



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

Journal of Combinatorial Theory, Series B 95 (2005) 318–332

Journal of  
Combinatorial  
Theory

Series B

[www.elsevier.com/locate/jctb](http://www.elsevier.com/locate/jctb)

# Tait's flyping conjecture for 4-regular graphs

Jörg Sawollek

*Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany*

Received 8 July 1998

Available online 19 July 2005

---

## Abstract

Tait's flyping conjecture, stating that two reduced, alternating, prime link diagrams can be connected by a finite sequence of flypes, is extended to reduced, alternating, prime diagrams of 4-regular graphs in  $\mathbb{S}^3$ . The proof of this version of the flyping conjecture is based on the fact that the equivalence classes with respect to ambient isotopy and rigid vertex isotopy of graph embeddings are identical on the class of diagrams considered.

© 2005 Elsevier Inc. All rights reserved.

*Keywords:* Knotted graph; Alternating diagram; Flyping conjecture

---

## 0. Introduction

Very early in the history of knot theory attention has been paid to alternating diagrams of knots and links. At the end of the 19th century Tait [21] stated several famous conjectures on alternating link diagrams that could not be verified for about a century. The conjectures concerning minimal crossing numbers of reduced, alternating link diagrams [15, Theorems A, B] have been proved independently by Thistlethwaite [22], Murasugi [15], and Kauffman [6]. Tait's flyping conjecture, claiming that two reduced, alternating, prime diagrams of a given link can be connected by a finite sequence of so-called *flypes* (see [4, p. 311] for Tait's original terminology), has been shown by Menasco and Thistlethwaite [14], and for a special case, namely, for well-connected diagrams, also by Schrijver [20].

The present article, as well as [19], deals with generalizations of Tait's conjectures to embeddings of 4-regular (topological) graphs into 3-space. In [19] it has been shown that

---

*E-mail address:* [sawollek@math.uni-dortmund.de](mailto:sawollek@math.uni-dortmund.de)

*URL:* <http://www.mathematik.uni-dortmund.de/lsv/sawollek>.

0095-8956/\$ - see front matter © 2005 Elsevier Inc. All rights reserved.

doi:10.1016/j.jctb.2005.05.001

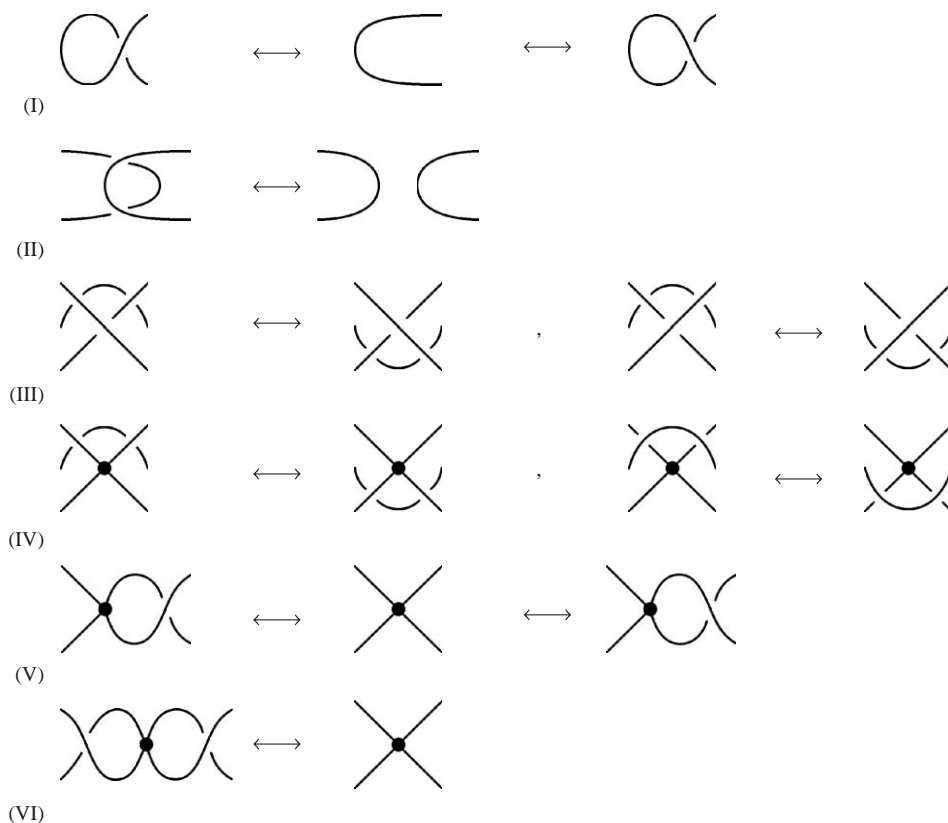


Fig. 1. Reidemeister moves.

a reduced, alternating graph diagram  $D$  has minimal crossing number. Furthermore, if  $D$  is prime in addition, then a non-alternating diagram that is equivalent to  $D$  cannot have the same crossing number as  $D$ .

The purpose of this paper is to prove Tait’s flying conjecture for reduced, alternating, prime diagrams of 4-regular graphs in the 3-sphere  $S^3$ . The result depends on a suitable definition of primality, see Section 3. Its proof is based on the fact that the equivalence classes with respect to rigid vertex isotopy and ambient isotopy of graph diagrams are identical on the class of diagrams under consideration.

Definitions of these two equivalence relations for graph diagrams are given in Section 1. Then, in Section 2, the notion of *tangles* is introduced to derive invariants of graph diagrams and to give, via transformation tangles, a description of a sequence of Reidemeister V moves (see Fig. 1) applied to a graph vertex. In Section 3 certain properties of graph diagrams are discussed, and two important theorems on reduced, alternating diagrams are stated. After, in Section 4, Tait’s flying conjecture has been shown to be an immediate consequence of the fact that the two equivalence classes mentioned above coincide for reduced, alternating, prime graph diagrams, the latter statement is finally proved in Section 5.

