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ON BALANCED COMPLEMENTATION FOR REGULAR t-WISE BALANCED DESIGNS*

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Vanstone has shown a procedure, called r-complementation, to construct a regular pairwise balanced design from an existing regular pairwise balanced design. In this paper, we give a generalization of r-complementation, called balanced complementation. Necessary and sufficient conditions for balanced complementation which gives a regular t-wise balanced design from an existing regular t-wise balanced design are shown. We characterize those aspects of designs which permit balanced complementation. Results obtained here will be applied to construct regular t-wise balanced designs which are useful in Statistics.

1. Introduction

A *t-wise balanced design* (denoted by *t-BD*) is a pair $(V, \mathscr{B})$, where V is a v-set (called points) and $\mathscr{B}$ is a collection of subsets of V (called blocks), satisfying the following condition:

> For any t-subset T of V, the number of blocks containing T is λ_t which is independent of the t-subset T chosen.

If, for any s-subset S $(s \leq t)$, the number of blocks containing S is λ_s which is independent of the s-subset S chosen, then the design is called a *regular t-wise balanced design*. When $t = 2$, the design is called a *regular pairwise balanced design* (regular *PBD*) or an (r, λ)-*design* $(r = \lambda_1,\ \lambda = \lambda_2)$.

Vanstone [4] has shown a procedure, called r-complementation, to construct a regular *PBD* from an existing regular *PBD*. The r-complementation is the procedure defined as follows:

Let $(V, \mathscr{B})$ be a regular *PBD*. For any point $x \in V$, let $\mathscr{B}_x$ be a collection of blocks containing x. Consider

$$V^* = V - \{x\}$$

and

$$\mathscr{B}^* = \{V - B : B \in \mathscr{B}_x\} \cup (\mathscr{B} - \mathscr{B}_x).$$

Then the pair $(V^*, \mathscr{B}^*)$ is also a regular *PBD* with new parameters $v^* = v - 1$, $r^* = 2(r - \lambda)$ and $\lambda^* = r - \lambda$.

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The r-complementation is useful to construct new (r, λ)-designs (see e.g. [3]).

In this paper, we give a generalization of r-complementation in Sections 2 and 3, called balanced complementation. Its definition is given in Section 2 for regular PBD's and in Section 3 for regular t-BD's ($t \geq 3$), respectively. Necessary and sufficient conditions for balanced complementation which gives a regular t-BD from an existing regular t-BD are shown in Section 2 for $t=2$ and in Section 3 for $t \geq 3$, respectively. In Section 3 we characterize those aspects of designs which permit balanced complementation. Results obtained here will be applied to construct regular t-BD's which are useful in Statistics (see e.g. [2]).

2. Balanced complementation for a regular *PBD*

We generalize r-complementation by the following theorem:

Theorem 2.1. *Let $(V, \mathcal{B})$ be a regular PBD. Consider $V^* = V$ and $\mathcal{B}^* = \{V - B : B \in \mathcal{B}'\} \cup (\mathcal{B} - \mathcal{B}')$, where $\mathcal{B}' \subset \mathcal{B}$. Then the pair $(V^*, \mathcal{B}^*)$ is also a regular PBD if and only if each point of V is contained in exactly the same number of blocks in $\mathcal{B}'$.*

Proof. Assume that each point of V is contained in exactly r' blocks in $\mathcal{B}'$. Let $|\mathcal{B}'| = b'$. It is easy to see that each point of V^* is contained in exactly $r + b' - 2r'$ blocks in $\mathcal{B}^*$. For any pair $\{x, y\}$ of V, let b_1 be the number of blocks in $\mathcal{B}'$ containing x and y and let b_2 be the number of blocks in $\mathcal{B}'$ containing neither x nor y, and let b_3 be the number of blocks in $\mathcal{B} - \mathcal{B}'$ containing x and y. Then we have

$$b_1 + b_3 = \lambda$$

and

$$b_2 - b_1 = b' - 2r'.$$

From these equations, we can show that each pair of V^* is contained in exactly $\lambda + b' - 2r'$ blocks in $\mathcal{B}^*$. Therefore, the above pair $(V^*, \mathcal{B}^*)$ is a regular PBD.

