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Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II

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ABSTRACT

We investigate the convergence rates for total variation regularization of the problem of identifying (i) the coefficient q in the Neumann problem for the elliptic equation $-\mathrm{div}(q\nabla u) = f$ in Ω, $q\partial u/\partial n = g$ on $\partial\Omega$, (ii) the coefficient a in the Neumann problem for the elliptic equation $-\Delta u + au = f$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$, $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, when u is imprecisely given by $z^\delta \in H^1(\Omega)$, $\|u - z^\delta\|_{H^1(\Omega)} \leqslant \delta$, $\delta > 0$. We regularize these problems by correspondingly minimizing the strictly convex functionals

$$\frac{1}{2}\int_\Omega q\left|\nabla\left(U(q) - z^\delta\right)\right|^2 dx + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right),$$

and

$$\frac{1}{2}\int_\Omega \left|\nabla\left(U(a) - z^\delta\right)\right|^2 dx + \frac{1}{2}\int_\Omega a\left(U(a) - z^\delta\right)^2 dx + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right)$$

over admissible sets, where $U(q)$ $(U(a))$ is the solution of the first (second) Neumann boundary value problem, $\rho > 0$ is the regularization parameter. Taking the solutions of these optimization problems as the regularized solutions to the corresponding identification problems, we obtain the convergence rates of them to the solution of the inverse problem in the sense of the Bregman distance and in the L^2-norm under relatively simple source conditions without the smallness requirement on the source functions.

1. Introduction

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geqslant 1$ with Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we continue the paper [19] on the investigation of total variation regularization for the problem of identifying the coefficient q in the Neumann problem for the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$q\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \tag{1.2}$$

or the coefficient a in the Neumann problem for the elliptic equation

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$$-\Delta u + au = f \quad \text{in } \Omega, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \tag{1.4}$$

from the observations $z^\delta \in H^1(\Omega)$ of the exact solution $\bar{u}$ of (1.1)–(1.2) (or (1.3)–(1.4)) with

$$\|\bar{u} - z^\delta\|_{H^1(\Omega)} \leqslant \delta, \tag{1.5}$$

$\delta > 0$ being given, while f and g are prescribed. For practical models and surveys on these problems we refer the reader to our recent papers [18–20] and the references therein.

We note that in our setting we assume to have observations $z^\delta \in L^2(\Omega)$, $\nabla z^\delta \in (L^2(\Omega))^d$ for the solution u and its gradient, respectively. In [20] we have discussed about this assumption and also about the ill-posedness of the above identification problems in the L^2 and L^∞ norms (see, more details in [2,6,8,26,32]). Recently, in [24] Knowles and LaRussa have shown that these problems are well-posed but in the weak L^2 topology on the recovered coefficients. Such a similar property in the $H^{-1}(\Omega)$ topology has also been noted by Kohn and Lowe in [26].

As the above identification problems are ill-posed in the L^2 and L^∞ norms, several authors applied Tikhonov regularization to stabilize them. However, as noted in [18,19], previously only Engl, Kunisch and Neubauer [12,11] considered the convergence rates of the method. In fact, these authors directly applied their theory of nonlinear ill-posed problems to the above inverse problems; and to obtain a convergence rate they have to require some smallness condition of the source functions which is very hard to verify. Recently, in [18,20], based on another approach, we got convergence rates for Tikhonov regularization with L^2-stabilization of the above inverse problems under rather simple source conditions without requiring a smallness condition of the source functions.

To ease the exposition, suppose that the coefficient q in (1.1)–(1.2) is given so that we can determine the unique solution u and thus define a nonlinear coefficient-to-solution map from q to the solution $u = u(q) := U(q)$. Then the inverse problem has the form: solve the nonlinear equation

$$U(q) = \bar{u} \quad \text{for } q \text{ with } \bar{u} \text{ being given.} \tag{1.6}$$

To estimate a possible discontinuous or highly oscillating coefficient q, some authors used the output least-squares method with total variation regularization (see, e.g., [5,17,30]). Their technique led to the *non-convex* optimization problem

$$\min_{q \in \mathcal{Q}} \int_\Omega \big(U(q) - z^\delta\big)^2\,dx + \rho \int_\Omega |\nabla q|. \tag{1.7}$$

Here $\rho > 0$ is a regularization parameter, $\mathcal{Q}$ is some admissible set of the coefficients, z^δ is the observed data of the exact data $\bar{u}$ and $\int_\Omega |\nabla q|$ is the total variation of the function q. However, these authors did not consider the convergence rate of the method. Furthermore, there are some difficulties with the least squares approach to (1.6). First, since the cost function appeared in (1.7) is not convex, it is difficult to find global minimizers. Second, it appeared that obtaining convergence rates for Tikhonov regularization (1.7) is still an open problem [29]. To overcome these, in [19] we apply the total variation regularization method to new *convex energy functionals* (see Lemmas 2.4 and 3.2 in [19]) for identifying q in (1.1)–(1.2) and a in (1.3)–(1.4), and obtain convergence rates for this approach. Namely, for identifying q in (1.1)–(1.2), we consider the *convex* minimization problem (see Lemmas 2.4 and 3.2 in [19])

$$\min_{q \in Q_{ad}} \frac{1}{2} \int_\Omega q \big|\nabla\big(U(q) - z^\delta\big)\big|^2\,dx + \rho \int_\Omega |\nabla q|, \tag{1.8}$$

and for identifying a in (1.3)–(1.4) the *convex* minimization problem

$$\min_{a \in A_{ad}} \frac{1}{2} \int_\Omega \big|\nabla\big(U(a) - z^\delta\big)\big|^2\,dx + \frac{1}{2} \int_\Omega a\big(U(a) - z^\delta\big)^2\,dx + \rho \int_\Omega |\nabla a|. \tag{1.9}$$

