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Applied Mathematics Letters 14 (2001) 105–108

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Local Search Structure in the Symmetric Travelling Salesperson Problem under a General Class of Rearrangement Neighborhoods

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(Received October 1999; accepted November 1999)

Abstract—The symmetric travelling salesperson problem with n cities (1-STSP) possesses no arbitrarily poor local optima for search neighborhoods defined by arbitrary unions of conjugacy classes in the symmetric group on n letters, $S(n)$. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—TSP, Symmetric group, Rearrangement neighborhood, Conjugacy class.

Colletti and Barnes [1] detailed how the conjugacy class of n cycles in $S(n)$ may be viewed as the set of all solutions, or tours, to the symmetric travelling salesperson problem with one agent (1-STSP). Large 1-STSPs overwhelm exact solution methods, causing recourse to heuristic methods like the tabu search metaheuristic [2] which iteratively chooses a new solution from among those in the neighborhood of a previously obtained incumbent solution. In this paper, we use group theory to reveal a useful property common to a *general class of rearrangement neighborhoods*.

In earlier work, Grover [3] and Codenotti and Margara [4,5] showed that four specific elementary 1-STSP neighborhoods—2-city swap, 2+3-new-change, 3-new-change, and 2-opt—satisfy a simple homogeneous linear difference equation

$$\nabla^2 f + \frac{k}{n} f = 0, \tag{1}$$

where n is the number of cities; $k > 0$ is a constant which depends on the neighborhood; and for tour p, $f(p)$ is the tourlength of p minus the average tourlength of all tours, μ. Finally, $\nabla^2 f(p)$ denotes the average of all $f(q) - f(p)$, where q is a neighbor of p. Grover [3] shows that the

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PII: S0893-9659(00)00120-8

tourlength of any local optimum of any duplicative 1-STSP neighborhood (not only the above four) does not exceed μ, i.e., arbitrarily poor local optima cannot exist for such neighborhoods. A duplicative neighborhood is one whose construction method admits duplicate members; and henceforth, we presume such neighborhoods.

Let $|C|$ denote the cardinality of $C \subseteq S(n)$. For any n-cycle $p \in S(n)$, define $p^C \equiv \{p^c = c^{-1}pc : c \in C\}$ to be the *C-rearrangement neighborhood* of p. Note, that each element of C contributes a neighbor, and so p^C is a duplicative neighborhood since there may be distinct $x, y \in C$ such that $p^x = p^y$. When C is a conjugacy class of $S(n)$, [1] showed that in the multiple asymmetric TSP (m-ATSP), the summed neighbor tourlengths—denoted weight[tourlength,p^C]—is linear in the tourlengths of p and its inverse. Thus, for the m-STSP and $n > 3$, this weight is linear in tourlength(p)

$$\text{weight}\left[\text{tourlength}, p^C\right] = (2c_2 + (n-4)c_4 + 2c_6)\,\text{Sum}\,D + [c_1 - 2c_2 + 2c_4 + c_5 - 2c_6]\,\text{tourlength}(p), \tag{2}$$

where Sum$D$ is the sum of all elements in the arbitrary symmetric zero-diagonal n by n distance matrix, and c_i is described in [1]. The neighborhood f-weight of p, denoted weight$[f, p^C]$, is the summed f values of p's neighbors.

A *conjugacy class* of $S(n)$ consists of all permutation having a given cycle structure. Thus, the set of all two-cycles in $S(4)$ is the conjugacy class $C_{(x,x)} \equiv \{(1,2),(1,3),(1.4),(2,3),(2,4),(3,4)\}$, while the conjugacy class of three-cycles is $C_{(x,x,x)} \equiv \{(2,3,4),(2,4,3),(1,2,3),(1,2,4),(1,3,2),(1,3,4),(1,4,2),(1,4,3)\}$. If p is a derangement in $S(n)$, i.e., a permutation that moves all n letters, then the neighborhood of p consisting of all possible two-letter swaps on p is given by $p^{C_{(x,x)}}$. Likewise, all possible three-letter swaps are given by $p^{C_{(x,x,x)}}$.

However, the set of all four-letter swaps is given by p^C, where $C = C_{(x,x,x,x)} \cup C_{(x,x)(x,x)}$. In general, p^C are all k-letter rearrangements on derangement p, where C is the union of all conjugacy classes whose cycle structures move k letters. Again, p^C may contain duplicates since each element in C creates a neighbor. Until otherwise stated, C henceforth denotes a single conjugacy class of $S(n)$.