Let $(V^*, \mathcal{B}^*)$ be a regular PBD. For some $x \in V$, let c_x be the number of blocks in $\mathcal{B}'$ containing x and let d_x be the number of blocks in $\mathcal{B} - \mathcal{B}'$ containing x. Since $(V, \mathcal{B})$ is a regular PBD, $c_x + d_x$ is independent of the chosen x. The number of blocks in $\mathcal{B}^*$ containing x is $b' - c_x + d_x$, which is also independent of the chosen x, since $(V^*, \mathcal{B}^*)$ is a regular PBD. Hence, each point of V is contained in exactly the same number of blocks in $\mathcal{B}'$. □

In this paper we call this procedure *balanced complementation*. A *spread* (or *resolution class*) of a PBD is a set of blocks, in which each point appears in exactly one block of the set. If the blocks of the design are partitioned into spreads, then the partition is called a *resolution* and the design is said to be *resolvable.* There are many examples of resolvable designs. We can apply Theorem 2.1 to designs with spreads.

Corollary 2.2. *Let $(V, \mathscr{B})$ be a regular PBD with m disjoint spreads. Then there exists a regular PBD $(V^*, \mathscr{B}^*)$ with parameters $v^* = v$, $r^* = r + b' - 2m$ and $\lambda^* = \lambda + b' - 2m$, where b' is the total number of blocks in the m spreads. (If block size of the design is a constant k, then $b' = mv/k$.)*

In a regular *PBD* $(V, \mathscr{B})$, $r - \lambda$ is called *order* and denoted by n. From the proof of Theorem 2.1, we have the following corollary:

Corollary 2.3. *The order $n = r - \lambda$ is invariant under any balanced complementation.*

3. Balanced complementation for a regular *t-BD*

Let $(V, \mathscr{B})$ be a pair, where V is a finite set (called points) and $\mathscr{B}$ is a collection of subsets of V (called blocks). For subsets T and S of V such that $S \subseteq T$, let $\lambda(T, S)$ be the number of blocks in $\mathscr{B}$ which contain S but do not contain any point of $T - S$. The following lemma is used throughout this section.

Lemma 3.1 (Basic Lemma). *Let T and S be subsets of V such that $S \subseteq T$. Then, for a point e of $V - T$, $\lambda(T, S) = \lambda(T \cup \{e\}, S \cup \{e\}) + \lambda(T \cup \{e\}, S)$ holds.*

Proof. Let $\mathscr{B}'$ be a collection of blocks which contain S but do not contain any point of $T - S$. $\mathscr{B}'$ will be partitioned into $\mathscr{B}_1$ and $\mathscr{B}_2$, where each block of $\mathscr{B}_1$ contains e and each of $\mathscr{B}_2$ does not contain e. The number of blocks of $\mathscr{B}'$ is $\lambda(T, S)$, the number of blocks of $\mathscr{B}_1$ is $\lambda(T \cup \{e\}, S \cup \{e\})$ and the number of blocks of $\mathscr{B}_2$ is $\lambda(T \cup \{e\}, S)$. □

We consider two properties which will be useful in our study of balanced complementation.

Definition. A pair $(V, \mathscr{B})$ is said to have the property $L(t, s)$ if for every t-subset T and s-subset S of V with $S \subseteq T$, $\lambda(T, S)$ is independent of T and S. We denote this constant by $\lambda_{t,s}$.

If a pair $(V, \mathscr{B})$ has the properties $L(i, i)$'s for $i \leq t$, then it is a regular *t-BD*.

The following lemma is an immediate consequence of the Basic Lemma.

Lemma 3.2. *If two of the properties $L(t, s)$, $L(t+1, s+1)$ and $L(t+1, s)$ are satisfied, then the rest of the properties is also satisfied.*

Note that, from Lemma 3.2, if the properties $L(i, i)$'s are satisfied for every $i \leq t$, then the properties $L(i, j)$'s are also satisfied for every $j \leq i \leq t$.

Definition. A pair $(V, \mathscr{B})$ is said to have the property $M(t, s)$ if for every t-subset T and s-subset S of V with $S \subseteq T$, $\lambda(T, S) - \lambda(T, T - S)$ is independent of T and S. We denote this constant by $\delta_{t,s}$.

If a pair $(V, \mathscr{B})$ is a regular t-BD, then it has the properties $M(i, j)$'s for $j \leq i \leq t$.

On the property $M(t, s)$, we will show some results.

Lemma 3.3. *If two of the properties $M(t, s)$, $M(t+1, s+1)$ and $M(t+1, s)$ are satisfied, then the rest of the properties is also satisfied.*

Proof. This is clear from the Basic Lemma. □

Note that $\delta_{t,s} = \delta_{t+1,s+1} + \delta_{t+1,s}$, when two of the properties $M(t, s)$, $M(t+1, s+1)$ and $M(t+1, s)$ are satisfied.

Lemma 3.4. *If the property $M(t, s)$ is satisfied, then the property $M(t, t-s)$ is also satisfied.*

Proof. This is also clear from the definition of the property $M(t, s)$. □

Note that $\delta_{t,s} + \delta_{t,t-s} = 0$, when the property $M(t, s)$ is satisfied.