Here, $U(q)$ and $U(a)$ are the coefficient-to-solution nonlinear maps for (1.1)–(1.2) and (1.3)–(1.4) with Q_{ad} and A_{ad} being the admissible sets, respectively. In [19], we obtain convergence rates of regularized solutions to the solution of the coefficient identification problems under source conditions which are easy to check (see Theorems 2.9 and 3.6 of [19]). However, our convergence rates in this approach are just in the sense of the Bregman distance which is in general not a metric. To enhance these results, in this paper we add an additional L^2-stabilization to the *convex energy functionals* (1.8) and (1.9) for respectively identifying q in (1.1)–(1.2) and a in (1.3)–(1.4), and obtain convergence rates not only in sense of the Bregman distance but also in the $L^2(\Omega)$-norm. That is, for identifying q in (1.1)–(1.2), we consider the *strictly convex* minimization problem

$$\min_{q \in Q_{ad}} \frac{1}{2} \int_\Omega q \big|\nabla\big(U(q) - z^\delta\big)\big|^2\,dx + \rho\bigg(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\bigg), \tag{1.10}$$

and for identifying a in (1.3)–(1.4) the *strictly convex* minimization problem

$$\min_{a\in A_{ad}} \frac{1}{2}\int_\Omega \left|\nabla\left(U(a)-z^\delta\right)\right|^2 dx + \frac{1}{2}\int_\Omega a\left(U(a)-z^\delta\right)^2 dx + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right). \tag{1.11}$$

To present our results, we briefly summarize the space of functions with bounded total variation and the notion of the Bregman distance; for more details, the reader may consult Attouch, Buttazzo and Michaille [1], Evans and Gariepy [13], Guisti [14], Burger and Osher [4], Resmerita and Scherzer [29] and the references therein.

A function $q \in L^1(\Omega)$ is said to be of bounded total variation if

$$TV(q) := \int_\Omega |\nabla q| := \sup\left\{\int_\Omega q \operatorname{div} g\, dx \;\Big|\; g \in C^1_c(\Omega)^d,\ \left|g(x)\right|_\infty \leqslant 1,\ x\in\Omega\right\} < \infty. \tag{1.12}$$

Here $|\cdot|_\infty$ denotes the ℓ_∞-norm on $\mathbb{R}^d$ defined by $|x|_\infty = \max_{1\leqslant i\leqslant d}|x_i|$. The space of all functions in $L^1(\Omega)$ with bounded total variation is denoted by

$$BV(\Omega) = \left\{q \in L^1(\Omega) \;\Big|\; \int_\Omega |\nabla q| < \infty\right\}.$$

It is the Banach space under the norm

$$\|q\|_{BV(\Omega)} := \|q\|_{L^1(\Omega)} + \int_\Omega |\nabla q|.$$

Further, if Ω is an open bounded set in $\mathbb{R}^d$ ($d \geqslant 1$) with Lipschitz boundary, then $W^{1,1}(\Omega) \subsetneq BV(\Omega)$ (Giusti [14, pp. 3–4]).

Let $\mathcal{H}$ be a Banach space with $\mathcal{H}^*$ being the dual space of it, $R : \mathcal{H} \to (-\infty, +\infty]$ is a proper convex functional and $\partial R(q)$ stands for the subdifferential of R at $q \in \operatorname{Dom} R := \{q \in \mathcal{H} \mid R(q) < +\infty\} \neq \emptyset$ defined by

$$\partial R(q) := \left\{q^* \in \mathcal{H}^* \;\big|\; R(p) \geqslant R(q) + \langle q^*, p-q\rangle_{(\mathcal{H}^*,\mathcal{H})} \text{ for all } p \in \mathcal{H}\right\}.$$

The set $\partial R(q)$ may be empty; however, if R is continuous at q, then it is nonempty. Further, $\partial R(q)$ is convex and weak* compact (see, [10], Propositions 5.1, 5.2, pp. 21–22). In case $\partial R(q) \neq \emptyset$, for any fixed $p \in \mathcal{H}$ we denote by

$$D_R(p,q) := \left\{R(p) - R(q) + \langle q^*, p-q\rangle_{(\mathcal{H}^*,\mathcal{H})} \;\big|\; q^* \in \partial R(q)\right\}.$$

Then for a fixed element $q^* \in \partial R(q)$,

$$D_R^{q^*}(p,q) := R(p) - R(q) + \langle q^*, p-q\rangle_{(\mathcal{H}^*,\mathcal{H})} \tag{1.13}$$

is called the *Bregman distance with respect to R and q^* of two elements $p, q \in \mathcal{H}$*.

The notion of Bregman distance was first given by Bregman [3] along with an iterative algorithm for minimizing (1.13) for Fréchet differentiable R and it was generalized by Kiwiel [21] to nonsmooth but strictly convex R. Burger and Osher [4] further generalized this notion for R being neither smooth, nor strictly convex. In general, the Bregman distance is not a metric on $\mathcal{H}$. However, for each $q^* \in \partial R(q)$ the $D_R^{q^*}(p,q) \geqslant 0$ for any $p \in \mathcal{H}$ and $D_R^{q^*}(q,q) = 0$. Further, in case R is a strictly convex function, $D_R^{q^*}(p,q) = 0$ if and only if $p = q$. In recent years, this notion was proved to be useful in getting the convergence rates of regularization methods in Banach spaces (see, e.g., [4,16,28,29] and the references therein).

Now we formulate our convergence results as follows.

