of μ' has always equicontinuous points since $S(\mu') = \{0,2\}^{\mathbb{Z}}$ and the restriction of F_s to $\{0,2\}^{\mathbb{Z}}$ is the identity map. Since it has more interest to consider a dynamical system $(S(\mu), F, \mu)$ instead of the system $(A^{\mathbb{Z}}, F, \mu)$ when we mix topological and measurable conditions, we will describe here a CA called F_e and a measure μ_c such that $(S(\mu_c), F_e)$ is sensitive and μ_c-equicontinuous.

Roughly, to have μ-equicontinuous points but no equicontinuous points requires that there exist some ‘perturbations’ that can move to infinity but the probability that these perturbations move to infinity is equal to zero. One way to get these properties for an automaton F and an invariant measure μ_c (obtained thanks to the results of Theorem 1) is that $(A^{\mathbb{Z}}, F)$ generates permanently “propagating structures” of all type of sizes. The “length of life” of the “propagating structures” depends on their sizes. This is roughly the dynamic of the following cellular automaton F_e.

3.3.1. Definition of the cellular automaton F_e

The automaton F_e we consider acts on $X = X^1 \times X^2$ where $X^1 = \{E_0, E_1, E_2, E_3, 0, R, L\}^{\mathbb{Z}}$ and $X^2 = \{0,1\}^{\mathbb{Z}}$. We define F_e as the composition of 3 other CA: $F_e = F_3 \circ F_2 \circ F_1$. To simplify, we write $\hat{E} = \{E_0, E_1, E_2, E_3\}$ and $\overline{E} = \{0, L, R\}$. We denote by $x = (x^1, x^2)$ any point $x \in X$ and by x_i^j the letter in position i of x^j $(1 \leqslant j \leqslant 2)$. Next we call $\mathbf{1}_S(x)$ the map which is equal to one if $x \in S$ and zero otherwise.

The automaton F_1 is the identity on X^1 and its restriction to X^2 came from the following block map f_1 of radius 3

$$f_1\big(x_{i-3}^2, x_{i-2}^2, x_{i-1}^2, x_i^2, x_{i+1}^2, x_{i+2}^2, x_{i+3}^2\big) = \mathbf{1}_{\{1\}}\big(x_{i-3}^2\big) \times \mathbf{1}_{\{1\}}\big(x_{i-2}^2\big) \times \mathbf{1}_{\{1\}}\big(x_{i-1}^2\big).$$

The automaton F_2 is still the identity on $X^0 \times X^1$ but its action on X^2 depends on X^1. The block map f_2 is defined by

$$f_2\begin{pmatrix} x_{i-2}^1, x_{i-1}^1, x_i^1, x_{i+1}^1, x_{i+2}^1 \\ x_{i-2}^2, x_{i-1}^2, x_i^2, x_{i+1}^2, x_{i+2}^2 \end{pmatrix} = \begin{pmatrix} x_i^1 \\ \bigvee_{j=0}^2 \mathbf{1}_{\{E_0\}}(x_{i-j}^1) \end{pmatrix}$$

where $\bigvee_{i=0}^2 \mathbf{1}_{\{E_0\}} x_{i-j}^1$ is equal to 1 if at least one x_{i-j}^1 is equal to 1 and equal to 0 in all the other cases.

The automaton F_3 is the identity map on X^2 and is defined thanks to a local rule f_3 on X^1.

When the central coordinate x_i is an element of $\overline{E}$, the block map f_3 is defined by the following rules:

$$\begin{aligned}
&f_3\big(x^1_{i-11},\ldots,x^1_i,\ldots,x^1_{i+11}\big)\\
&\quad = R \quad \text{if } x^1_{i-10},\ldots,x^1_i,\ldots,x^1_{i+m} = R0^{11+m} \text{ where } m=\min\big\{10,\min\{k-1 \mid x_{i+k}\in\hat{E}\}\big\}\\
&\quad = L \quad \text{if } x^1_{i-m},\ldots,x^1_i,\ldots,x^1_{i+10} = 0^{11+m}L \text{ where } m=\min\big\{10,\min\{k-1 \mid x_{i-k}\in\hat{E}\}\big\}\\
&\quad = R \quad \text{if } \exists 0\leqslant k,j\leqslant 9 \text{ such that } x^1_{i-j-k},\ldots,x^1_i,\ldots,x^1_{i+10} = E^*0^kL0^{j+10}\\
&\qquad \text{with } j+2k+1=10 \text{ and } E^*\in\hat{E}\\
&\quad = L \quad \text{if } \exists 0\leqslant k,j\leqslant 9 \text{ such that } x^1_{i-10},\ldots,x^1_i,\ldots,x^1_{i+j+k} = 0^{10+j}R0^kE^*\\
&\qquad \text{with } j+2k+1=10 \text{ and } E^*\in\hat{E}.
\end{aligned}$$

If the central coordinate x_i is an element of $\hat{E}$, for each $i \in \{0,1,2,3\}$, one has

$$\begin{aligned}
&f_3\big(x^1_{i-10},\ldots,E_i,\ldots,x^1_{i+10}\big) = E_{i+1}\\
&\quad \text{if } x^1_{i-k},\ldots,x^1_i = R0^{k-1}E_i \text{ with } 0\leqslant k\leqslant 9 \text{ where the addition `}i+1\text{' is modulo 4.}
\end{aligned}$$

For all the other cases where the central coordinate x_i is an element E^* in $\hat{E}$, we have $f_3(x^1_{i-10},\ldots,E^*,\ldots,x^1_{i+10}) = E^*$. In all other cases not described above we have, $f_3(x^1_{i-10},\ldots,x^1_i,\ldots,x^1_{i+10}) = 0$.

