ELSEVIER

Discrete Mathematics 191 (1998) 57–64

DISCRETE MATHEMATICS

Minimal 2-connected non-hamiltonian claw-free graphs

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Received 5 May 1996; revised 27 November 1997; accepted 23 December 1997

Abstract

We say a graph G is *minimal* with respect to a property Q if there exists no proper induced subgraph G' of G with property Q. In this paper we characterize all minimal graphs with respect to the property 'to be 2-connected, non-hamiltonian and claw-free'. Several sufficient forbidden subgraph conditions are obtained as corollaries.

1. Introduction

Throughout the paper, a *graph* will be a finite undirected graph $G=(V(G),E(G))$ without loops and multiple edges. We follow the most common graph-theoretical terminology and notation, and for concepts not defined here we refer to [2].

If $M \subset V(G)$ then $\langle M \rangle$ denotes the induced subgraph on M. If $0 \neq M \neq V(G)$, we say that $\langle M \rangle$ is a *proper induced subgraph* . A graph G is *minimal* with respect to a property Q if there exists no proper induced subgraph G' of G with property Q, but G has property Q. G is *H-free* if G contains no copy of H as an induced subgraph.

Throughout the paper we will use fixed terms and notation for some graphs, see Fig. 1.

We define $\mathscr{P}$ to be the class of graphs obtained by taking two vertex-disjoint triangles $\langle\{v_1, v_2, v_3\}\rangle$, $\langle\{w_1, w_2, w_3\}\rangle$ and by joining every pair of vertices $\{v_i, w_i\}$ by a path P_n for $n \geqslant 3$ or by a triangle. We denote graphs from the class $\mathscr{P}$ by P_{x_1,x_2,x_3}, where $x_i = n$ if v_i, w_i are joined by a P_n, and $x_i = T$ if v_i, w_i are joined by a triangle (see Fig. 2).

We say that a set $M \subset V(G)$ is *independent* if the graph $\langle M \rangle$ has no edges. The *independence number* $\alpha(G)$ of G is the size of a largest independent set in G.

We say that a vertex w is a *neighbor* of a vertex v if $vw \in E(G)$. The set of neighbors of a vertex v is denoted by $N(v)$. The *neighborhood* of a vertex v is the graph $\langle N(v) \rangle$. We say the graph G is *locally connected* if $\langle N(v) \rangle$ is connected for every $v \in V(G)$.

PII S0012-365X(98)00093-4

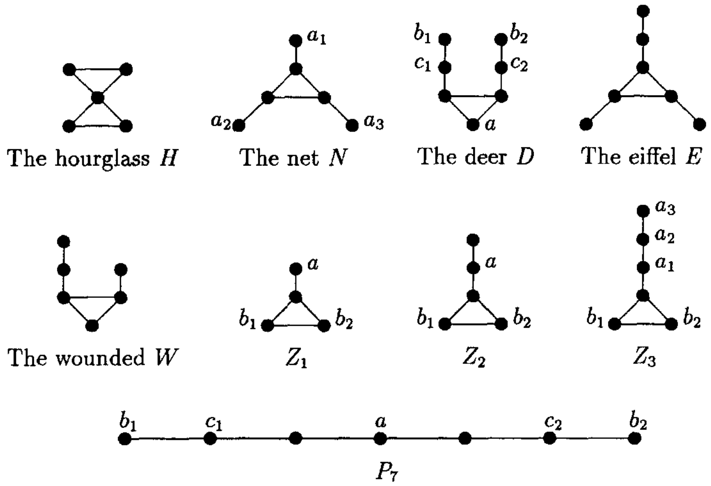

Fig. 1.

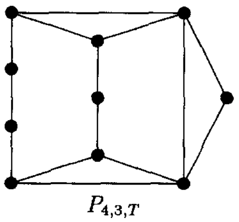

Fig. 2.