Set $R(\cdot) = \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|$. Denote by q_ρ^δ the solution of (1.10), $q^\dagger$ the R-minimizing norm solution of the problem of identifying q in (1.1)–(1.2) (see Section 2.1). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ (see page 596 for the definition of $H^1_\diamond(\Omega)$) such that

$$U'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger)$$

for some element ℓ in $\partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$. Here $U'(q^\dagger)^*$ is the adjoint to the Fréchet derivative of $U(q^\dagger)$. Then, we have the convergence rates

$$\left\|q_\rho^\delta - q^\dagger\right\|^2_{L^2(\Omega)} + D^\ell_{TV}\left(q_\rho^\delta, q^\dagger\right) = \mathcal{O}(\delta) \quad\text{and}\quad \left\|U\left(q_\rho^\delta\right) - z^\delta\right\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Similarly, set $T(\cdot) := \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|$. Denote by a_ρ^δ the solution of (1.11), $a^\dagger$ the T-minimizing norm solution of the problem of identifying a in problem (1.3)–(1.4) (see Section 3.1). Assume that there exists a function $w^* \in H^1(\Omega)^*$ such that

$$U'\left(a^\dagger\right)^* w^* = a^\dagger + \lambda \in \partial T\left(a^\dagger\right)$$

for some element λ in $\partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$. Here, $U'(a^\dagger)^*$ is the adjoint to the Fréchet derivative of $U(a^\dagger)$. Then, we have the convergence rates

$$\|a_\rho^\delta - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}\left(a^\delta_\rho, a^\dagger\right) = \mathcal{O}(\delta) \quad \text{and} \quad \|U\left(a^\delta_\rho\right) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Our above source conditions are easy to check and much weaker than the related ones, since we remove the so-called small enough condition on the source functions which is popularized and very hard to check in the theory of regularization of nonlinear ill-posed problems [12,29]. We also note that, to our knowledge, up to now there is only the paper by Chavent and Kunisch [7] devoted to convergence rates for such a total variation regularization of a certain *linear* ill-posed problem. Besides, the use of the convex energy functionals in our identification problems is dated back to Knowles [25,22,23] and Zou [34].

This paper is organized as follows. In Section 2 we will prove our result on convergence rates for total variation regularization combining with additional L^2-stabilization of the diffusion coefficient identification problem (1.1)–(1.2) and in Section 3 the related result for the reaction coefficient identification problem (1.3)–(1.4). In Section 4 we present the related result for identifying the diffusion coefficient in problems with Dirichlet or mixed boundary conditions. The discussion on our source conditions is given at the end of each section.

In the whole paper we assume that Ω is an open bounded connected domain in $\mathbb{R}^d, d \geqslant 1$ with Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ in (1.1) and (1.3), and $g \in L^2(\partial\Omega)$ in (1.2) and (1.4) are given. We use the standard notion of Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H^1_0(\Omega \cup \Gamma)$ and $W^{1,\infty}(\Omega)$ from the books [27,31]. Moreover, for the simplicity of notation, as there will be no ambiguity, we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

2. The diffusion coefficient identification problem

In this section we investigate the following coefficient identification problem in the Neumann problem for elliptic partial differential equations.

Find the coefficient q in the problem (1.1)–(1.2) *subject to the constraints*

$$q \in Q := \left\{q \in L^\infty(\Omega) \;\middle|\; 0 < \underline{q} \leqslant q(x) \leqslant \bar{q} \text{ a.e. on } \Omega\right\} \tag{2.1}$$

with $\underline{q}$ and $\bar{q}$ being given positive constants, when the solution u is imprecisely given in Ω.

2.1. Problem setting and regularization

We consider the problem (1.1)–(1.2) assuming that the functions f and g satisfy the compatibility condition

$$\int_\Omega f + \int_{\partial\Omega} g = 0.$$

Then a function u in $H^1_\diamond(\Omega)$, the closed subspace of $H^1(\Omega)$ consisting all the functions $u \in H^1(\Omega)$ with mean-zero:

$$H^1_\diamond(\Omega) := \left\{u \in H^1(\Omega) \;\middle|\; \int_\Omega u\,dx = 0\right\},$$

is said to be a weak solution of the problem (1.1)–(1.2), if

$$\int_\Omega q\nabla u \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv, \quad \forall v \in H^1_\diamond(\Omega). \tag{2.2}$$

By the aid of the Poincaré–Friedrichs inequality in $H^1_\diamond(\Omega)$, we obtain that there exists a positive constant α depending only on $\underline{q}$ and the domain Ω such that the following coercivity condition is fulfilled

$$\int_\Omega q|\nabla u|^2 \geqslant \alpha \|u\|^2_{H^1(\Omega)} \quad \text{for all } u \in H^1_\diamond(\Omega) \text{ and } q \in Q. \tag{2.3}$$

Here,

$$\alpha := \frac{\underline{q} C_\Omega}{1 + C_\Omega} > 0 \tag{2.4}$$

with C_Ω being the positive constant, depending only on Ω, appeared in the Poincaré–Friedrichs inequality:

$$C_\Omega \int\limits_\Omega v^2 \leqslant \int\limits_\Omega |\nabla v|^2 \quad \text{for all } v \in H^1_\diamond(\Omega).$$

It follows from the inequality (2.3) and the Lax–Milgram lemma that for all $q \in Q$, there is a unique weak solution in $H^1_\diamond(\Omega)$ of (1.1)–(1.2) which satisfies the inequality

$$\|u\|_{H^1(\Omega)} \leqslant \Lambda_\alpha \big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big),$$

where Λ_α is a positive constant depending only on α.

Thus, in the direct problem we defined the nonlinear coefficient-to-solution operator $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ which maps the coefficient $q \in Q$ to the solution $U(q) \in H^1_\diamond(\Omega)$ of the problem (1.1)–(1.2). The inverse problem is stated as follows: given $\bar{u} := U(q) \in H^1_\diamond(\Omega)$ find $q \in Q$.