3.3.2. The invariant measure μ_c

Let S^1 be a subshift of finite type defined by the following list of accepted words: words of type LE_j0^k with ($1\leqslant j\leqslant 3$ and $k=170$); words of type $0E_l0^k$ with ($0\leqslant l\leqslant 3$ and $k=170$); words of type 0^l with $l\in\mathbb{N}$. A typical configuration in S^1 is

$$\ldots 00E_2000\ldots 00LE_000\ldots 00LE_100\ldots 00E_100\ldots 00E_300\ldots 00LE_000\ldots.$$

We denote by μ_1 the Parry measure on S^1 (see for example [4]), by $\delta_{\infty 0^\infty}$ the Dirac measure on the point ${}^\infty 0^\infty = \ldots 00000\ldots$ and we call μ_I the measure $\mu_1\times\delta_{\infty 0^\infty}$ on X. Since μ_I is a shift ergodic measure, using Proposition 1, Theorems 1 and 2, F_e is a μ_c-equicontinuous CA (with $\mu_c = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu_I\circ F_e^i$) if there exists a point x such that $\mu_I(B_r(x))>0$. Remark that $S^1\times^\infty 0^\infty$ is not an invariant set for F_e. In Section 3.3.3 we will characterize the topological support of μ_c by describing the action of F_3 on the non-invariant set S^1.

3.3.3. The dynamic of F_e

In this subsection, we describe the global dynamic of F_e by showing in first place the contribution of each of the 3 cellular automata F_1, F_2 and F_3.

The dynamic of F_1

The action of F_1 on X^2 is only the shift of consecutive sequences of letters “1” (that we call “trains of 1”) of one coordinate to the right and the destruction of the two last letters “1” at the left side of this *train*.

Action of F_1 on a configuration of X^2:

$$\ldots 01111110000\ldots \xmapsto{F_1} \ldots 00001111000\ldots \xmapsto{F_1} \ldots 00000001100\ldots \xmapsto{F_1} \ldots 0000000000\ldots.$$

Remark that a *train* of 1 with a length $2k+1$ will move of k coordinates to the right before collapsing.

The dynamic of F_2

The action of the cellular automaton F_2 is to 'create' a sequence of three letters '1' in X^2 when there is a letter E_0 in x_i^1.

Example:

$$x = \begin{matrix} x^1 & \ldots E_0 * * \ldots\ldots \\ x^2 & \ldots 00000 \ldots \end{matrix} \overset{F_2(x)}{\longmapsto} \begin{matrix} \ldots E_0 * * \ldots\ldots \\ \ldots 11100 \ldots \end{matrix} \overset{F_2(x)}{\longmapsto} \begin{matrix} \ldots E_0 * * \ldots\ldots \\ \ldots 11100 \ldots \end{matrix}.$$

The symbol $*$ replaces any letter in $\{0, L, R\}$.

Action of $F_2 \circ F_1$

If $F_e^i(x^1) = E_0$ for $0 \leqslant i \leqslant n-1$, then there is at least a *train of* 1 of length $n+3$ moving to the left of at least $\lceil \frac{n+3}{2} \rceil$ coordinates.

Example of the action of $F_2 \circ F_1$ on $x = \begin{smallmatrix} x^1 \\ x^2 \end{smallmatrix}$

$$\begin{matrix} x^1 = {}^\infty 0 E_0 * * * * * * * \ldots \\ x^2 = {}^\infty 0000000000 \ldots . \end{matrix} \overset{(F_2 \circ F_1)^n}{\Longrightarrow} \begin{matrix} {}^\infty 0 E_0 * * * * * * * \ldots & = F^n(x)^1 \\ \underbrace{{}^\infty 01111\ldots 1}_{n+3 \text{ times}} 000 \cdots & = F^n(x)^2 \end{matrix}.$$

Here the symbol $*$ replaces any letter in $\hat{E} \cup \overline{E}$.