## 1. Diagrams of 4-regular graphs

Embedding graphs into  $\mathbb{S}^3$  extends, in a natural way, the classical knot theoretical problem of embedding one or more disjoint copies of the 1-sphere  $\mathbb{S}^1$  into  $\mathbb{S}^3$  where the resulting images are called *knots* and *links*, respectively. Classical terminology of knot theory can be found in [1] or [17], see [9,11,12,16] for more recent introductions to the field.

A *topological graph* is a one-dimensional cell complex which is related to an abstract graph in the obvious way. In the following, always 4-regular graphs, which are allowed to have multiple edges or loops, are considered. Vertices of degree two may occur but are neglected since they are uninteresting for a topological treatment.

If  $G$  is a topological graph, then a *graph  $\mathcal{G}$  in  $\mathbb{S}^3$*  is the image of an embedding of  $G$  into  $\mathbb{S}^3$ . Two graphs  $\mathcal{G}_1, \mathcal{G}_2$  in  $\mathbb{S}^3$  are called *equivalent with respect to ambient isotopy* or *ambient isotopic* if there exists an orientation preserving autohomeomorphism of  $\mathbb{S}^3$  which maps  $\mathcal{G}_1$  onto  $\mathcal{G}_2$ . Embeddings of topological graphs in  $\mathbb{S}^3$  can be examined via *regular graph diagrams*, i.e., images under regular projections to an appropriate sphere equipped with over-under information at double points. Two graph diagrams  $D$  and  $D'$  are called *equivalent with respect to ambient isotopy* or *ambient isotopic* if one can be transformed into the other by a finite sequence of Reidemeister moves I–V (see Fig. 1) combined with orientation preserving homeomorphisms of the sphere to itself. Two graphs in  $\mathbb{S}^3$  are ambient isotopic if and only if they have diagrams that are ambient isotopic (see [7] or [24]).

Soon after the discovery of polynomial link invariants which fulfil certain recurrence formulas, such as the Jones polynomial and its generalizations, it had been tried to extend these invariants to graphs in 3-space. Only quite recently such an invariant for arbitrary topological graphs has been found by Yokota [25] (see [19] for a different approach in the case of 4-regular graphs). Yokota's invariant manages the difficulty to be invariant under Reidemeister move V which before had been the main obstacle for a full generalization of combinatorial link invariants to knotted graphs.

Besides ambient isotopy there is a further equivalence relation for graph diagrams which avoids Reidemeister move V and which will be important for the purposes of this paper: two graph diagrams  $D$  and  $D'$  are called *equivalent with respect to rigid vertex isotopy* or *rigid vertex isotopic* if one can be transformed into the other by a finite sequence of Reidemeister moves I–IV and VI (see Fig. 1) combined with orientation preserving homeomorphisms of the sphere to itself. Rigid vertex isotopy corresponds to an equivalence relation on graphs in  $\mathbb{S}^3$  where a small neighbourhood of each graph vertex is contained inside a disk, and only those orientation preserving autohomeomorphisms of  $\mathbb{S}^3$  are considered that respect these disks (see [5,7,23]). Observe that Reidemeister moves I–V imply Reidemeister move VI, thus rigid vertex isotopic graph diagrams are ambient isotopic. See Fig. 8 for two ambient isotopic diagrams which are not rigid vertex isotopic.

For the sake of shortness, the phrase *graph diagram* will always mean (*regular*) *diagram of a 4-regular graph in  $\mathbb{S}^3$*  in the subsequent text, and throughout the article a *link* will be considered as 4-regular graph in  $\mathbb{S}^3$  without vertices of degree four. Of course, the equivalence classes with respect to rigid vertex isotopy and ambient isotopy coincide for links.

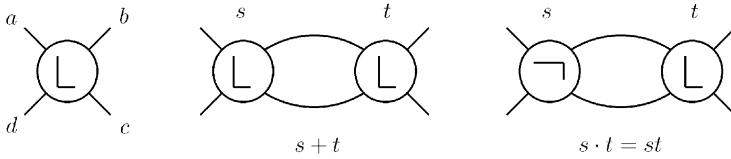


Fig. 2. A tangle, and tangle operations.

### 2. Tangles, and invariants of graph diagrams

In this section, the notion *rational tangle* is introduced for two purposes: to define invariants of graph diagrams with respect to ambient isotopy, and to describe the effect of a sequence of Reidemeister moves applied to a graph vertex. The definitions and notations used here are due to Conway [2].

A *tangle* is a part of a link diagram in form of a disk with four arcs emerging from it, see Fig. 2 (left), where the tangle’s position is indicated by an L-shaped symbol and its emerging arcs are labeled with letters *a, b, c, d* in a clockwise ordering (or simply one of them with “*a*”). Two tangles are said to be *equivalent (with respect to ambient isotopy)* if one can be transformed into the other by a finite sequence of Reidemeister moves of type I–III and autohomeomorphisms of the disk which keep the boundary fixed. In the following, a notational difference between a tangle and the corresponding equivalence class will be avoided. Some basic tangles are  $0 = \frown, \infty = \smile, 1 = \times, \bar{1} = -1 = \bar{\times}$ .

For tangles *s* and *t*, the operations  $+$  and  $\cdot$  are defined as depicted in Fig. 2. The tangles  $0, n = 1 + \dots + 1$ , and  $\bar{n} = \bar{1} + \dots + \bar{1}$  are called *integer tangles*. If a tangle *t* is of the form  $t = a_1 \dots a_n$  with integer tangles  $a_1, \dots, a_n$  or if  $t = \infty$ , then *t* is called *rational tangle*. Let  $\mathcal{K}$  denote the set of all (equivalence classes of) rational tangles.