Now we assume that $\bar{u}$ is the exact solution of (1.1)–(1.2). It means that there exists some $q \in Q$ such that $\bar{u} = U(q)$. We assume that instead of the exact $\bar{u}$ we have only its observations $z^\delta \in H^1_\diamond(\Omega)$ such that

$$\big\|\bar{u} - z^\delta\big\|_{H^1(\Omega)} \leqslant \delta \tag{2.5}$$

with $\delta > 0$. Our problem is to reconstruct q from z^δ. For solving this ill-posed problem we minimize the *strictly convex* functional

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int\limits_\Omega |\nabla q|\right), \tag{$P^q_{\rho,\delta}$}$$

where $Q_{ad} := Q \cap BV(\Omega)$ is the admissible set, $\rho > 0$ is the regularization parameter and

$$J_{z^\delta}(q) := \frac{1}{2}\int\limits_\Omega q\big|\nabla\big(U(q) - z^\delta\big)\big|^2, \quad q \in Q. \tag{2.6}$$

In the following we will see that the problem $(P^q_{\rho,\delta})$ has a *unique solution* q^δ_ρ on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm set Q_{ad}, which is called *regularized solution* to our inverse problem (see Theorem 2.7). Due to the nonempty convexity, closedness and boundedness in the $L^2(\Omega)$-norm of the set

$$\Pi_{Q_{ad}}(\bar{u}) := \big\{q \in Q_{ad} \;\big|\; U(q) = \bar{u}\big\} \tag{2.7}$$

(see Lemma 2.5), we can conclude that there is a *unique solution* $q^\dagger$ of the problem

$$\min_{q \in \Pi_{Q_{ad}}(\bar{u})} \left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int\limits_\Omega |\nabla q|\right), \tag{Π^q}$$

which we call *R-minimizing norm solution* to our inverse problem, where $R(\cdot) := \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|$.

Our aim in this section is to investigate convergence rates of q^δ_ρ to the R-minimizing norm solution $q^\dagger$ of the equation $U(q) = \bar{u}$.

The following results are useful.

Lemma 2.1. *(See [14, pp. 7–17].)* (i) *Let (q_n) be a bounded sequence in the $BV(\Omega)$-norm. Then, there exist a subsequence (q_{k_n}) of it and an element $q \in BV(\Omega)$ such that (q_{k_n}) converges to q in the $L^1(\Omega)$-norm.*

(ii) *Let (q_n) be a sequence in $BV(\Omega)$ which converges to q in the $L^1(\Omega)$-norm. Then, $q \in BV(\Omega)$ and*

$$\int\limits_\Omega |\nabla q| \leqslant \liminf_n \int\limits_\Omega |\nabla q_n|.$$

Lemma 2.2. *(See [19, Lemma 2.2].) The total variation is continuous on $BV(\Omega)$, i.e., if $(q_n) \subset BV(\Omega)$ converges to $q \in BV(\Omega)$, then*

$$\lim_n \int\limits_\Omega |\nabla q_n| = \int\limits_\Omega |\nabla q|.$$

Lemma 2.3. *(See [15, Theorem 2.4], [19, Lemma 2.3].) The coefficient-to-solution operator $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is continuously Fréchet differentiable on the set Q. For each $q \in Q$, the Fréchet derivative $U'(q)$ of $U(q)$ has the property that the differential $\eta := U'(q)h$ with $h \in L^\infty(\Omega)$ is the (unique) weak solution in $H^1_\diamond(\Omega)$ of the Neumann problem*

$$-\operatorname{div}(q\nabla\eta) = \operatorname{div}\big(h\nabla U(q)\big) \quad \text{in } \Omega, \qquad q\frac{\partial\eta}{\partial n} = -h\frac{\partial U(q)}{\partial n} \quad \text{on } \partial\Omega$$

in the sense that it satisfies the equation

$$\int_\Omega q\nabla\eta\nabla v = -\int_\Omega h\nabla U(q)\nabla v \tag{2.8}$$

for all $v \in H^1_\diamond(\Omega)$. *Moreover,*

$$\|\eta\|_{H^1(\Omega)} \leqslant \frac{\Lambda_\alpha}{\alpha}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big)\|h\|_{L^\infty(\Omega)}$$

for all $h \in L^\infty(\Omega)$.

Lemma 2.4. *The functional* $J_{z^\delta}(\cdot)$ *defined by* (2.6) *is continuous and convex on the convex set* Q_{ad} *with respect to the* $L^2(\Omega)$*-norm.*

Proof. Suppose that the sequence $(q_n) \subset Q_{ad}$ converges to q in the $L^2(\Omega)$-norm. It follows from Lemma 2.1 that $q \in Q_{ad}$. By the same reasonings as in the proof of Theorem 2.1 in [18], we conclude that $J_{z^\delta}(q_n) \to J_{z^\delta}(q)$ as $n \to \infty$. Besides, the proof of the fact that $J_{z^\delta}(\cdot)$ is convex on the set Q_{ad} is based on the similar reasonings as in [15, § 3.1] and [18], Lemma 2.3. We note that in [15] Gockenbach and Khan proved a similar result but for the L^∞-norm. □

Lemma 2.5. *The set* $\Pi_{Q_{ad}}(\bar{u})$ *in* (2.7) *is nonempty, convex, closed and bounded in the* $L^2(\Omega)$*-norm.*

Proof. The proof of this lemma is based on the same reasonings of Lemma 2.1 in [18]. □

Lemma 2.6. *Let* $\hat{\mathcal{Q}} \subset BV(\Omega)$ *be nonempty, convex, closed and bounded in the* $L^2(\Omega)$*-norm. Suppose that* Ξ *is a non-negative, strictly convex and continuous function on* $\hat{\mathcal{Q}}$ *in the* $L^2(\Omega)$*-norm. Then, the problem*

$$\min_{q\in\hat{\mathcal{Q}}} \Xi(q) + \int_\Omega |\nabla q| \tag{2.9}$$

has a unique solution.