Lemma 3.5. *If the properties $M(i, i)$'s are satisfied for every $i \leq t$, then $\delta_{2d,d} = 0$, for $d = 0, 1, \ldots, [\frac{1}{2}t]$, where $[a]$ denotes the largest integer $\leq a$.*

Proof. Since the properties $M(i, i)$'s are satisfied for every $i \leq t$, the properties $M(i, j)$'s are also satisfied for every $j \leq i \leq t$, from Lemma 3.3. Then, from the note of Lemma 3.4, we have $\delta_{2d,d} = 0$ for $d \leq [t/2]$. □

Theorem 3.6. *If the properties $M(t-1, j)$'s are satisfied for every $j \leq t-1$ and t is even, then the properties $M(t, s)$'s are also satisfied for every $s \leq t$.*

Proof. Let $S_0, S_1, \ldots, S_t$ be subsets of V such that $S_0 (= \phi) \subset S_1 \subset \cdots \subset S_t$ with $|S_j| = j$, $j = 0, 1, \ldots, t$, respectively. Define variables d_j as

$$d_j = \lambda(S_t, S_j) - \lambda(S_t, S_t - S_j).$$

Since the properties $M(t-1, j)$'s are satisfied for every $j \leq t-1$, we have, from the Basic Lemma,

$$d_j + d_{j+1} = \delta_{t-1,j},$$

for $j = 0, 1, \ldots, t-1$. Since t is even, from these equations, we have

$$\sum_{j=0}^{t-1} (-1)^j \delta_{t-1,j} = d_0 - d_t$$
$$= 2\{\lambda(S_t, \phi) - \lambda(S_t, S_t)\}.$$

This implies that the property $M(t, 0)$ is satisfied and $\delta_{t,0} = \frac{1}{2}\sum_{j=0}^{t-1}(-1)^j\delta_{t-1,j}$. Thus, from Lemma 3.3, the properties $M(t, s)$'s are satisfied for every $s \leqslant t$. □

When block size is constant, it is well known that, if the property $L(t, t)$ is satisfied, then the properties $L(i, j)$'s are also satisfied for every $j \leqslant i \leqslant t$. But, for the property $M(i, j)$, such a result is unknown. We can only make the following statement.

Lemma 3.7. *If the property $M(t, s)$ is satisfied and block size is $k = \frac{1}{2}v$ ($\geqslant s$), then the property $M(t-1, s-1)$ is also satisfied.*

Proof. Let T and S be a $(t-1)$-subset and an $(s-1)$-subset of V, respectively, such that $S \subseteq T$. Since $M(t, s)$ is satisfied, we have

$$\lambda(T \cup \{e\}, S \cup \{e\}) - \lambda(T \cup \{e\}, T - S) = \delta_{t,s},$$

for any point e of $V - T$. Let $\mathbb{B}_e$ and $\mathbb{C}_e$ be a collection of blocks counted in the first term and in the second term of the above equation, respectively. Since block size is a constant k, we have $|B - T| = k - (s-1)$ for a block B which contains S but does not contain any point of $T - S$. Such a block appears in exactly $k - (s-1)$ collections of $\mathbb{B}_{e_1}, \mathbb{B}_{e_2}, \ldots, \mathbb{B}_{e_{v-(t-1)}}$, where $V - T = \{e_1, e_2, \ldots, e_{v-(t-1)}\}$. Similarly, if a block B appears in one of the collections $\mathbb{C}_{e_1}, \mathbb{C}_{e_2}, \ldots, \mathbb{C}_{e_{v-(t-1)}}$, then B is contained in exactly $v - k - (s-1)$ collections of $\mathbb{C}_{e_1}, \mathbb{C}_{e_2}, \ldots, \mathbb{C}_{e_{v-(t-1)}}$. Thus we have

$$\{k-(s-1)\}\lambda(T, S) - \{v-k-(s-1)\}\lambda(T, T-S) = \{v-(t-1)\}\delta_{t,s}.$$

Substituting the equation into $\lambda(T, S) - \lambda(T, T-S)$, we have

$$\lambda(T, S) - \lambda(T, T-S) = \frac{\{(v-t+1)\delta_{t,s} + (v-2k)\lambda(T, T-S)\}}{k-s+1}.$$

So, if $v = 2k$, then $\lambda(T, S) - \lambda(T, T-S)$ is independent of the $(t-1)$-subset T and the $(s-1)$-subset S chosen. This implies that the property $M(t-1, s-1)$ is satisfied. □

From Lemmas 3.3, 3.4 and 3.7, we have the following theorem:

Theorem 3.8. *If the property $M(t, s)$ is satisfied and block size is $k = \frac{1}{2}v$ ($\geqslant s$), then the properties $M(i, j)$'s are also satisfied for every $j \leqslant i \leqslant t$.*

Now we consider balanced complementation for a regular t-BD.