The oscillator dynamic of F_3 and the subshifts S^1 and $S(\mu_c)$

First remark that $S(\mu_c) \subset w(S^1 \times^\infty 0^\infty, F_e)$ where

$$w\big(S^1 \times^\infty 0^\infty, F_e\big) = \lim_{n \to \infty} \bigcap_{j=0}^{n} \bigcup_{i=j}^{\infty} F_e^i\big(S^1 \times^\infty 0^\infty\big).$$

When we apply F_3 to a finite configuration of the subshift S^1, it is easy to see that only finite configurations of the type $0^k L E_j 0^l$ will be affected by the first iteration of the automaton F_3 where $k, l \in \mathbb{N}^2$ and $j \in [0..3]$. We have $F_3(\ldots 0^k L E_j 0^{l+m} \ldots) = \ldots L 0^{10} E_j 0^l \ldots$ and $F_3^2(\ldots 0^k L E_j 0^l \ldots) = \ldots L 0^{20} E_j 0^l \ldots$ where $m, k \geqslant 170$, $l \in \mathbb{N}$ and $0 \leqslant j \leqslant 3$. Remark that the L moves to the left indefinitely unless it would approach a letter E^* in $\hat{E}$. If there is less than nine 0 between the L and the E^*, the L disappears and an R appears at i coordinates to the right of E^* if the L was at $10-i$ coordinates to the right of the E^*. A similar process occurs when after some iterations, the R's appears and move to the right until they encounter a letter $E^* \in \hat{E}$. The letter R disappears, it appears an L which return to the left and the letter $E^* = E_j$ in the neighborhood become $E_{j+1 \bmod 4}$.

Let us see a typical evolution of an *oscillator*:

$$\ldots \underbrace{E^* 0 \ldots 0 \overset{\ldots\hookrightarrow}{R} 0 \ldots 0 E_i \ldots}_{\text{oscillator size } 50n} \overset{F_3^n}{\Longrightarrow} \ldots \underbrace{E^* 0 \ldots R 0^k E_i}_{\text{same oscillator}} \ldots \overset{F_3}{\Longrightarrow} \ldots \underbrace{E^* 0 \ldots 0 \overset{\hookleftarrow\ldots}{L} 0^l E_{i+1 \bmod 4}}_{\text{same oscillator}} \cdots$$

where $k \leqslant 9$, $n \geqslant 4$ and $l + k = 10$.

We will describe only the projection on X^1 of the subshift $S(\mu_c) = w(S^1, F_e) \times w({}^\infty 0^\infty, F_e)$. In $w(S^1, F_e)$:

(i) There is at least 170 letters in $\overline{E}$ between two E^*.
(ii) There is at most one letter $M \in \{L, R\}$ between two E^*.

(iii) At the right side of the last E^* to the right (if it exists) there are only letters 0.
(iv) At the left side of the last E^* (if it exists), there are only letters 0.
(v) There is no configurations of the type $E^*0^m E_0 *^l E^*$ where $*$ is in place of any letter in $\{0, R, L\}$ and $l \geqslant 170$.

We remark that the configurations of $S(\mu_c)$ can be generated by a finite automata which means that $S(\mu_c)$ is a sofic subshift.

Action of F_e

Since the action of F_e on letters $E^* \in \hat{E}$ is the identity or a permutation in $\hat{E}$, the set of configurations that contains an infinite number of letters in $\hat{E}$ has measure one in $S(\mu_c)$.

Under the action of $F_2 \circ F_1$, a configuration of the type $\ldots E_0 \ldots \times^\infty 0^\infty$ will generate in X^2 (at the same coordinate that E_0) a *train of* 1 until the E_0 will change in E_i (with $i \geqslant 1$). Under the action of F_e (see action of $F_2 \circ F_1$) a configuration $\ldots E^*0^l M0^k E_0 \ldots \times^\infty 0^\infty$ ($E^* \in \hat{E}$, $M \in \{R, L\}$) will produce a *train of* 1 of length $\mathcal{L}$ such that $\lfloor \frac{m}{5} \rfloor + 2 \leqslant \mathcal{L} \leqslant \lfloor \frac{m}{5} \rfloor + 4$ with $m = l + k + 1$. In the following we will choose $\lfloor \frac{m}{5} \rfloor + 2$ or $\lfloor \frac{m}{5} \rfloor + 4$ for the length $\mathcal{L}$ of this *train* of 1 according to the context.

Let see a typical dynamic of an *oscillator transmitter*

$$x = \begin{array}{l} {}^\infty 0 \ldots \underbrace{E^*0 \overset{\hookrightarrow}{\ldots\ldots} R0^9}_{\text{size } l} E_3 \ldots\ldots\ldots \\ {}^\infty 0 \ldots 0000 \ldots\ldots\ldots 00000 \ldots \end{array} \overset{F_e}{\Longrightarrow} \begin{array}{l} {}^\infty 0 \ldots \underbrace{E^*0 \overset{\hookleftarrow}{\ldots\ldots} L0E_0}_{\text{size } l} \ldots\ldots\ldots \\ {}^\infty 0 \ldots 0000 \ldots\ldots\ldots 00000 \ldots \end{array}$$

$$= \begin{array}{l} {}^\infty 0 \ldots \underbrace{E^*0 \overset{\hookleftarrow}{\ldots} L0}_{\text{size } l} E_0 \ldots\ldots\ldots \\ {}^\infty 0 \ldots 0000 \ldots\ldots 0000 \ldots \end{array} \overset{F_e^{\lceil (l/5) \rceil}}{\Longrightarrow} \begin{array}{l} {}^\infty 0 \ldots \underbrace{E^*0 \overset{\hookleftarrow}{\ldots\ldots} L00E_1}_{\text{size } l} \ldots\ldots\ldots\ldots \\ {}^\infty 0 \ldots 000 \ldots\ldots 0000 \underbrace{\overset{\cdots\Rightarrow}{111 \ldots 11}}_{\lfloor \frac{l}{5} \rfloor + 4} 00 \ldots \end{array}.$$