Proof. Let (q_n) be a sequence in $\hat{\mathcal{Q}}$ such that

$$\lim_n\left(\Xi(q_n) + \int_\Omega |\nabla q_n|\right) = \inf_{q\in\hat{\mathcal{Q}}}\left(\Xi(q) + \int_\Omega |\nabla q|\right).$$

It follows that the set $(\int_\Omega |\nabla q_n|)_{n\in\mathbb{N}}$ is bounded. Since (q_n) is bounded in the $L^2(\Omega)$-norm and $\operatorname{mes}(\Omega) < \infty$, it is bounded in the $L^1(\Omega)$-norm. Hence (q_n) is bounded in the $BV(\Omega)$-norm. By Lemma 2.1, we conclude that there exist a subsequence (q_{1_n}) of (q_n) and an element $\hat{q} \in \hat{\mathcal{Q}}$ such that (q_{1_n}) converges to $\hat{q}$ in the $L^1(\Omega)$-norm, weakly in $L^2(\Omega)$ and $\int_\Omega |\nabla\hat{q}| \leqslant \liminf_n \int_\Omega |\nabla q_{1_n}|$. Since Ξ is convex and continuous on $\hat{\mathcal{Q}}$ in the $L^2(\Omega)$-norm, it is weakly lower semicontinuous in $L^2(\Omega)$. Therefore,

$$\Xi(\hat{q}) + \int_\Omega |\nabla\hat{q}| \leqslant \liminf_n\left(\Xi(q_{1_n}) + \int_\Omega |\nabla q_{1_n}|\right) = \inf_{q\in\hat{\mathcal{Q}}}\left(\Xi(q) + \int_\Omega |\nabla q|\right).$$

This means that $\hat{q}$ is the (unique) solution of the problem (2.9). □

Theorem 2.7. (i) *There exists a unique solution* q^δ_ρ *of the problem* $(P^q_{\rho,\delta})$.

(ii) *There exists a unique solution* $q^\dagger$ *of the problem* (Π^q).

Proof. The proposition of the theorem directly follows from Lemmas 2.4–2.6. □

In the following we denote by

$$\mathfrak{X} := L^\infty(\Omega) \cap BV(\Omega).$$

Then, $\mathfrak{X}$ is a Banach space with the norm

$$\|q\|_{\mathfrak{X}} := \|q\|_{L^\infty(\Omega)} + \|q\|_{BV(\Omega)}.$$

Further,

$$L^\infty(\Omega)^* \subset \mathfrak{X}^* \quad \text{and} \quad BV(\Omega)^* \subset \mathfrak{X}^*.$$

On the other hand, we will write $\mathfrak{X}_{BV(\Omega)} := (\mathfrak{X}, \|\cdot\|_{BV(\Omega)})$ ($\mathfrak{X}_{L^\infty(\Omega)} := (\mathfrak{X}, \|\cdot\|_{L^\infty(\Omega)})$) to denote the space $\mathfrak{X}$ with respect to the $BV(\Omega)$-norm ($L^\infty(\Omega)$-norm).

The functional $J_{z^\delta}(\cdot)$ in (2.6) is Fréchet differentiable on Q in the $L^\infty(\Omega)$-norm and for each $q \in Q$

$$J'_{z^\delta}(q)h = \langle J'_{z^\delta}(q), h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle J'_{z^\delta}(q), h\rangle_{(\mathfrak{X}^*, \mathfrak{X})}, \quad \forall h \in \mathfrak{X}. \tag{2.10}$$

Furthermore, for any $\ell \in \mathfrak{X}^*_{BV(\Omega)}$

$$\langle \ell, h\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle \ell, h\rangle_{(\mathfrak{X}^*, \mathfrak{X})}, \quad \forall h \in \mathfrak{X}. \tag{2.11}$$

Besides, for each $q \in Q$ and any $h \in \mathfrak{X}$, since

$$\left|\langle q, h\rangle_{L^2(\Omega)}\right| \leqslant \|q\|_{L^\infty(\Omega)}\|h\|_{L^1(\Omega)} \leqslant \|q\|_{L^\infty(\Omega)}\|h\|_{\mathfrak{X}_{BV(\Omega)}} \leqslant \|q\|_{L^\infty(\Omega)}\|h\|_{\mathfrak{X}},$$

we get

$$\langle q, h\rangle_{L^2(\Omega)} = \langle q, h\rangle_{(L^1(\Omega)^*, L^1(\Omega))} = \langle q, h\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle q, h\rangle_{(\mathfrak{X}^*, \mathfrak{X})}. \tag{2.12}$$

Using the same arguments as in the proof of Lemma 2.6 of [19] we get the necessary and sufficient optimality condition for the problems $(P^q_{\rho,\delta})$ and (Π^q).

Lemma 2.8. (i) *Let $q^\delta_\rho \in Q_{ad}$. Then q^δ_ρ is a (unique) solution of $(P^q_{\rho,\delta})$ if and only if for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\delta_\rho)$, the inequality*

$$J'_{z^\delta}(q^\delta_\rho)(q - q^\delta_\rho) + \rho\langle q^\delta_\rho, q - q^\delta_\rho\rangle_{L^2(\Omega)} + \rho\langle \ell, q - q^\delta_\rho\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \geqslant 0 \tag{2.13}$$

is satisfied for all q in Q_{ad}.

(ii) *Let $q^\dagger \in \Pi_{Q_{ad}}(\bar{u})$. Then $q^\dagger$ is a (unique) solution of (Π^q) if and only if for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$, the inequality*

$$\langle q^\dagger, q - q^\dagger\rangle_{L^2(\Omega)} + \langle \ell, q - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \geqslant 0$$

holds for all q in $\Pi_{Q_{ad}}(\bar{u})$.