Observation A. A graph $G=(V,E)$ is claw-free if and only if $\alpha(N(v))<3$ for every $v\in V(G)$.

The following theorem was proved by Fouquet:

Theorem A (Fouquet [8]). *Let G be a claw-free graph with $\alpha(G)>2$. Then for every $v\in V(G)$, the graph $\langle N(v)\rangle$ either can be covered by two cliques or contains an induced C_5.*

We say that a cycle $C\subset G$ is *non-extendable* if there is no cycle C' in G such that $V(C)\subsetneq V(C')$. A vertex $v\in V(C)$ such that $N(v)\not\subset V(C)$ is called a *contact-vertex* of C.

We consider every cycle to be oriented. For any $x\in C$ we denote by x^- or x^{--} the first or second predecessor of x, and by x^+ or x^{++} the first or second successor of x on C in this orientation, respectively. An *arc* is a sequence of consecutive vertices on C.

An arc with first vertex x and last vertex y (in the given orientation of C) will be denoted by xCy.

We will use the following known results on claw-free graphs:

ST Lemma (Standard techniques). *Let G be a claw-free graph, let C be a non-extendable cycle on G. Let $\langle V(G)\backslash V(C)\rangle$ be a connected graph, let v, w be contact-vertices of C. Then*

1. $v^-v^+ \in E(G)$,
2. $vw^- \notin E(G)$,
3. $v^-w^- \notin E(G)$,
4. $v^-w^{--} \notin E(G)$.

CVL Lemma (Faudree et al. [7]). (Contact vertex lemma). *Let G be a claw-free graph and let C be a non-extendable cycle in G, let $v \in V(C)$ be a contact-vertex of C. Then $\langle N(v)\rangle$ is disconnected.*

Remark. Moreover since $\alpha(\langle N(v)\rangle)=2$, $\langle N(v)\rangle$ consist of two vertex-disjoint cliques.

Theorem B (Bedrossian [1]). *Let X and Y be connected graphs, $X, Y \not\cong P_3$, and let G be a 2-connected graph that is not a cycle. Then G being XY-free implies G is hamiltonian if and only if one of X, Y is a claw and the other one is isomorphic to P_6, N, W or to an induced subgraph of some of them.*

Theorem C (Broersma and Veldman, Faudree et al. [4,6]).

1. *Every 2-connected CHP_7-free graph is hamiltonian.*
2. *Every 2-connected CDP_7-free graph is hamiltonian.*

The main result of this paper extends Theorem C.

2. Lemmas

Lemma 1 (First reduction). *Let G be a minimal 2-connected non-hamiltonian claw-free graph, let C be a non-extendable cycle in G. Then C contains only two contact-vertices.*

Proof. Let G be a minimal 2-connected non-hamiltonian claw-free graph, let C be a non-extendable cycle in G with more than two contact vertices. We show G is not minimal.

Let $v_1,\dots,v_k$, $k \geqslant 3$, be contact vertices of C. Consider the graph H such that

$$V(H)=(V(G)\backslash V(C))\cup\{v_1,\dots,v_k\},$$

$$E(H)=E(\langle V(H)\rangle)\backslash E(\langle v_1,\dots,v_k\rangle).$$

Let v_i, v_j be a pair of contact-vertices such that $d_H(v_i, v_j) \leqslant d_H(v_l, v_m)$ for any $l, m \in 1, 2, \ldots, k$ (where $d_H(v_i, v_j)$ denotes the distance of v_i and v_j in H). We distinguish two cases.