We call *oscillator transmitter* of size $l + m + 1$, any pattern of the form $E^*0^l M0^m E^*$ and *void oscillator* any pattern of the form $E^*0^k E'^*$ where $k, l + m + 1 \geqslant 170$, each E^* belongs to $\hat{E}$, each $E'^* \in \{E_1, E_2, E_3\}$ and $M \in \{R, L\}$. Remark that *oscillators* of the type $E^*0^k E_0$ do not belong to the language of $S(\mu_c)$.

For each $l \in \mathbb{N}$, denote by $\mathcal{C}_l[i]$ the union of all the sets $[U]_i \times X^2 \subset X$ where $U = [E^*0^j M0^k E^*]_i$ is a cylinder in X^1, M replaces one letter in $\{L, R\}$, each E^* are any letters in $\hat{E}$ and j, k, l verify $j + k + 1 = l$. Let $\overline{\mathcal{C}_k[i]}$ be the union of sets $([E^*0^k E'^*]_i \times X^2)$ where E^* replaces any letter in $\hat{E}$, $k \geqslant 170$ and $E'^* \in \{E_1, E_2, E_3\}$. We call respectively *oscillators transmitter* in position i and *void oscillators* in positions i the sets $\mathcal{C}_l[i]$ and $\overline{\mathcal{C}_l[i]}$. Remark that $\mu_I(\mathcal{C}_l[i]) = \mu_I([E^*0^{i-1} RE^*] \times^\infty 0^\infty)$. Without taking into consideration the position of the *oscillators*, we will call respectively $\mathcal{C}_l$ and $\overline{\mathcal{C}_l}$ the *oscillators transmitter* and *void oscillators* of size l.

Propagation of trains of 1 *generated by oscillator transmitter*

Since an *oscillator transmitter* $\mathcal{C}_l$ generate a *train of* 1 with a length $\mathcal{L}$ at most equal to $\lfloor \frac{l}{5} \rfloor + 4$ and that *train* looses 2 elements when it moves of 1 coordinate, it can influences some patterns situated at $(\lfloor \frac{l}{5} \rfloor + 4) + (\lfloor \frac{l}{5} \rfloor + 4)/2 = \lfloor \frac{3l}{10} \rfloor + 6$ coordinates to the right of the right extremity of the *oscillator* if there is no concatenation process with another train created by other *oscillators transmitter*. Since the proofs of Propositions 7 and 8 only require the understanding of the propagation of *trains of* 1 when the initial configuration is in $S^1 \times^\infty 0^\infty$, we will only consider concatenation process with trains generated by *oscillators transmitter* situated to the right side. For two consecutive oscillators $\mathcal{C}_l \mathcal{C}_m$, a *train of* 1 generated by the first one $\mathcal{C}_l$ will reach the coordinates situated under the beginning of $\mathcal{C}_m$ only if $m \leqslant \lfloor \frac{3l}{10} \rfloor + 6$ (iq1).

When a *train* will cross the l coordinates of another *oscillator*, it will loose $2l$ elements and can incorporate at most $\lfloor 2l/5 \rfloor + 8$ others ($\lfloor l/5 \rfloor + 4$ at the beginning and $\lfloor l/5 \rfloor + 4$ at the end of the *train*). Now consider a sequence of three *oscillators* $\mathcal{C}_l \mathcal{C}_m \mathcal{C}_n$ where m and n are fixed and l is the