Now, we state and briefly prove stability results for total variation regularization method combining with additional L^2-stabilization of the diffusion coefficient identification problem.

Theorem 2.9. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence in $H^1_\diamond(\Omega)$ which converges to z^δ in the $H^1(\Omega)$-norm and $(q^{\delta_n}_\rho)$ be the unique minimizers of the problems*

$$\min_{q \in Q_{ad}} \frac{1}{2}\int_\Omega q\left|\nabla\big(U(q) - z^{\delta_n}\big)\right|^2 + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right).$$

Then, $(q^{\delta_n}_\rho)$ converges to the unique solution q^δ_ρ of $(P^q_{\rho,\delta})$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega \left|\nabla q^{\delta_n}_\rho\right| = \int_\Omega \left|\nabla q^\delta_\rho\right|. \tag{2.14}$$

Theorem 2.10. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \quad \text{and} \quad \frac{\delta_n^2}{\rho_n} \to 0 \quad \text{as } n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence in $H^1_\diamond(\Omega)$ satisfying $\|\bar{u} - z^{\delta_n}\|_{H^1(\Omega)} \leqslant \delta_n$ and $(q^{\delta_n}_{\rho_n})$ be the unique minimizers of the problems

$$\min_{q \in Q_{ad}} \frac{1}{2}\int_\Omega q\left|\nabla\big(U(q) - z^{\delta_n}\big)\right|^2 + \rho_n\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right).$$

Then, $(q^{\delta_n}_{\rho_n})$ converges to the unique solution $q^\dagger$ of the problem (Π^q) in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega \left|\nabla q^{\delta_n}_{\rho_n}\right| = \int_\Omega \left|\nabla q^\dagger\right| \quad \text{and} \quad \lim_n D^\ell_{TV}\big(q^{\delta_n}_{\rho_n}, q^\dagger\big) = 0$$

for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$.

To prove these results we remark that if $\mathfrak{F}$ is a normed linear space and (f_n) is a sequence in $\mathfrak{F}$ such that for each subsequence (f_{1_n}) of (f_n) there exist a subsequence (f_{2_n}) of (f_{1_n}) which converges to a fixed element $f_0 \in \mathfrak{F}$, then the whole sequence (f_n) also converges to f_0.

Proof of Theorem 2.9. For all $n \in \mathbb{N}$ and $q \in Q_{ad}$, by the definition of $q_\rho^{\delta_n}$, we have

$$J_{z^{\delta_n}}\left(q_\rho^{\delta_n}\right) + \rho\left(\frac{1}{2}\left\|q_\rho^{\delta_n}\right\|^2_{L^2(\Omega)} + \int_\Omega \left|\nabla q_\rho^{\delta_n}\right|\right) \leqslant J_{z^{\delta_n}}(q) + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right). \tag{2.15}$$

It follows from the last inequality that $(q_\rho^{\delta_n})$ is bounded in the $L^2(\Omega)$-norm (and so in the $L^1(\Omega)$-norm, since $\operatorname{mes}(\Omega) < \infty$) and the sequence $(\int_\Omega |\nabla q_\rho^{\delta_n}|)$ is bounded, too. By Lemma 2.1, there exist a subsequence $(q_\rho^{\delta_{1n}})$ of $(q_\rho^{\delta_n})$ and $q_\rho^\delta \in Q_{ad}$ such that

$$\left(q_\rho^{\delta_{1n}}\right) \quad \text{converges to } q_\rho^\delta \quad \text{in } L^1(\Omega), \tag{2.16}$$

$$\left(q_\rho^{\delta_{1n}}\right) \quad \text{weakly converges to } q_\rho^\delta \quad \text{in } L^2(\Omega), \quad \text{and} \tag{2.17}$$

$$\int_\Omega \left|\nabla q_\rho^\delta\right| \leqslant \liminf_n \int_\Omega \left|\nabla q_\rho^{\delta_{1n}}\right|. \tag{2.18}$$

By (2.16), we see that there exists a subsequence $(q_\rho^{\delta_{2n}})$ of $(q_\rho^{\delta_{1n}})$ such that $U(q_\rho^{\delta_{2n}})$ weakly converges to $U(q_\rho^\delta)$ in $H^1(\Omega)$ (see the proof of Theorem 2.1 in [18]). This and the hypothesis that (z^{δ_n}) converges to z^δ in the $H^1(\Omega)$-norm lead to

$$\lim_n J_{z^{\delta_{2n}}}\left(q_\rho^{\delta_{2n}}\right) = J_{z^\delta}\left(q_\rho^\delta\right). \tag{2.19}$$

On the other hand, it follows from (2.17) that

$$\left\|q_\rho^\delta\right\|^2_{L^2(\Omega)} \leqslant \liminf_n \left\|q_\rho^{\delta_{2n}}\right\|^2_{L^2(\Omega)}. \tag{2.20}$$

Therefore, by (2.18)–(2.20) and (2.15),

$$\begin{aligned}
J_{z^\delta}\left(q_\rho^\delta\right) + \rho\left(\frac{1}{2}\left\|q_\rho^\delta\right\|^2_{L^2(\Omega)} + \int_\Omega \left|\nabla q_\rho^\delta\right|\right) &\leqslant \liminf_n\left(J_{z^{\delta_{2n}}}\left(q_\rho^{\delta_{2n}}\right) + \frac{\rho}{2}\left\|q_\rho^{\delta_{2n}}\right\|^2_{L^2(\Omega)} + \rho\int_\Omega \left|\nabla q_\rho^{\delta_{2n}}\right|\right) \\
&\leqslant \limsup_n\left(J_{z^{\delta_{2n}}}\left(q_\rho^{\delta_{2n}}\right) + \frac{\rho}{2}\left\|q_\rho^{\delta_{2n}}\right\|^2_{L^2(\Omega)} + \rho\int_\Omega \left|\nabla q_\rho^{\delta_{2n}}\right|\right) \\
&\leqslant \limsup_n\left(J_{z^{\delta_{2n}}}(q) + \frac{\rho}{2}\|q\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q|\right) \\
&= J_{z^\delta}(q) + \frac{\rho}{2}\|q\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q|
\end{aligned} \tag{2.21}$$

for all $q \in Q_{ad}$. This means that q_ρ^δ is a (unique) solution to $(P^q_{\rho,\delta})$.