Case 1. $d_H(v_i, v_j) > 2$

Let $v_i, w_1, \ldots, w_p, v_j$ $(p \geqslant 2)$ be any shortest path between v_i and v_j in H. Let x be a contact-vertex of C different from v_i, v_j. If $xw_1 \in E(H)$ then $d_H(x, v_i) = 2 < d_H(v_i, v_j)$. If $xw_i \in E(H)$ for some $i > 1$, then $d_H(x, v_j) < d_H(v_i, v_j)$. Thus $w_1, \ldots, w_p \cap N(x) = \emptyset$. But then, since $(V(G) \backslash V(C)) \cap N(x) \neq \emptyset$, necessarily $\{w_1, \ldots, w_p\} \subsetneq V(G) \backslash V(C)$ and the graph $G' = \langle V(C) \cup \{w_1, \ldots, w_p\} \rangle$ is a proper induced subgraph of G. It is easy to see that G' is 2-connected, claw-free and C is a non-extendable cycle in G'. Thus G is not minimal.

Case 2. $d_H(v_i, v_j) = 2$

Let w be a common neighbor of v_i, v_j in $V(G) \backslash V(C)$. If $V(G) \backslash V(C) \neq \{w\}$, then the graph $G' = \langle V(C) \cup \{w\} \rangle$ contradicts the minimality of G. Thus, $V(G) \backslash V(C) = \{w\}$. By the ST lemma, $\{w, v_i^+, v_j^+\}$ is an independent set. By theorem A, the neighborhood of any vertex of G can either be covered by two cliques or contains an induced C_5. Let $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ be an induced $C_5 \subset \langle N(w) \rangle$. In this case $N(w)$ consists of only contact-vertices and by the ST-lemma, $\langle x_1, x_1^-, x_2, x_5 \rangle$ is an induced claw.

Hence, $N(w)$ can be covered by two cliques. Let $\{z_{1,1}, \ldots, z_{1,r}\}, \{z_{2,1}, \ldots, z_{2,s}\} (0 < r, s;$ $r + s = k)$ be such a partition of $N(w)$.

Let $G' = \langle V(G) \backslash \{z_{i,j}, i = 1, 2, j \neq 1\} \rangle$. If $r > 1$ or $s > 1$, then G' is a proper induced subgraph of G, and G' is obviously 2-connected and claw-free. Suppose G' is hamiltonian. Since $N(w)|_{G'} = \{z_{1,1}, z_{2,1}\}$, any hamiltonian cycle in G' must contain the path $z_{1,1} w z_{2,1}$. If we replace this path by the path $z_{1,1} \ldots z_{1,r} w z_{2,s} \ldots z_{2,1}$, we obtain a hamiltonian cycle in G. Thus G' is not hamiltonian, which contradicts the minimality of G. Hence $r = 1$ and $s = 1$, which completes the proof. □

Corollary. *Let G be a minimal 2-connected non-hamiltonian claw-free graph. Then*

(a) *$\{v_i, v_j\}$ is a biarticulation.*

(b) *$H = \langle V(G) \backslash V(C) \cup \{v_i, v_j\} \rangle$ is a path or a triangle.*

Remark. In what follows we shall use for these two contact-vertices a fixed notation v, w.

Lemma 2 (Second reduction). *Let G be a minimal 2-connected non-hamiltonian claw-free graph, let C be a non-extendable cycle in G with contact-vertices v, w. Then each of the vertices v^-, v^+, w^-, w^+ is in some biarticulation of G.*

Proof. By the CVL lemma, $N(v)$ consists of two vertex-disjoint cliques. Let x be a neighbor of v in $V(G) \backslash V(C)$. The clique containing x is either the vertex x alone (if $H = \langle (V(G) \backslash V(C)) \cup \{v, w\} \rangle$ is a path) or $K = \langle \{x, w\} \rangle$ (if H is a triangle). The second clique in $N(v)$ contains only vertices on C different from w. Consider the graph $G' = \langle V(G)\backslash\{v^-\}\rangle$. Obviously, G' is claw-free. Suppose G' is hamiltonian. Any hamiltonian cycle in G' must contain the path xvy where, by the CVL lemma, y is a common neighbor of v and v^-. If we replace this path by the path xvv^-y, we obtain a hamiltonian cycle in G — a contradiction. Thus G' is not hamiltonian and, by the minimality of G, G' cannot be 2-connected. Hence, v^- is in a biarticulation. The proof for v^+, w^- and w^+ follows by symmetry. □