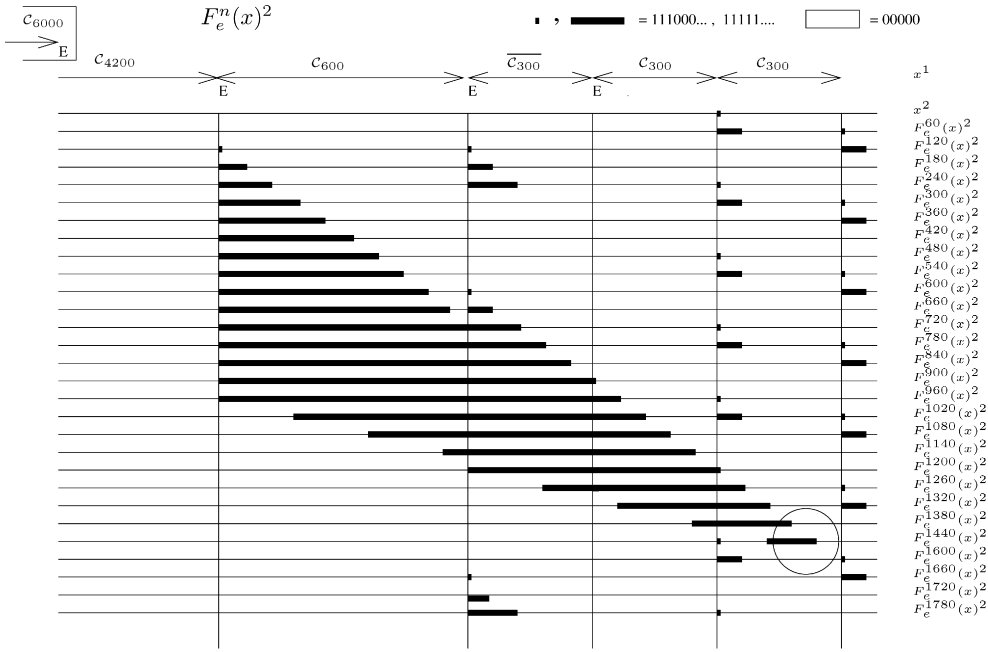

Fig. 1. An illustration of the dynamic of F_e on 5 *oscillators* $\mathcal{C}_{4200}$, $\mathcal{C}_{600}$ and three $\mathcal{C}_{300}$ and the resulting dynamic of *train of* 1 in X^2. For simplification, we do not specify the states of the *oscillators* and their evolution because the interesting part of their dynamic can be deduced from the evolution of the *train of* 1 in X^2. Each line represents the sequence of images of x^2 after every 60 iterations of F_e. The black horizontal lines represent the *trains of* 1 and blank ones, the sequences of 0. The extremity of the *oscillators* are delimited by arrows. The first *oscillator* of size 300 is a *void oscillator* which never generates any *trains* of 1 in X^2. To the left side, there is one large *oscillator transmitter* of size 6000 which is in a non-emitting state (the last letter is not an E_0). Remark that the non 'emitting period' of the *oscillators transmitter* last 3 times more than the 'emitting' one. The circle shows the end of the propagation of the train generated by $\mathcal{C}_{4200}$.

minimum size of the first *oscillator* in order that its train will reach $\mathcal{C}_n$. The *train* of length at most $\lfloor l/5\rfloor + 4$ will gain at most $\lfloor 2m/5\rfloor + 8$ and will loose at least $2m + 2(n - (\lfloor m/5\rfloor + 4))$ elements when the head of the train has crossed the two *oscillators* (we take into consideration that the head of the train may eventually progress of $\lfloor m/5\rfloor + 4$ coordinates when it passes under $\mathcal{C}_m$). We obtain that $l \geqslant 6m + 10n - 100$. More generally for all sequences of $n+1$ consecutive *oscillators* $\mathcal{C}_{l_n}\mathcal{C}_{l_{n-1}}\dots\mathcal{C}_{l_0}$, any change of the state of the first *oscillator* $\mathcal{C}_{l_n}$ will affect the *train of* 1 situated in the *oscillator* $\mathcal{C}_{l_0}$ if $l_n \geqslant \sum_{i=n-1}^{1} 6l_i + 10l_0 - 80n + 60$ (iq2).

Recall that 170 is the minimum size of an *oscillator*. This minimum size is required to simplify the proof of Proposition 8 which uses quantitative arguments on "flows of trains of 1." For a typical illustration of this dynamic see Fig. 1.

3.3.4. *The topological and measurable properties of F_e*

Proposition 7. *The dynamical system $(S(\mu_c), F_e)$ is a μ_c-equicontinuous cellular automaton.*

Proof. From the discussion of Section 3.3.2, we only need to show that there exist a point x and an integer $m \geqslant r$ such that $\mu_I(B_m(x)) > 0$. Remark that since $\mu_I(X) = \mu_I(S^1 \times^\infty 0^\infty)$ we will take into consideration only configuration in $S^1 \times^\infty 0^\infty$ in this proof. Let $x_0 = ({}^\infty 0^\infty, {}^\infty 0^\infty)$ and for each integer $k \geqslant 1$, pick a point $x_k \in \overline{\mathcal{C}_k[-k-1-r]} \cap B_r(x_0) \cap X^1 \times^\infty 0^\infty$. We will prove that there exist integers $k > 0$ such that $\mu_I(B_r(x_k)) > 0$ by showing that $\mu_I(\overline{\mathcal{C}_k[-k-1-r]} \cap B_r(x_k)^{\complement}) < \mu_I(\overline{\mathcal{C}_k[-k-1-r]})$ (where $B_r(x_k)^{\complement}$ is the complement of $B_r(x_k)$). The set $\overline{\mathcal{C}_k[-k-1-r]} \cap B_r(x_k)^{\complement}$ is the set of points that contains *oscillators transmitter* in the left side of $\overline{\mathcal{C}_k[-k-1-r]}$ that are able to generate *trains* *of* 1 which move to the right and cross this *void oscillator* $\overline{\mathcal{C}_k[-k-1-r]}$ (the *trains* of "1" may enter in the central coordinates $([-r,r])$ and in this case the point does not belong to $B_m(x_0) = B_m(x_k)$). Now, consider an *oscillator transmitter* of size l in position $-p-l-k-r-1$: $\mathcal{C}_l[-p-l-k-1-r]$. Denote by $\mathbf{S}(p,k)$ the minimum size of the *oscillator* $\mathcal{C}_l[-p-l-k-1-r]$ (whose the right extremity is situated at p coordinates to the left of $\overline{\mathcal{C}_k[-k-1-r]}$) in order that it can produce *trains of* 1 that cross completely the *void oscillator* $\overline{\mathcal{C}_k[-k-1-r]}$. From the discussion in Section 3.3.3 about the propagation of *train* of 1, it follows that $\overline{\mathcal{C}_k[-k-1-r]} \cap B_r(x_k)^{\complement}$ is equal to $\mathbb{S}_k = \{\bigcup_{i=\mathbf{S}(0,k)}^{\infty} \mathcal{C}_i[-i-k-1-r] \bigcup_{p=l_{\mathbf{m}}}^{\infty} \{\bigcup_{j=S(p,k)}^{\infty} \mathcal{C}_j[-j-p-k-1-r]\}\} \cap \overline{\mathcal{C}_k[-k-1-r]}$ where $l_{\mathbf{m}} = 170$ be the minimum size of the *oscillators* in $S(\mu_I)$.