When applied to the n-city 1-STSP, equation (2) may be used to prove *any* C-rearrangement neighborhood satisfies equation (1), and so has no arbitrarily poor local optima. To see this, simply manipulate equation (1) to obtain

$$k = n - \frac{n}{\nu}\frac{\text{weight}\left[f, p^C\right]}{f(p)} > 0, \tag{3}$$

where $\nu = |p^C| = |C|$, a constant $\forall p \in S(n)$. Because equation (1) presumes k is constant, then so is

$$\rho = \frac{\text{weight}\left[f, p^C\right]}{f(p)} < \nu. \tag{4}$$

Since this argument can be reversed, it follows that a constant $\rho < \nu$ implies the 1-STSP C-rearrangement neighborhood satisfies equation (1).

To show that any C-rearrangement neighborhood respects equation (1), we need only show its ρ value satisfies equation (4). First, it is well known that SUM$D = (n-1)\mu$, and so

$$\begin{aligned}\text{weight}\left[f, p^C\right] = \text{weight}\left[\text{tourlength}, p^C\right] - |C|\mu = [2c_2 + (n-4)c_4 + 2c_6]\,(n-1)\mu \\ + [c_1 - 2c_2 + 2c_4 + c_5 - 2c_6]\,\text{tourlength}(p) - |C|\mu,\end{aligned} \tag{5}$$

where as stated earlier

$$f(p) = \text{tourlength}(p) - \mu. \tag{6}$$

For the reader's convenience, the arc transformation table from [1] is given in Table 1.

Table 1. Arc transformation table.

Transformed arc	τ	$[x, \neq x^p]$	$[\neq x, x^p]$	$[\notin \{x, x^p\}$ $\notin \{x, x^p\}]$	$[x^p, x]$	$[x^p, \neq x]$	$[\neq x^p, x]$
# C-elements	c_1	c_2	$c_3 (= c_2)$	c_4	c_5	c_6	$c_7 (= c_6)$
Arc variates	1	$n-2$	$n-2$	$P(n-2,2)$	1	$n-2$	$n-2$

Table 1 presents the seven transformation of p-arc $\tau = [x, x^p]$ under conjugation by C. c_k elements in C transform τ into a specific column header arc α, and the bottom row gives the number of variants of α. For example, if $\tau = [1,2]$, then c_2 elements in C change τ into $\alpha = [1,3]$, one of the $n-2$ variants $\{[1,3],[1,4],\dots,[1,n]\}$. $P(n,m)$ denotes the number of permutations on n choose m letters.

Thus, $|C|$ is the summed pairwise products of corresponding elements from the second and third rows of Table 1

$$\begin{aligned} |C| &= 1^* c_1 + (n-2)^* c_2 + (n-2)^* c_3 + P(n-2,2)^* c_4 + 1^* c_5 + (n-2)^* c_6 + (n-2)^* c_7 \\ &= c_1 + 2(n-2)c_2 + (n-2)(n-3)c_4 + c_5 + 2(n-2)c_6. \end{aligned} \tag{7}$$

Substituting equations (5)–(7) into equation (4) and simplifying yields the constant

$$\rho = c_1 - 2c_2 + 2c_4 + c_5 - 2c_6 < |C|. \tag{8}$$

Thus, any 1-STSP C-rearrangement neighborhood satisfies equation (1) with an associated k coefficient

$$k = n\left[1 + \frac{2(c_2 - c_4 + c_6) - c_1 - c_5}{|C|}\right] > 0. \tag{9}$$

In turn, this neighborhood has no arbitrarily poor local optima. Now, if C is the union of arbitrary conjugacy classes $\{C_i\}_{i \in I}$ in $S(n)$, then

$$\frac{\text{weight}\left[f, p^C\right]}{f(p)} = \sum_{i \in I} \frac{\text{weight}\left[f, p^{C_i}\right]}{f(p)}. \tag{10}$$

Since each summand in equation (10) is constant and less than its $|C_i|$, their sum must be constant and less than $|C|$, the sum of all the $|C_i|$. Thus, this C-rearrangement neighborhood satisfies equation (1) with k coefficient

$$k = n - n\frac{\text{weight}\left[f, p^C\right]/f(p)}{\sum_{i \in I} |C_i|} = n\left[1 - \frac{\sum_{i \in I}\left(\text{weight}\left[f, p^{C_i}\right]/f(p)\right)}{\sum_{i \in I} |C_i|}\right]. \tag{11}$$

This result generalizes upon and proves the (specific) conjecture of [4] that says the 1-STSP $\{2 + \cdots + m\}$-letter rearrangement move satisfies equation (1).

In closing, results found here are subsumed by more general m-STSP results presented in [6]. For example, equation (2) holds for the m-STSP, i.e., p need not be an n-cycle (a solution to the 1-TSP). All other equations hold for any deranged m-STSP, i.e., p is any derangement (an n-cycle is a special type of derangement). The authors focused upon the more familiar 1-STSP because once understood, this paper's application to the m-STSP becomes clear.

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