Theorem 3.9. *Let $(V, \mathscr{B})$ be a regular t-BD. Consider $V^* = V$ and $\mathscr{B}^* = \{V - B : B \in \mathscr{B}'\} \cup (\mathscr{B} - \mathscr{B}')$, where $\mathscr{B}' \subset \mathscr{B}$. Then the pair $(V^*, \mathscr{B}^*)$ is also a regular t-BD if and only if the pair $(V, \mathscr{B}')$ has the properties $M(t, s)$'s for $s \leq t$.*

Proof. Let $\mathscr{B}_1 = \{V - B : B \in \mathscr{B}'\}$ and $\mathscr{B}_2 = \mathscr{B} - \mathscr{B}'$. For subsets T and S of V such that $S \subseteq T$, let $\lambda^{(i)}(T, S)$ be the number of blocks in $\mathscr{B}_i$ which contain S but do not contain any point of $T - S$. Since $(V, \mathscr{B})$ is a regular t-BD, it has the properties $L(t, s)$'s; that is,

$$\lambda^{(1)}(T, T-S) + \lambda^{(2)}(T, S) = \lambda_{t,s},$$

for $s \leq t$, where $t = |T|$ and $s = |S|$.

If $(V^*, \mathscr{B}^*)$ is a regular t-BD, then it has the properties $L(t, s)$'s; that is,

$$\lambda^{(1)}(T, S) + \lambda^{(2)}(T, S) = \lambda^*_{t,s},$$

say, for $s \leq t$. Therefore, we have

$$\lambda^{(1)}(T, T-S) - \lambda^{(1)}(T, S) = \lambda_{t,s} - \lambda^*_{t,s},$$

for $s \leq t$. This implies that the pair $(V, \mathscr{B}')$ has the properties $M(t, s)$'s for $s \leq t$.

If $(V, \mathscr{B}')$ has the properties $M(t, s)$'s for $s \leq t$, then we have

$$\lambda^{(1)}(T, T-S) - \lambda^{(1)}(T, S) = \delta^{(1)}_{t,s},$$

say, for $s \leq t$. Therefore, we have

$$\lambda^{(1)}(T, S) + \lambda^{(2)}(T, S) = \lambda_{t,s} - \delta^{(1)}_{t,s},$$

for $s \leq t$. This implies that the pair $(V^*, \mathscr{B}^*)$ has the properties $L(t, s)$'s for $s \leq t$ and it is a regular t-BD. □

It is easily seen, from the above proof, that $\lambda^*_{i,j} = \lambda_{i,j} - \delta^{(1)}_{i,j}$ for $j \leq i \leq t$, when $(V^*, \mathscr{B}^*)$ is a regular t-BD. Especially, from Lemma 3.5, we have $\lambda^*_{2d,d} = \lambda_{2d,d}$ for $d \leq [\frac{1}{2}t]$.

From Theorems 3.6 and 3.9, we have the following theorem:

Theorem 3.10. *If $(V, \mathscr{B})$ is a regular t-BD with a subdesign which is a regular $(t-1)$-BD $(V, \mathscr{B}')$, $\mathscr{B}' \subset \mathscr{B}$, and t is even, then $(V^*, \mathscr{B}^*)$ is also a regular t-BD, where $(V^*, \mathscr{B}^*)$ is defined in Theorem* 3.9.

Let $Q(v)$ be the complete design of block size 4 with v points. Lindner [1] has shown that $Q(4p)$ contains at least $3p$ mutually disjoint Steiner quadruple systems as subdesigns, where $p \equiv 2$ or $4 \pmod 6$, $p \geq 8$. Therefore, from Theorem 3.10, there exists a regular 4-*BD* with parameters

$$r = \tfrac{1}{3}(2p-1)(4p-1)(4p-3) + \tfrac{1}{3}l(p-2)(2p-1)(4p-1),$$
$$\lambda_2 = (2p-1)(4p-3) + \tfrac{1}{3}l(p-2)(2p-1)(4p-1),$$
$$\lambda_3 = 4p-3 + \tfrac{1}{3}l(p-2)(8p^2-14p+9),$$

and

$$\lambda_4 = 1 + \tfrac{1}{3}l(p-2)(8p^2-22p+17),$$

for $1 \leqslant l \leqslant 3p$.

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