For a rational tangle  $a_1 \dots a_n$  the evaluation of the continued fraction

$$a_n + \frac{1}{a_{n-1} + \dots + \frac{1}{a_2 + \frac{1}{a_1}}}$$

gives a number in  $\mathbb{Q} \cup \{\infty\}$  where the obvious rules for handling “ $\infty$ ”, such as  $\frac{1}{\infty} = 0$ , are applied, if necessary, during the calculation. It is known that two rational tangles are equivalent if and only if the values of the corresponding continued fractions are identical (see [2,1] for classical proofs, and [3] for an elementary combinatorial proof). Therefore, a rational tangle *r* can be identified with this number, thus let *r* denote an element of  $\mathcal{K}$  as well as the corresponding value in  $\mathbb{Q} \cup \{\infty\}$ , and let  $|r|$  denote the tangle’s *crossing number*, i.e., the minimal number of crossings in any diagram belonging to the equivalence class represented by *r*. Furthermore, a rational tangle contained in  $\mathcal{K} \setminus \{0, \infty, 1, \bar{1}\}$  can be expressed in a uniquely determined *normal form*  $a_1 \dots a_n$  such that  $|a_1| \geq 2, a_2 \neq 0, \dots, a_{n-1} \neq 0$ , and all  $a_i \geq 0$  or all  $a_i \leq 0$ . The normal forms of the remaining four tangles are the obvious ones. Tangles in normal form have minimal number of crossings.

From a given graph diagram *D* there can be obtained link diagrams by substituting rational tangles for the graph vertices (see [18]). To do this in a well-defined way, it is necessary to

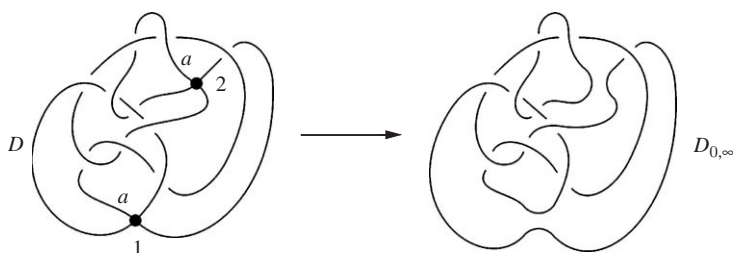


Fig. 3. Cutting open vertices.

give an ordering to the  $k \geq 0$  graph vertices, i.e., a bijection from the set  $\{1, \dots, k\}$  to the set of graph vertices contained in  $D$  called *vertex-enumeration*, and an *orientation* to each vertex, i.e., labeling an edge incident to the vertex with the letter  $a$ . In the following, mainly the rational tangles  $0$ ,  $\infty$ , and  $1$  will be needed. Substituting  $0$  or  $\infty$  for graph vertices is done as depicted in Fig. 3, that is to say, vertices are cut open in one of the two possible ways determined by the vertex-orientation. Substituting the tangle  $1$  corresponds to replacing a graph vertex with a crossing  $\times$  with respect to the vertex-orientation given.

Replacing vertices of a graph diagram with all rational tangles, or, to be precise, with a representative of each equivalence class (e.g., all rational tangles in normal form), gives an invariant of diagrams with respect to ambient isotopy that consists of infinitely many link diagrams (see [18]).

Getting invariants with respect to rigid vertex isotopy is much easier: just substitute one or more arbitrary (but fixed) tangles for each vertex and get rid of the ambiguity arising from different vertex-orientations by considering all choices of such orientations. It is readily checked that this gives well-defined invariants of graph diagrams under Reidemeister moves I–IV and VI. For example, the unordered tuple  $(D_{0,0}, D_{0,\infty}, D_{\infty,0}, D_{\infty,\infty})$  defines a rigid vertex invariant of the graph belonging to the diagram  $D$  depicted in Fig. 3. Observe that the invariant  $C(G)$  defined in [7] and denoted by  $\mathcal{C}(G)$  in [10] is a special version of this type of invariants, induced by the tangles  $0$  and  $1$ , where sets are used instead of—more informative—unordered tuples.

Another point of view in considering ambient isotopic graph diagrams is to observe that the Reidemeister moves of type V, applied to a vertex during a transformation of one graph diagram into another, can be collected to a rational tangle. For example, the transformation depicted in Fig. 4 can be described by the tangle  $t = 2\ 1\ 0 \in \mathcal{K}$ .

**Definition 1.** Let  $D, D'$  be ambient isotopic graph diagrams with  $k \geq 1$  vertices and given vertex-enumerations and -orientations. If  $D_{0,\dots,0} = D'_{t_1,\dots,t_k}$  with tangles  $t_1, \dots, t_k$  then  $t_j$ , for  $j \in \{1, \dots, k\}$ , is called  *$j$ th transformation tangle* of the transformation from  $D$  into  $D'$ . If  $r_1, \dots, r_k$  are rational tangles then let  $r_j * t_j$  denote the tangle into which  $r_j$  is transformed when replacing the  $j$ th vertex of  $D$  with  $r_j$ , that is to say,  $D_{r_1,\dots,r_k} = D'_{r_1 * t_1, \dots, r_k * t_k}$ .

Considering Reidemeister move VI, it may be assumed, without loss of generality, that only the *admissible* Reidemeister moves of type V depicted in Fig. 5 have to be applied during a transformation. Furthermore, if the orientation of a vertex is chosen appropriately,

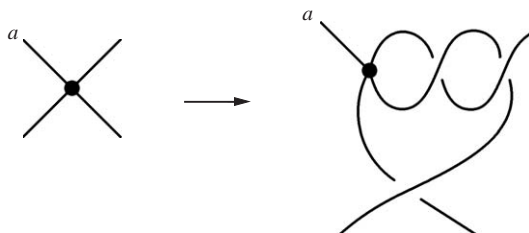


Fig. 4. Applying a sequence of Reidemeister V moves.

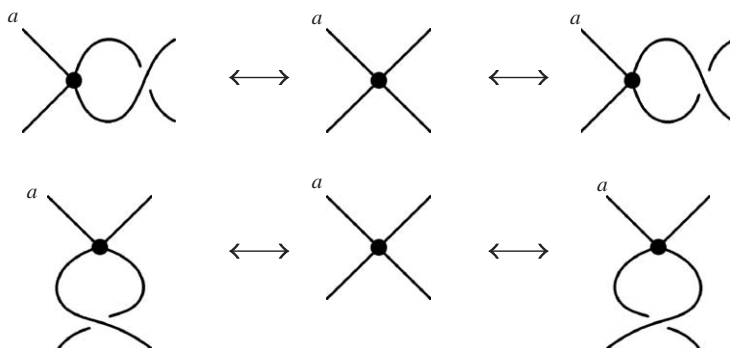


Fig. 5. Admissible Reidemeister moves of type V.