Next we claim that there exists a real $M \geqslant 0$ such that for all integers $k \geqslant l_{\mathbf{m}}$ one has $\mu_I(\overline{\mathcal{C}_k[-k-1-r]} \cap B_r(x_k)^{\complement}) = \mu_I(\mathbb{S}_k) \leqslant \mu_I(\overline{\mathcal{C}_k[-k-1-r]}) \times \eta(k) \times M$ with $\lim_{k\to\infty} \eta(k) = 0$. Remark that since $\mu_I(\overline{\mathcal{C}_k[-k-1-r]}) > 0$, the proof of this claim will finish the proof.

From Section 3.3.3 a *train* of 1 generated by an *oscillator* $\mathcal{C}_{l_n}$ will cross the $n-1$ *oscillator* $\mathcal{C}_{l_{n-1}} \ldots \mathcal{C}_{l_1}$ and reach $\overline{\mathcal{C}_{l_0}}$ if $l_n \geqslant \sum_{i=n-1}^{1} 6l_i + 10l_0 - 80n + 60$ (iq2). Remark that the last *oscillator* is $\overline{\mathcal{C}_k[-k-1-r]}$ which implies $l_0 = k$ and that the number of *oscillators* $n \leqslant \frac{p}{l_{\mathbf{m}}} = \frac{p}{170}$. Using (iq2) we obtain that $\mathbf{S}(p,k) \geqslant 6p + 10k - 80(\frac{p}{l_{\mathbf{m}}}) + 60 \geqslant 5p + 10k + 60$ (iq3).

Since μ_I is the product of the Parry measure on the mixing subshift of finite type S^1 (see [4]) and the Dirac measure on ${}^{\infty}0^{\infty}$, then there exist a real $0 < q < 1$ and a positive integer $\mathcal{L}$ such that $\forall m \geqslant 0$ one has $\mu_I([u_0 \ldots u_m] \times^{\infty} 0^{\infty}) \leqslant q^{\lfloor \frac{m+1}{\mathcal{L}} \rfloor}$.

To prove the claim put $\eta(k) = \mu_I(\mathcal{C}_{\mathbf{S}(0,k)}[-S(0) - k - 1 - r])$ and using the lower bound (iq3) for $\mathbf{S}(0,k)$, we obtain that $\eta(k) \leqslant \mu_I(\mathcal{C}_{10k+60}[-10k - 60 - k - 1 - r]) \leqslant q^{\lfloor \frac{11k+59}{\mathcal{L}} \rfloor}$ which implies that $\lim_{k\to\infty} \eta(k) = 0$. Using again (iq3) for $\mathbf{S}(p,k)$, we obtain that $\mu_I(\bigcup_{i=\mathbf{S}(0,k)}^{\infty} \mathcal{C}_i[-i-k-1-r]) \leqslant \eta(k) \sum_{i=1}^{+\infty} q^{\lfloor \frac{i}{\mathcal{L}} \rfloor}$. Remark that the last and the following inequality will give us an upper bound for the measure of $\mathbb{S}_k$:

$$\mu_I\left(\bigcup_{p=l_{\mathbf{m}}}^{\infty}\left\{\bigcup_{j=\mathbf{S}(p,k)}^{\infty} \mathcal{C}_j[-j-p-k-1-r]\right\}\right) \leqslant \eta(k) \sum_{i=1}^{+\infty} q^{\lfloor \frac{i}{\mathcal{L}} \rfloor}\left(\sum_{j=1}^{+\infty} q^{\lfloor \frac{j}{\mathcal{L}} \rfloor}\right).$$

It follows that we can prove the claim and consequently finish the proof taking

$$M = \sum_{i=1}^{+\infty} q^{\lfloor \frac{i}{\mathcal{L}} \rfloor} + \sum_{i=1}^{+\infty} q^{\lfloor \frac{i}{\mathcal{L}} \rfloor}\left(\sum_{j=1}^{+\infty} q^{\lfloor \frac{j}{\mathcal{L}} \rfloor}\right) < +\infty. \qquad \square$$

Proposition 8. *The cellular automaton* $(S(\mu_c), F_e)$ *is sensitive* (*has no equicontinuous points in the topological support* $S(\mu_c)$).