By contradiction we show that $(q_\rho^{\delta_{2n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. In fact, assume that $(q_\rho^{\delta_{2n}}) \nrightarrow q_\rho^\delta$ in the $L^2(\Omega)$-norm. This and (2.20) follow that

$$\epsilon := \limsup_n \left\|q_\rho^{\delta_{2n}}\right\|^2_{L^2(\Omega)} > \left\|q_\rho^\delta\right\|^2_{L^2(\Omega)}. \tag{2.22}$$

Therefore, there exists a subsequence $(q_\rho^{\delta_{3n}})$ of $(q_\rho^{\delta_{2n}})$ such that

$$q_\rho^{\delta_{3n}} \to q_\rho^\delta \quad \text{weakly in } L^2(\Omega), \quad \text{and} \quad \left\|q_\rho^{\delta_{3n}}\right\|^2_{L^2(\Omega)} \to \epsilon. \tag{2.23}$$

Choosing $q = q_\rho^\delta$ in (2.21), we get

$$\lim_n\left(J_{z^{\delta_{2n}}}\left(q_\rho^{\delta_{2n}}\right) + \frac{\rho}{2}\left\|q_\rho^{\delta_{2n}}\right\|^2_{L^2(\Omega)} + \rho\int_\Omega \left|\nabla q_\rho^{\delta_{2n}}\right|\right) = J_{z^\delta}\left(q_\rho^\delta\right) + \frac{\rho}{2}\left\|q_\rho^\delta\right\|^2_{L^2(\Omega)} + \rho\int_\Omega \left|\nabla q_\rho^\delta\right|. \tag{2.24}$$

It follows from (2.22), (2.23) and (2.19) that

$$
\begin{aligned}
J_{z^\delta}\big(q_\rho^\delta\big) + \frac{\rho}{2}\big\|q_\rho^\delta\big\|^2_{L^2(\Omega)} + \rho \limsup_n \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big| &< J_{z^\delta}\big(q_\rho^\delta\big) + \frac{\rho}{2}\epsilon + \rho \limsup_n \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big| \\
&= \lim_n J_{z^{\delta_{3n}}}\big(q_\rho^{\delta_{3n}}\big) + \frac{\rho}{2}\lim_n \big\|q_\rho^{\delta_{3n}}\big\|^2_{L^2(\Omega)} + \rho \limsup_n \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big| \\
&= \limsup_n \bigg(J_{z^{\delta_{3n}}}\big(q_\rho^{\delta_{3n}}\big) + \frac{\rho}{2}\big\|q_\rho^{\delta_{3n}}\big\|^2_{L^2(\Omega)} + \rho \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big| \bigg).
\end{aligned}
$$

By (2.24), the last inequality leads to

$$
\limsup_n \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big| < \int_\Omega \big|\nabla q_\rho^\delta\big| \leqslant \liminf_n \int_\Omega \big|\nabla q_\rho^{\delta_{1n}}\big| \quad \big(\text{by (2.18)}\big) \leqslant \liminf_n \int_\Omega \big|\nabla q_\rho^{\delta_{3n}}\big|,
$$

which is a contradiction. Thus, $(q_\rho^{\delta_{2n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. Hence the whole sequence $(q_\rho^{\delta_n})$ also converges to q_ρ^δ in the $L^2(\Omega)$-norm.

Now, from this and (2.15) it follows that

$$
\begin{aligned}
&J_{z^\delta}\big(q_\rho^\delta\big) + \frac{\rho}{2}\big\|q_\rho^\delta\big\|^2_{L^2(\Omega)} + \rho \limsup_n \int_\Omega \big|\nabla q_\rho^{\delta_{2n}}\big| \\
&\quad = \limsup_n \bigg(J_{z^{\delta_{2n}}}\big(q_\rho^{\delta_{2n}}\big) + \frac{\rho}{2}\big\|q_\rho^{\delta_{2n}}\big\|^2_{L^2(\Omega)} + \rho \int_\Omega \big|\nabla q_\rho^{\delta_{2n}}\big| \bigg) \\
&\quad \leqslant \limsup_n \bigg(J_{z^{\delta_{2n}}}\big(q_\rho^\delta\big) + \frac{\rho}{2}\big\|q_\rho^\delta\big\|^2_{L^2(\Omega)} + \rho \int_\Omega \big|\nabla q_\rho^\delta\big| \bigg) \\
&\quad = J_{z^\delta}\big(q_\rho^\delta\big) + \frac{\rho}{2}\big\|q_\rho^\delta\big\|^2_{L^2(\Omega)} + \rho \int_\Omega \big|\nabla q_\rho^\delta\big|.
\end{aligned}
$$

By (2.18), it follows from the last estimate that $\lim_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}| = \int_\Omega |\nabla q_\rho^\delta|$ and so (2.14) holds. The proof of Theorem 2.9 is now completed. □