Fig. 6. Replacing a graph vertex with 0 and  $\infty$ .

the corresponding transformation tangle  $t$  can be written as  $t = b_1 \dots b_s \neq \infty$  with integer tangles  $b_1, \dots, b_s$  such that  $b_1 \dots b_s$  is in normal form. A rational tangle  $r = a_1 \dots a_l$  is transformed by the transformation tangle  $t$  into the tangle  $r * t = a_1 \dots a_{l-1} (a_l + b_1) b_2 \dots b_s$ . Especially, the tangles 0 and  $\infty$  are transformed into  $t$  and into a tangle equivalent to  $b_2 \dots b_s$ , respectively, see Fig. 6.

**Remark 2.** It is not difficult to see that the inverse of a transformation described by a tangle  $t = b_1 \dots b_s$  is given by  $t' = \overline{b_s \dots b_1}$ , i.e., if  $D_0 = D_t$  then  $D'_0 = D_{t'}$  for ambient isotopic graph diagrams  $D$  and  $D'$  in which corresponding vertices have been replaced.

### 3. Properties of graph diagrams

A link diagram is said to be *alternating* if over- and undercrossings are alternating with each other while walking along any link component in the diagram. A link diagram  $D$  is called *reduced* if it contains no *isthmus*, i.e., a crossing  $p$  such that  $D \setminus \{p\}$  has more components than  $D$ . A connected link diagram is said to be *prime* if it cannot be written as connected sum of two link diagrams both of which having at least one crossing.

**Definition 3.** A graph diagram  $D$  is said to be *alternating/reduced/prime* if, corresponding to an arbitrarily chosen vertex-enumeration and -orientation, the link diagrams  $D_{i_1, \dots, i_k}$  are alternating/reduced/prime for every choice of  $i_1, \dots, i_k \in \{0, \infty\}$ .

An example of a graph diagram that is alternating, reduced, and prime is depicted in Fig. 3.

#### Remark 4.

- (1) The definition of primality for graph diagrams does not seem to be the natural one, but it is the one that fits into the context (see the counterexample contained in the remark at the end of the next section).
- (2) In contrast to the case of link diagrams, a graph diagram that is not reduced may be *irreducible*, i.e., the number of the diagram's crossings is minimal. See [19] for examples.
- (3) A prime graph diagram which has more than one crossing is reduced.
- (4) The definition of “alternating” for graph diagrams has been adapted to the definitions of “prime” and “reduced”. Of course, a graph diagram  $D$  is alternating if and only if there is a choice of vertex-orientations such that  $D_{1, \dots, 1}$  is alternating.

It is remarkable that, after introducing an appropriate definition of primality (see [19]), for a 4-regular graph in  $\mathbb{S}^3$  which possesses a reduced, alternating diagram the property to be prime can be deduced from the corresponding property of its diagram. A proof of this fact for link diagrams is due to Menasco [13], and it can easily be extended to 4-regular graphs in  $\mathbb{S}^3$ , see [19, Theorem 8]. The following statement is an immediate consequence.

**Theorem 5.** *Let  $D$  and  $D'$  be ambient isotopic graph diagrams that are alternating and reduced. Then:*

- (a)  $D$  is connected if and only if  $D'$  is connected.
- (b)  $D$  is prime if and only if  $D'$  is prime.

To prove Tait's flying conjecture for graph diagrams, a generalization of a Tait conjecture concerning minimal crossing numbers, cited in the introduction, will be needed. A proof can be found in [19, Theorem 9].

**Theorem 6.** *Let  $\mathcal{G}$  be a 4-regular graph in  $\mathbb{S}^3$ , and let  $D$  be a reduced, alternating, prime diagram of  $\mathcal{G}$  with  $n$  crossings. Then there is no diagram of  $\mathcal{G}$  having less than  $n$  crossings, and any non-alternating diagram of  $\mathcal{G}$  has more than  $n$  crossings.*

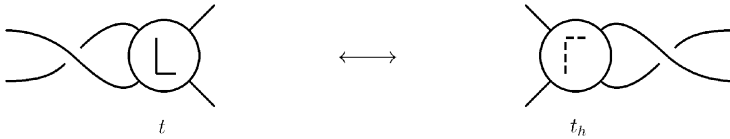


Fig. 7. A flype.

#### 4. Tait’s flying conjecture

The definition of a tangle as part of a link diagram can be extended to graph diagrams in the obvious way, and appropriate equivalence relations can be introduced likewise. In the following, always those generalized tangles are considered.

**Definition 7.** A flype is a local change in a graph diagram as depicted in Fig. 7. Using Conway’s notation [2], this corresponds to a transformation of the form  $1 + t \leftrightarrow t_h + 1$  or  $\bar{1} + t \leftrightarrow t_h + \bar{1}$ .

In [14] Tait’s flying conjecture is proved for diagrams of 4-regular rigid vertex graphs to obtain the validity of Tait’s original conjecture. This result is stated in the following theorem where, as well as in the subsequent text, the phrase *sequence of flypes* is an abbreviation for *sequence of flypes plus orientation preserving autohomeomorphisms of the sphere*. It should be mentioned that the notion of primality for graph diagrams used in [14] is different from the one used in the present text. Indeed, a graph diagram that is prime with respect to the definition given above is prime in the sense of [14], too, and thus the result from [14] can be adopted here.

**Theorem 8.** Let  $D$  and  $D'$  be rigid vertex isotopic graph diagrams that are reduced, alternating, and prime. Then there exists a finite sequence of flypes which transforms  $D$  into  $D'$ .

Considering ambient isotopy of graph diagrams, the desired extension of Tait’s flying conjecture to 4-regular graphs can be deduced immediately from the next theorem. The proof of the theorem will be given in Section 5.

**Theorem 9.** Let  $D$  and  $D'$  be graph diagrams that are reduced, alternating, and prime. Then  $D$  and  $D'$  are equivalent with respect to ambient isotopy if and only if they are equivalent with respect to rigid vertex isotopy.