Lemma 3. *Let G be a minimal 2-connected non-hamiltonian claw-free graph, let C be a non-extendable cycle in G with contact-vertices v, w. Then v has no neighbors on C except v^+, v^- (and possibly w).*

Proof. Without loss of generality suppose that a neighbor of v different from v^+, v^- lies on v^+Cw^-. Let y be the first neighbor of v on this arc. Since v^+ is in a biarticulation, we have $y \neq v^{++}$. Consider the graph $\langle\{y, y^+, y^-, v\}\rangle$. By the choice of y we have $yy^- \in E(G), yy^+ \in E(G), yv \in E(G)$ and $y^-v \notin E(G)$.

Suppose $y^-y^+ \in E(G)$. Then the graph $G' = \langle(V(G))\backslash\{y\}\rangle$ is 2-connected and claw-free. Suppose that G' is hamiltonian. A hamiltonian cycle in G' must contain the path xvz, where $x \in N(v) \cap (V(G)\backslash V(C))$. By the CVL-lemma, z is a common neighbor of v and y. If we replace this path by the path $xvyz$, we obtain a hamiltonian cycle in G — a contradiction. Hence, $y^-y^+ \notin E(G)$.

Suppose $y^+v \in E(G)$. Then the graph G' constructed from G by removing the arc v^+Cy is 2-connected and claw-free. Suppose G' is hamiltonian. A hamiltonian cycle in G' must contain the path xvz, where z is a common neighbor of v, v^+ and y^+, and where x is a neighbor of v in $V(G)\backslash V(C)$. If we replace this path by the path $xvyCv^+z$, we obtain a hamiltonian cycle in G — a contradiction. Hence, $y^-v \notin E(G)$ also, which implies that $\langle\{y, y^+, y^-, v\}\rangle$ is an induced claw. This contradiction completes the proof. □

Corollary. *Let G be a minimal 2-connected non-hamiltonian claw-free graph. Let C be a non-extendable cycle in G with contact-vertices v and w. Then*
1. *v^{--} has no neighbors on the arc v^+Cw^-,*
2. *v^- has no neighbors on the arc v^+Cw^- and every neighbor of v^- on w^+Cv^- is adjacent to v^{--}.*

Proof. 1. If we suppose that v^{--} has a neighbor on v^+Cw^-, then we have a contradiction with Lemma 2.

2. Let y be a neighbor of v^-. Then, since $\langle\{v^-, v, v^{--}, y\rangle\}$ cannot be a claw, Lemma 3 implies $v^{--}y \in E(G)$ and, by Corollary 1, $y \in w^+Cv^-$. □

3. Main result

Theorem 1. *A graph G is a minimal 2-connected non-hamiltonian claw-free graph if and only if $G \in \mathscr{P}$.*

Proof. It is easy to see that every graph from $\mathscr{P}$ is a minimal 2-connected non-hamiltonian claw-free graph.

Let G be a minimal 2-connected non-hamiltonian claw-free graph. Every such graph must have the properties given in Lemmas 1, 2, 3 and Corollaries 1, 2.

Let C be a non-extendable cycle in G.

Denote by $X=x_1,\dots,x_p$ a path between v and w with interior vertices in $V(G)\setminus V(C)$, by $Y=y_1,\dots,y_s$ a shortest path between v^+ and w^- on the arc v^+Cw^- with respect to the sequence of vertices on C that does not contain the edge v^+w^-, and by $Z=z_1,\dots,z_t$ the shortest path between v^- and w^+ on the arc v^-Cw^+ defined analogous to $y_1,\dots,y_s$.

By the First reduction lemma there are no edges between X and Y and between X and Z. Now we distinguish two cases.