Proof. If we suppose that there exists an equicontinuous point $x \in S(\mu_c)$, there must exist an integer m such that $C_m(x) \subset B_0(x)$. First suppose that if for all positive integers n, there exist integers $i > n$, such that $(F^i(x))_0^2 = 0$, then there exists $y \in C_m(x)$ such that $(F^i(y))_0^2 = 1$ (it is always possible to choose a point $y \in C_m(x)$ that contains a large enough *oscillator transmitter* at the left side of $-m$ that sent a *train of* 1 that can arrived at coordinate 0 after i iterations). This contradict the hypothesis $C_m(x) \subset B_0(x)$ and it follows that if there exist $x \in X$ and $m > 0$ such that $C_m(x) \subset B_0(x)$, then there exists a positive integer n such that for all $y \in C_m(x)$ and $i \in \mathbb{N}$ one has $(F^{i+n}(y))_0^2 = 1$ (condition $(*)$). Then suppose that there exists $m \in \mathbb{N}$ such that $C_m(x) \subset B_0(x)$. Since the point $z := {}^{\infty}0^{\infty}(-\infty, -m-1)x(-m, \infty)$ belongs to $C_m(x)$ and by hypothesis $(F^{i+n}(z))_0^2 = 1$ for all $i \in \mathbb{N}$, we obtain that the pattern $x(-m, -1)$ contains a finite sequence of consecutive *oscillators* $\mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}}, \ldots, \mathcal{C}_{l_0}$ that generates a "continuous flow of letters 1." To finish the proof we need to show that this finite sequence of *oscillators* does not exist. In order to do that, we will consider the propagation of *trains of* 1 generated by a finite sequence of k *oscillators* $(k \in \mathbb{N})$. (See Fig. 2.)

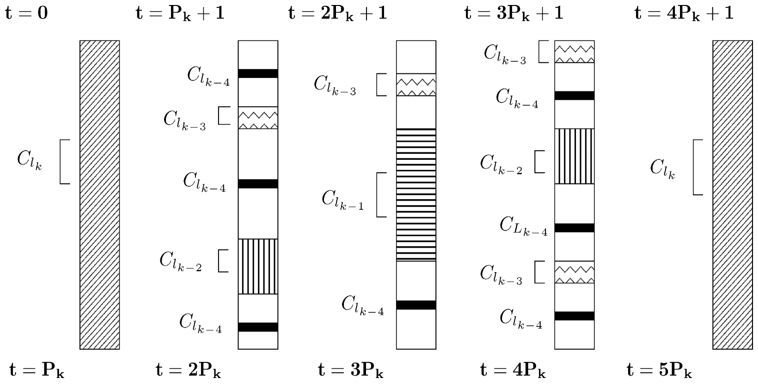

Fig. 2. The flow of 1 generated by a sequence of k *oscillators*. The parts not in white color represents the successive flows of letters 1 due to the trains which come from the oscillators transmitters $\mathcal{C}_{l_k}, \mathcal{C}_{l_{k-1}}, \ldots, \mathcal{C}_{l_{k-4}}$.