**Corollary 10.** Let  $D$  and  $D'$  be ambient isotopic graph diagrams that are reduced, alternating, and prime. Then there exists a finite sequence of flypes which transforms  $D$  into  $D'$ .

**Remark 11.** Theorem 9 does not hold in general without assuming primality. A counterexample is depicted in Fig. 8: the two (alternating and reduced) graph diagrams obviously are equivalent with respect to ambient isotopy, but they are not rigid vertex isotopic because



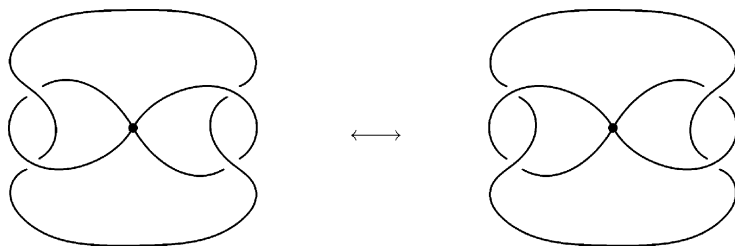


Fig. 8. Ambient isotopic diagrams that are not rigid vertex isotopic.

cutting open vertices in the two possible ways gives diagrams of 2- and 3-component links, respectively, where the diagrams belonging to the 3-component link are equivalent, and the diagrams belonging to the 2-component link correspond to different mirror images of a chiral link.

## 5. Proof of Theorem 9

Theorem 9 is proved by induction on the number of graph vertices. The main ingredient for the induction step comes from the fact that transformation tangles which describe a transformation between reduced, alternating, prime diagrams always are trivial, and thus replacing a vertex in both diagrams with the same tangle, for corresponding vertices and with respect to appropriately chosen vertex-orientations, gives ambient isotopic graph diagrams with one vertex less. As a main tool for showing this claim, properties of the *Kauffman polynomial* [8] of link diagrams and its relations to a diagram's crossing number are used.

**Definition 12.** The *Kauffman polynomial*  $\Gamma_D(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$  of a link diagram  $D$  is defined by the following properties:

- (i)  $\Gamma_D(a, z) = 1$  if  $D$  is a simple closed curve.
- (ii)  $\Gamma_{D \searrow} = a\Gamma_{D \swarrow}$  and  $\Gamma_{D \swarrow} = a^{-1}\Gamma_{D \searrow}$ ,
- (iii)  $\Gamma_{D \nearrow} + \Gamma_{D \nwarrow} = z(\Gamma_{D \searrow} + \Gamma_{D \swarrow})$ .

The highest exponent in the variable  $z$  is called  $z$ -degree of  $\Gamma_D$ .

In the following, an  $n$ -bridge  $b$  in a link diagram  $D$  is an arc of  $D$  that contains only overcrossings or only undercrossings, and the number of these crossings is  $n$ , the *length* of  $b$ .

The  $z$ -degree of the Kauffman polynomial heavily depends on the length of the longest bridge contained in a diagram. A precise formulation of this fact is given in the next theorem, for a proof see [22, Theorems 4, 5].

**Theorem 13.** Let  $D$  be a link diagram with  $n \geq 1$  crossings, and let  $b \geq 1$  be the length of an arbitrary bridge contained in  $D$ . Then:

- (a)  $z\text{-degree}(\Gamma_D) \leq n - b \leq n - 1$ ,
- (b)  $z\text{-degree}(\Gamma_D) = n - 1$  if and only if  $D$  is reduced, alternating, prime.

Now two technical lemmas on Kauffman polynomials are stated which are needed to deduce the crucial Lemma 19. For a proof of the first lemma see [18, Lemmas 8, 10].

**Lemma 14.** *Let  $D$  be a reduced, alternating, prime graph diagram with  $k \geq 1$  vertices and  $n$  crossings, and let  $r$  be a rational tangle in normal form. Furthermore, let  $i_1, \dots, i_{k-1} \in \{0, \infty\}$ . Then, corresponding to an arbitrarily chosen vertex-enumeration and -orientation, the following holds for the link diagram  $D' = D_{i_1, \dots, i_j, r, i_{j+1}, \dots, i_{k-1}}$  with  $j \in \{1, \dots, k\}$ .*

$$z\text{-degree}(\Gamma_{D'}) = \begin{cases} n + |r| - 1 & \text{if } D' \text{ is alternating,} \\ n + |r| - 2 & \text{if } D' \text{ is not alternating and } |r| \neq 1. \end{cases}$$

**Definition 15.** Let  $D$  be an alternating graph diagram with  $k \geq 1$  vertices and  $n$  crossings, supplied with an arbitrarily chosen vertex-enumeration and a vertex-orientation such that  $D_{\bar{1}, \dots, \bar{1}}$  is alternating. Then  $D$  is called *degree-reducing* if  $z\text{-degree}(\Gamma_{D_{\varepsilon_1, \dots, \varepsilon_k}}) \leq n - 2$  for every choice of  $\varepsilon_1, \dots, \varepsilon_k \in \{0, \infty, 1\}$  with  $\varepsilon_j = 1$  for at least one index  $j \in \{1, \dots, k\}$ .

**Lemma 16.** *Let  $D$  be a reduced, alternating, prime, degree-reducing graph diagram with  $k \geq 1$  vertices and  $n$  crossings, supplied with an arbitrarily chosen vertex-enumeration and a vertex-orientation such that  $D_{\bar{1}, \dots, \bar{1}}$  is alternating. If  $r_1, \dots, r_k$  are rational tangles in normal form having at least two crossings each, then*

$$z\text{-degree}(\Gamma_{D_{r_1, \dots, r_k}}) = n + |r_1| + \dots + |r_k| - t_+ - 1$$

holds, where  $t_+$  denotes the number of indices  $j$  with  $r_j > 0$ .