Case 1. $v^-w^+\in E(G)$

By Corollary 2 $v^{--}w^+\in E(G)$. Hence $t=1$ and $z_1=v^{--}$. By Corollary 1 there are no edges between Y and Z and hence $G\in\mathscr{P}$.

Case 2. $v^-w^+\notin E(G)$

By symmetry, we can also suppose without loss of generality that also $v^+w^-\notin E(G)$. If there are no edges between Y and Z, then $G\in\mathscr{P}$. Let thus y_i be the first vertex on Y which has a neighbor on Z, and let z_j be the first such neighbor. We distinguish four subcases.

Subcase A. $i>1, j>1$

The graph $\langle\{z_j,z_{j-1},z_{j+1},y_i\}\rangle$ cannot be a claw. Since $y_iz_{j-1}\in E(G)$ contradicts the choice of the edge y_iz_j, and $z_{j-1}z_{j+1}\in E(G)$ contradicts the choice of Z, we have $y_iz_{j+1}\in E(G)$. Consider the graph $G'=\langle\{v^+,y_1,\dots,y_i,v\}\cup X\cup\{w,v^-\}\cup Z\cup\{w^+\}\rangle$ if $wv\notin E(G)$ and $G'=\langle\{v^+,y_1,\dots,y_i,v,w,v^-\}\cup Z\cup\{w^+\}\rangle$ if $wv\in E(G)$.

In G' we have two triangles $\langle\{v^+,v,v^-\}\rangle$ and $\langle\{y_i,z_j,z_{j+1}\}\rangle$. Vertices v^+,y_i are joined by the path $y_1,\dots,y_{i-1}$; vertices v^-,z_j by the path $z_1,\dots,z_{j-1}$ and vertices v,z_{j+1} by the path $z_{j+2},\dots,z_t,w^+,w$ and, if $vw\notin E(G)$, possibly X. To show that $G'\in\mathscr{P}$ it remains to prove that in G' there are no other edges except these two triangles and three paths.

Edge between		Contradiction with
X	any	First reduction
v	w^+	ST lemma
v	any other	Lemma 3
v^+	$y_k, k>1$	choice of Y
v^+	Z	Corollary 2
v^+	w^+,w	ST lemma
v^-	Y	Corollary 2
v^-	$z_k, k>1$	choice of Z
v^-	w^+	the assumption of Case 2
v^-	w	ST lemma
$y_k, k<i$	Z	choice of y_i

$y_k, k<i$	w^+	Corollary 2
$y_k, k<i$	w	Lemma 3
y_i	$z_k, k<j$	choice of y_iz_j
y_i	$z_k, k>j+1$	see below
y_i	w^+	Corollary 2
y_i	w	Lemma 3
z_k	$z_l, l>k+1$	choice of Z
z_k	w^+	choice of Z
z_k	w	Lemma 3

In the case $y_iz_k \in E(G)$, for $k>j+1$ we have $z_jz_k \notin E(G)$ (by the choice of Z), and $N(y_{i-1} \cap Z) = \emptyset$ (by the choice of y_i), and hence $\langle\{y_i, y_{i-1}, z_j, z_k\}\rangle$ is an induced claw.