Let P_i ($0 \leqslant i \leqslant k$) be an upper bound of the number of iterations needed for a *train of* 1 generated by the *oscillator* $\mathcal{C}_{l_i}$ to cross completely the coordinate 0. We consider that there is no concatenations with other *trains* coming from the left but possible concatenations with *trains* generated by *oscillators* situated to the right. The value of P_i depends on the difference between the lost and gained letters when the left extremity of the *train* arrived in coordinate 0. It follows that $P_i \geqslant \lfloor \frac{l_i}{5} \rfloor + 4 - s_i$ where $s_i = \sum_{j=0}^{i-1} 2l_j - \sum_{j=0}^{i-1} 2(\lfloor \frac{l_j}{5} \rfloor + 4) = \sum_{j=0}^{i-1} \lfloor \frac{8}{5} l_j \rfloor - 8i$. Remark that $s_i \geqslant 0$ if $\forall 0 \leqslant j \leqslant i-1$, $l_j \geqslant 5$. As $l_{\mathbf{m}} = 170$ we obtain that $P_i \leqslant \lfloor \frac{l_i}{5} \rfloor + 4$ for all $0 \leqslant i \leqslant k$. Without loosing generalities, we can suppose that the *train of* 1 generated by the first *oscillator* of size l_k arrive in coordinate 0 at $t = 0$ and last at most P_k iterations. For time $t = P_k + 1$ to $4 \times P_k$, there is no *train of* 1 due to this first *oscillator* that passes through the central coordinate (because $P_i \geqslant \lfloor \frac{l_k}{5} \rfloor + 2$ and the *oscillator* $\mathcal{C}_{l_k}$ stop to generate *train of* 1 for at least $3(\lfloor \frac{l_k}{5} \rfloor + 2)$ iterations). The *train* of 1 generated by the second oscillator from the left: $\mathcal{C}_{l_{k-1}}$ last at most P_{k-1} iterations and its effect stops for a period of $3P_{k-1}$ in the interval time $t = P_k + 1$ to $t = 4P_k$. Clearly, if P_{k-1} is small enough, between $t = P_k + 1$ and $t = 4P_k$, there is at least one interval of length at least $3P_{k-1}$ that will be not affected by the two first *oscillators*. This interval is minimum when the *train* of 1 generated by the second *oscillator* pass exactly in the middle of the interval $[P_k + 1, 4P_k]$, between 2 *trains* of the first *oscillator*. In this case the condition of non-existence of "continuous flow" is that $3P_k - (2 \times 3 + 1)(P_{k-1}) \geqslant 0 \Leftrightarrow P_{k-1} \leqslant \frac{3P_k}{7}$. Hence repeating the same process if $P_{i-1} \leqslant \frac{3P_i}{7}$ ($0 \leqslant i \leqslant k$), there will always remain a blank interval and the "flow of 1" would not be continuous. Since for all $0 \leqslant i \leqslant k$ one has $P_i \leqslant \lfloor \frac{l_i}{5} \rfloor + 4$, it follows that the condition $\lfloor \frac{l_{i-1}}{5} \rfloor + 4 \leqslant \frac{3}{7}(\lfloor \frac{l_i}{5} \rfloor + 2)$ (iq4) also implies that the "flow of 1" is not continuous. Remark that we take a lower bound for P_i and upper bound for P_{i-1} which leads to a stronger condition on the minimum size l_m. Then by simplification of (iq4) we get that $3l_i - 7l_{i-1} \geqslant 110$. Using (iq1) $\equiv (l_{i-1} \leqslant \lfloor \frac{3l_i}{10} \rfloor + 6)$ that gives the minimum condition for a train generated by an *oscillator* $\mathcal{C}_{l_i}$ to cross the following one $\mathcal{C}_{l_{i-1}}$ we obtain that if $3l_i - 7(\lfloor \frac{3l_i}{10} \rfloor + 6) \geqslant 110$ (iq5) then the condition $3l_i - 7l_{i-1} \geqslant 110$ (iq6) remains true. The simplification of inequality (iq5) leads to $l_i \geqslant \frac{1520}{9} \approx 168.88$. Since we have chosen $l_{\mathbf{m}} = 170$ as the minimum size of the *oscillators*, there is no equicontinuous points in $(S(\mu_c), F_e)$ which finishes the proof. $\square$

The dynamical system (X, F_e) has equicontinuous points since patterns of the type $E^* 0^k E_0$ (with $E^* \in \hat{E}$) will produce a continuous flow of letters 1. Note that we can construct F'_e a CA similar to F_e with a more complex local rule such that the initial measure μ'_I is the uniform measure on X and the invariant measure obtained by Cesàro means is similar to the measure μ_c used in our example. In this case (X, F'_e, μ_c) and $(S(\mu_c), F'_e, \mu_c)$ both have μ_c-equicontinuous points but no equicontinuous points.

3.4. Questions

- Is it possible to find a sensitive, μ-equicontinuous and μ-invariant CA when μ is the uniform measure? Or more generally when the topological support $S(\mu)$ of the F-invariant measure is a mixing subshift of finite type?
- To simplify the proof of Proposition 8, we have taken a lot of upper bounds (for example the use of (iq1) is very strong). We wonder what is the minimum size for the *oscillators* such that $(S(\mu), F_e)$ cannot produce equicontinuous points?
- From inequality (iq3) in Proposition 7, it is possible to see that for each point x and $m \in \mathbb{N}$ such that $\mu_c(B_m(x)) > 0$, we have $B_m^{\mu_c}(x) = B_m(x)$. What are the conditions on the minimum size of the counters l_m in order to loose this property? In this case is it possible that there are no equicontinuous points? Remark that for some points x, $B_m^{\mu}(x) \neq B_m(x)$ in the Gilman's example of μ-equicontinuous CA given in the beginning of Section 3.3.
- Is there exist a μ-invariant and μ-equicontinuous CA such that $(S(\mu), F)$ has no equicontinuous points and there exist $m \in \mathbb{N}$ and a point $x \in A^{\mathbb{Z}}$ with $B_m(x) \neq B_m^{\mu}(x)$ and $\mu(B_m(x)) > 0$?
- More generally, what type of dynamic characterizes sensitive and μ-equicontinuous CA $(S(\mu), F)$ and how common is this behavior that seems to appear in different simulations of one-dimensional CA?

Acknowledgments

The author wishes to acknowledge the NSERC Discovery Grant #562620, the CNPq and the Department of Mathematics at Trent University in which a part of the work have been done.

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