**Proof.** As a consequence of Lemma 14, replacing a vertex of  $D$  with a negative tangle  $r$  yields a graph diagram  $D'$  with  $k - 1$  vertices and  $n + |r|$  crossings that is reduced, alternating, prime ( $z\text{-degree}(\Gamma_{D'_{\varepsilon_1, \dots, \varepsilon_{k-1}}}) = n + |r| - 1$  with  $\varepsilon_i \in \{0, \infty\}$  can only be fulfilled if  $D'$  possesses all three properties). Thus it may be assumed that  $t_+ = k$ . In the following, it is shown that

$$z\text{-degree}(\Gamma_{D_{r_1, \dots, r_k}}) \begin{cases} = n + |r_1| + \dots + |r_k| - k - 1 & \text{if } r_j \neq 1 \text{ for all } j, \\ \leq n + |r_1| + \dots + |r_k| - k - 2 & \text{otherwise,} \end{cases}$$

holds for positive rational tangles  $r_1, \dots, r_k$  having at least one crossing each. Since Lemma 14 and Theorem 13 (if  $r_k = 1$ ) immediately give the result for the case  $k = 1$ , let  $k \geq 2$  for the rest of the proof.

Let  $l$  denote the number of indices  $j$  with  $r_j = 1$ . The proof is done by induction on  $k - l \geq 0$ . If  $k - l = 0$  then  $r_1 = \dots = r_k = 1$ , and therefore  $z\text{-degree}(\Gamma_{D_{r_1, \dots, r_k}}) \leq n - 2$  since  $D$  is degree-reducing. Thus let  $k - l \geq 1$ . Then  $r_j \neq 1$  for at least one index  $j \in \{1, \dots, k\}$ , and without loss of generality let  $r_k = a_1 \dots a_s \neq 1$ .

At first consider the case that  $r_k = 2$ . Then the recurrence formula for the Kauffman polynomial gives

$$\Gamma_{D_{r_1, \dots, r_{k-1}, 2}} = \Gamma_{D_{r_1, \dots, r_{k-1}, 0}} + z(a\Gamma_{D_{r_1, \dots, r_{k-1}, \infty}} + \Gamma_{D_{r_1, \dots, r_{k-1}, 1}}).$$

Applying the induction hypothesis immediately yields the desired inequality if  $l \geq 1$  and the desired equality if  $l = 0$ .

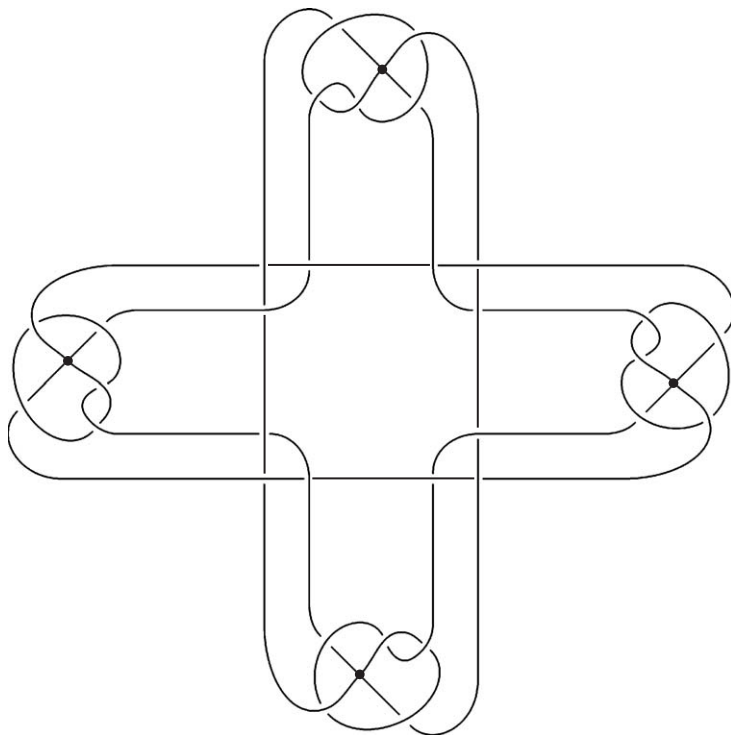


Fig. 9. A reduced, alternating, prime, vertex-separating graph diagram.

If  $r_k = a_1$  is integral then the claimed result can be verified inductively by considering the corresponding recurrence formula, and a further induction on  $s$  completes the proof.  $\square$

An important class of degree-reducing diagrams is defined next.

**Definition 17.** An alternating graph diagram  $D$  with  $k \geq 1$  vertices is called *vertex-separating* if there exist disjoint tangles  $t_1, \dots, t_k$  in  $D$  such that each tangle  $t_i$  contains exactly one graph vertex and replacing this vertex with a crossing  $\times$ , corresponding to an appropriately chosen vertex-orientation, yields a 3-bridge inside  $t_i$ .

An example of an alternating graph diagram that is vertex-separating is depicted in Fig. 9.

**Lemma 18.** A vertex-separating graph diagram is degree-reducing.

**Proof.** Considering the *numerator formula for the Kauffman polynomial* that has been deduced in [18, p. 733], it can easily be shown via induction on  $l$  that

$$z\text{-degree}(\Gamma_{D_{\varepsilon_1, \dots, \varepsilon_k}}) \leq n - l - 1 \leq n - 2$$

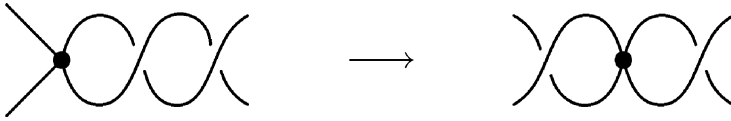


Fig. 10. Flying a vertex.

holds for a vertex-separating graph diagram  $D$  with  $k$  vertices and  $n$  crossings where  $\varepsilon_1, \dots, \varepsilon_k \in \{0, \infty, 1\}$  have been replaced for the vertices and  $l \geq 1$  denotes the number of indices  $j$  with  $\varepsilon_j = 1$ .  $\square$

**Lemma 19.** *Let  $D$  and  $D'$  be ambient isotopic graph diagrams with  $k \geq 1$  vertices, and let  $t_1, \dots, t_k$  denote the transformation tangles belonging to a sequence of Reidemeister moves which realizes the equivalence between  $D$  and  $D'$ , corresponding to chosen vertex-enumerations and -orientations of the diagrams. If  $D$  and  $D'$  both are reduced, alternating, and prime then  $|t_1| = \dots = |t_k| = 0$ .*

**Proof.** Obviously, a prime, reduced graph diagram with  $k \geq 1$  vertices has at least two crossings, and, applying Theorem 6, it is clear that  $D$  and  $D'$  have identical crossing number  $n \geq 2$ . Without loss of generality, let vertex-enumerations be chosen such that the  $j$ th vertex of  $D$  is mapped to the  $j$ th vertex of  $D'$  when the sequence of Reidemeister moves which transforms  $D$  into  $D'$  is applied. For the sake of notational convenience, assume that  $D_{\bar{1}, \dots, \bar{1}}$  and  $D'_{\bar{1}, \dots, \bar{1}}$  are alternating.