Subcase B. $i>1, j=1$

The graph $\langle\{z_1, v^-, y_i, z_2\}\rangle$ cannot be a claw. If $v^-y_i \in E(G)$ then we have a contradiction with Corollary 2; If $v^-z_2 \in E(G)$, there for $z_2 = w^+$ we are in Case 1 and for $z_2 \neq w^+$ we get a contradiction with the choice of Z. Thus $y_iz_2 \in E(G)$. Consider the graph $G' = \langle\{v^+, y_1, \ldots, y_i, v\} \cup X \cup \{w, v^-\} \cup Z \cup \{w^+, v^{--}\}$ if $uv \notin E(G)$ and the graph $G' = \langle\{v^+, y_1, \ldots, y_i, v, w, v^-\} \cup Z \cup \{w^+, v^{--}\}\rangle$ if $uv \in E(G)$ and take two triangles $\langle\{v, v^+, v^-\}\rangle$ and $\langle\{y_i, z_1, z_2\}\rangle$. By Corollary 2 we have $z_1 \neq v^{--}$ and $v^{--}z_1 \in E(G)$. Vertices v^+, y_i are joined by the path $y_1, \ldots, y_{i-1}$, vertices v^-, z_1 by the triangle $\langle\{v^-, v^{--}, z_1\}\rangle$ and vertices v, z_2 by the path $(X), w, w^+, z_t, \ldots, z_3$. If there are no other edges in G', then $G' \in \mathscr{P}$. We have to prove that v^{--} has no neighbors in G', except v^- and z_1. Other edges can be excluded similarly as in Subcase A.

Neighbor of v^{--}	Contradiction with
X	First reduction lemma
v, v^+, Y	Corollary 1
$z_j, j>2; w^+$	choice of Z
w	ST lemma

We must still prove that $v^{--}z_2 \notin E(G)$. Let, on the contrary $v^{--}z_2 \in E(G)$. The graph $\langle\{z_2, v^{--}, z_3, y_i\}\rangle$ cannot be a claw. Note that, by Corollary 2 $z_2 \neq w^+$, but possibly $z_3 = w^+$. Now $v^{--}y_i \notin E(G)$ by Corollary 1. Let $v^{--}z_3 \in E(G)$. If we replace the sequence $v^-z_1z_2z_3$ by the sequence $v^-v^{--}z_3$, we have a contradiction with the choice of Z. Thus $y_iz_3 \in E(G)$. But in this case $\langle\{y_i, y_{i-1}, z_1, z_3\}\rangle$ is neccessarily a claw. The Subcase B is proved.

Subcase C. $i=1, j>1$

Let $G' = \langle\{v, v^+, v^{++}, y_1, w\} \cup X \cup \{v, v^-\} \cup Z \cup \{w^+\}\rangle$ if $uv \notin E(G)$ or $G' = \langle\{v, v^+, v^{++}, y_1, w, v, v^-\} \cup Z \cup \{w^+\}\rangle$ if $uv \in E(G)\rangle$. By the symmetry we can argue analogously as in Subcase B.

Subcase D. $i=1, j=1$

Let $G' = \langle \{v^+, v^{++}, y_1, v\} \cup X \cup \{w, v^-, v^{--}\} \cup Z \cup \{w^+\} \rangle$ if $uv \notin E(G)$ or $G' = \langle \{v^+, v^{++}, y_1, v, w, v^-, v^{--}\} \cup Z \cup \{w^+\} \rangle$ if $uv \in E(G)$. Using the preceding lemmas and subcases it is easy to check that $G' \in \mathscr{P}$. □

4. Corollaries

Theorem 2A. *Every 2-connected non-hamiltonian claw-free graph contains a graph from $\mathscr{P}$ as an induced subgraph.*

Theorem 2B. *Every 2-connected $C, \mathscr{P}$ – free graph is hamiltonian.*

Proof. Follows immediately from Theorem 1. □

Corollary 3. *Let G be a 2-connected claw-free graph. If, moreover, G is net-free or wounded-free or P_6-free then G is hamiltonian.*

Proof. Let G be a 2-connected non-hamiltonian claw-free graph. Then G must contain a graph from $\mathscr{P}$ as an induced subgraph. It is easy to check that every graph from $\mathscr{P}$ contains all graphs N, W, P_6 and hence G must also contain all these graphs. □

Corollary 4. *Let G be a 2-connected claw-free graph. Moreover, if, G satisfies at least one of the following assumptions:*

(1) *G is $DP_{3,3,3}$ — free,*
(2) *G is $EP_{T,T,T}$ — free,*
(3) *G is $P_7P_{T,T,T}$ — free,*

then G is hamiltonian.

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