Suppose that some of the transformation tangles, which may assumed to be in normal form, have more than one crossing. Define  $\varepsilon_i := \infty$  if  $|t_i| = 1$  and  $\varepsilon_i := 0$  if  $|t_i| \neq 1$ . Then on the one hand,  $z\text{-degree}(\Gamma_{D_{\varepsilon_1, \dots, \varepsilon_k}}) = n - 1$  since  $D$  is reduced, alternating, prime. On the other hand,  $z\text{-degree}(\Gamma_{D'_{\varepsilon_1 * t_1, \dots, \varepsilon_k * t_k}}) \geq n$ , by Lemma 16, since  $\varepsilon_i * t_i = \infty$  if  $|t_i| = 1$  (perform a Reidemeister move I) and  $D'_{\varepsilon_1 * t_1, \dots, \varepsilon_k * t_k}$  can be thought to arise from a vertex-separating graph diagram, namely, the diagram in which the vertices  $v_i$  with  $|t_i| \neq 1$  not have been replaced (if there is no 3-bridge after replacing one of these vertices by a crossing then perform a flype as depicted in Fig. 10). This gives a contradiction because  $D_{\varepsilon_1, \dots, \varepsilon_k} = D'_{\varepsilon_1 * t_1, \dots, \varepsilon_k * t_k}$ . Thus  $|t_i| \leq 1$  for all indices  $i$ .

Now suppose there is an index  $j$  with  $|t_j| = 1$ . Define  $\varepsilon_j := 0$  and  $\varepsilon_i := \infty$  for  $i \neq j$ . Then  $z\text{-degree}(\Gamma_{D_{\varepsilon_1, \dots, \varepsilon_k}}) = n - 1$  as in the previous case, but it is  $\varepsilon_i * t_i = \infty$  for  $i \neq j$  and therefore either  $z\text{-degree}(\Gamma_{D'_{\varepsilon_1 * t_1, \dots, \varepsilon_k * t_k}}) \leq n - 2$  if  $t_j = 1$  because the diagram contains a 3-bridge (otherwise there would be a contradiction to reducedness and primality of the diagram), or  $z\text{-degree}(\Gamma_{D'_{\varepsilon_1 * t_1, \dots, \varepsilon_k * t_k}}) = n$  if  $t_j = \bar{1}$  because the diagram is reduced, alternating, and prime. Again a contradiction in both cases, thus  $|t_i| = 0$  for all indices  $i$ .  $\square$

Lemma 19 shows that transformation tangles belonging to two ambient isotopic graph diagrams always are trivial if both diagrams are reduced, alternating, and prime. Thus replacing vertices of such diagrams with tangles gives an ambient isotopy invariant of graph diagrams up to a choice of vertex-orientations. Especially:

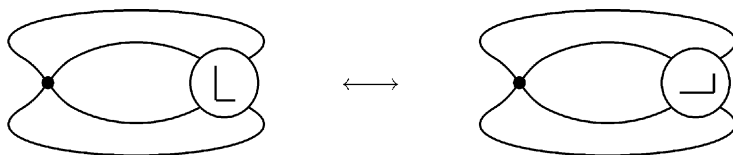


Fig. 11. Rigid vertex isotopic diagrams.

**Corollary 20.** *Let  $D$  and  $D'$  be ambient isotopic graph diagrams with  $k \geq 1$  vertices that are reduced, alternating, and prime. Let  $D_1, D'_1$  denote diagrams that arise from substituting the tangle  $l$  for corresponding vertices of  $D$  and  $D'$ , respectively, and let vertex-orientations be chosen such that  $D_1$  and  $D'_1$  both are alternating. Then  $D_1$  and  $D'_1$  are equivalent with respect to ambient isotopy.*

**Remark 21.** Observe that it is not clear, at this stage, that the diagrams  $D_1$  and  $D'_1$  of Corollary 20 are rigid vertex isotopic since the sequence of Reidemeister moves of type V applied to a vertex leads to a transformation tangle which is equivalent to a trivial tangle but not necessarily equal to it.

**Lemma 22.** *The graph diagrams that are depicted in Fig. 11 are rigid vertex isotopic.*

**Proof.** Perform the transformations depicted in Fig. 12, where “R.” is an abbreviation for “Reidemeister”, and observe that move 4 consists of pushing the diagram’s upper arc upwards beyond “ $\infty$ ”.  $\square$

Now let  $D$  and  $D'$  be ambient isotopic diagrams which fulfil the assumptions of Theorem 9. Of course, if the corresponding graph has no vertices then the statement follows from Theorem 8.

For the induction step, an arbitrary graph vertex of  $D$  and the corresponding vertex of  $D'$  are replaced with a crossing each such that the diagrams arising, which shall be denoted by  $D_1$  and  $D'_1$ , respectively, are alternating. Then it follows from Corollary 20 and the induction hypothesis that  $D_1$  and  $D'_1$  are rigid vertex isotopic and thus can be connected by a finite sequence of flypes, by Theorem 8.

Considering the defining Fig. 7, there are three different types of flype that can be applied to  $D_1$ : either the substituted crossing lies outside the depicted part of the diagram, or it lies inside the tangle  $t$ , or it is identical to the crossing next to  $t$ . Call the latter one *essential* flype. Because of the obvious one-to-one correspondence between the crossings before and after applying a flype it is possible to keep track of the substituted crossing during the whole sequence of flypes connecting  $D_1$  and  $D'_1$ . Obviously, if the sequence contains no essential flypes then it gives rise to a sequence of flypes that can be applied to the diagram  $D$  and has  $D'$  as final diagram, and the induction step is done.

Now assume that the flying sequence contains an essential flype. Since  $D$  is prime, cutting open the substituted crossing in either way must yield a prime graph diagram, and therefore an essential flype can, essentially, only be applied to  $D_1$  as depicted in the first two pictures of Fig. 13, i.e., the whole diagram is involved in the flying move. Applying Lemma

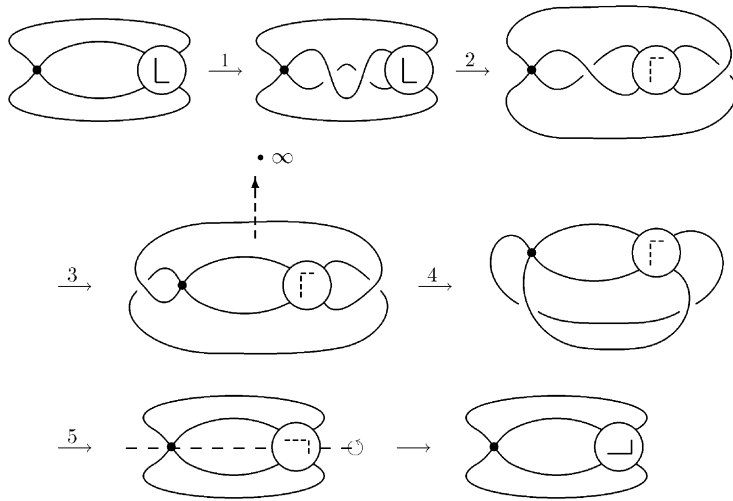


Fig. 12. 1 = R. II; 2 = flype; 3 = R. VI; 4 = planar isotopy; 5 = R. II.

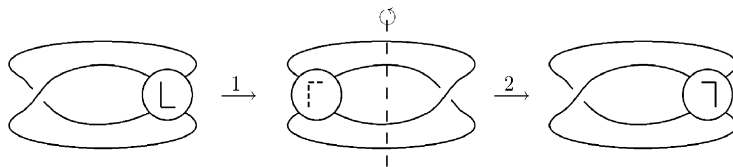


Fig. 13. 1 = flype; 2 = rotation around dashed axis.

22 twice shows that there arise rigid vertex isotopic diagrams from re-inserting vertices for crossings in first and last diagram of Fig. 13. Therefore an essential flype applied to  $D_1$  is related to a rigid vertex isotopy of the diagram  $D$ , and an induction on the number of essential flypes in the flying sequence completes the proof of Theorem 9.

### Acknowledgments

The author would like to thank the referee for inspiring him to prove the main result in full generality.

### References

- [1] G. Burde, H. Zieschang, *Knots*, de Gruyter, Berlin, 1985.
- [2] J.H. Conway, An enumeration of knots and links, and some of their algebraic properties, in: J. Leech (Ed.), *Computational Problems in Abstract Algebra*, Pergamon Press, Oxford, 1970, pp. 329–358.
- [3] J.R. Goldman, L.H. Kauffman, Rational Tangles, *Adv. Appl. Math.* 18 (1997) 300–332.
- [4] P. de la Harpe, M. Kervaire, C. Weber, On the Jones polynomial, *Enseign. Math.* 32 (1986) 271–335.

- [5] D. Jonish, K.C. Millett, Isotopy invariants of graphs, *Trans. Amer. Math. Soc.* 327 (1991) 655–702.
- [6] L.H. Kauffman, State models and the Jones polynomial, *Topology* 26 (1987) 395–407.
- [7] L.H. Kauffman, Invariants of graphs in three-space, *Trans. Amer. Math. Soc.* 311 (1989) 697–710.
- [8] L.H. Kauffman, An invariant of regular isotopy, *Trans. Amer. Math. Soc.* 318 (1990) 417–471.
- [9] L.H. Kauffman, *Knots and Physics*, World Scientific, Singapore, 1991.
- [10] L.H. Kauffman, P. Vogel, Link polynomials and a graphical calculus, *J. Knot Theory Ramifications* 1 (1992) 59–104.
- [11] A. Kawachi, *A Survey of Knot Theory*, Birkhäuser, Boston, 1996.
- [12] W.B.R. Lickorish, *An Introduction to Knot Theory*, Springer, New York, 1997.
- [13] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, *Topology* 23 (1984) 37–44.
- [14] W. Menasco, M. Thistlethwaite, The classification of alternating links, *Ann. of Math.* 138 (1993) 113–171.
- [15] K. Murasugi, Jones polynomials and classical conjectures in knot theory, *Topology* 26 (1987) 187–194.
- [16] K. Murasugi, *Knot Theory and Its Applications*, Birkhäuser, Boston, 1996.
- [17] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, second printing, 1990.
- [18] J. Sawollek, Embeddings of 4-Regular Graphs into 3-Space, *J. Knot Theory Ramifications* 6 (1997) 727–749.
- [19] J. Sawollek, Alternating Diagrams of 4-Regular Graphs in 3-Space, *Topology Appl.* 93 (1999) 261–273.
- [20] A. Schrijver, Tait’s flying conjecture for well-connected links, *J. Combin. Theory Ser. B* 58 (1993) 65–146.
- [21] P.G. Tait, On knots I, II, III, *Scientific Papers*, Vol. I, Cambridge University Press, London, 1898, pp. 273–347.
- [22] M.B. Thistlethwaite, Kauffman’s polynomial and alternating links, *Topology* 27 (1988) 311–318.
- [23] S. Yamada, An Invariant of Spatial Graphs, *J. Graph Theory* 13 (1989) 537–551.
- [24] D.N. Yetter, Category Theoretic Representations of Knotted Graphs in  $\mathbb{S}^3$ , *Adv. Math.* 77 (1989) 137–155.
- [25] Y. Yokota, Topological invariants of graphs in 3-space, *Topology* 35 (1996) 77–87.