Proof of Theorem 2.10. For all $n \in \mathbb{N}$, by the definition of $q_{\rho_n}^{\delta_n}$, we have

$$
\begin{aligned}
J_{z^{\delta_n}}\big(q_{\rho_n}^{\delta_n}\big) + \rho_n\bigg(\frac{1}{2}\big\|q_{\rho_n}^{\delta_n}\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla q_{\rho_n}^{\delta_n}\big|\bigg) &\leqslant J_{z^{\delta_n}}\big(q^\dagger\big) + \rho_n\bigg(\frac{1}{2}\big\|q^\dagger\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla q^\dagger\big|\bigg) \\
&\leqslant \frac{\bar{q}}{2}\delta_n^2 + \rho_n\bigg(\frac{1}{2}\big\|q^\dagger\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla q^\dagger\big|\bigg).
\end{aligned}
\tag{2.25}
$$

By the assumption $\delta_n^2/\rho_n \to 0$, the last inequality yields

$$
\limsup_n \bigg(\frac{1}{2}\big\|q_{\rho_n}^{\delta_n}\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla q_{\rho_n}^{\delta_n}\big|\bigg) \leqslant \frac{1}{2}\big\|q^\dagger\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla q^\dagger\big|. \tag{2.26}
$$

Thus, since $\operatorname{mes}(\Omega) < +\infty$,

$$
\sup_{n\in\mathbb{N}} \big\|q_{\rho_n}^{\delta_n}\big\|^2_{L^2(\Omega)} < +\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \bigg(\big\|q_{\rho_n}^{\delta_n}\big\|_{L^1(\Omega)} + \int_\Omega \big|\nabla q_{\rho_n}^{\delta_n}\big|\bigg) < +\infty.
$$

It follows from the last estimates that there exist a subsequence $(q_{\rho_{1n}}^{\delta_{1n}})$ of $(q_{\rho_n}^{\delta_n})$ and $\hat{q} \in Q_{ad}$ such that

$$
\big(q_{\rho_{1n}}^{\delta_{1n}}\big) \quad \text{converges to } \hat{q} \quad \text{in } L^1(\Omega), \tag{2.27}
$$

$$
\big(q_{\rho_{1n}}^{\delta_{1n}}\big) \quad \text{weakly converges to } \hat{q} \quad \text{in } L^2(\Omega), \quad \text{and} \tag{2.28}
$$

$$
\int_\Omega |\nabla \hat{q}| \leqslant \liminf_n \int_\Omega \big|\nabla q_{\rho_{1n}}^{\delta_{1n}}\big|. \tag{2.29}
$$

On the other hand, since $\|\cdot\|_{L^2(\Omega)}$ and $J_{\bar u}(\cdot)$ are weakly lower semicontinuous, it follows from (2.28) that

$$\|\hat q\|^2_{L^2(\Omega)} \leqslant \liminf_n \big\|q^{\delta_{1n}}_{\rho_{1n}}\big\|^2_{L^2(\Omega)} \tag{2.30}$$

and

$$J_{\bar u}(\hat q) \leqslant \liminf_n J_{\bar u}\big(q^{\delta_{1n}}_{\rho_{1n}}\big).$$

In virtue of the Poincaré–Friedrichs inequality, the last estimate follows that

$$\frac{\alpha}{2}\big\|U(\hat q)-\bar u\big\|^2_{H^1(\Omega)} \leqslant J_{\bar u}(\hat q) \leqslant \liminf_n J_{\bar u}\big(q^{\delta_{1n}}_{\rho_{1n}}\big) \tag{2.31}$$

with the positive constant α defined by (2.4). Now, we have

$$\begin{aligned} J_{\bar u}\big(q^{\delta_{1n}}_{\rho_{1n}}\big) &= \frac{1}{2}\int_\Omega q^{\delta_{1n}}_{\rho_{1n}}\big|\nabla\big(U\big(q^{\delta_{1n}}_{\rho_{1n}}\big)-z^{\delta_{1n}}\big)+\nabla\big(z^{\delta_{1n}}-\bar u\big)\big|^2 \\ &= \frac{1}{2}\int_\Omega q^{\delta_{1n}}_{\rho_{1n}}\big|\nabla\big(z^{\delta_{1n}}-\bar u\big)\big|^2 + \int_\Omega q^{\delta_{1n}}_{\rho_{1n}}\nabla\big(U\big(q^{\delta_{1n}}_{\rho_{1n}}\big)-z^{\delta_{1n}}\big)\cdot\nabla\big(z^{\delta_{1n}}-\bar u\big) \\ &\quad + \frac{1}{2}\int_\Omega q^{\delta_{1n}}_{\rho_{1n}}\big|\nabla\big(U\big(q^{\delta_{1n}}_{\rho_{1n}}\big)-z^{\delta_{1n}}\big)\big|^2. \end{aligned} \tag{2.32}$$

The first two terms in the right-hand side of (2.32) tend to zero as $n\to\infty$, since $z^{\delta_{1n}}$ converging to $\bar u$ in the $H^1(\Omega)$-norm. Thus,

$$\begin{aligned} \liminf_n J_{\bar u}\big(q^{\delta_{1n}}_{\rho_{1n}}\big) &= \liminf_n \frac{1}{2}\int_\Omega q^{\delta_{1n}}_{\rho_{1n}}\big|\nabla\big(U\big(q^{\delta_{1n}}_{\rho_{1n}}\big)-z^{\delta_{1n}}\big)\big|^2 \\ &= \liminf_n J_{z^{\delta_{1n}}}\big(q^{\delta_{1n}}_{\rho_{1n}}\big) \\ &\leqslant \liminf_n\bigg(\frac{\bar q}{2}\delta^2_{1n} + \frac{\rho_{1n}}{2}\big\|q^\dagger\big\|^2_{L^2(\Omega)} + \rho_{1n}\int_\Omega\big|\nabla q^\dagger\big|\bigg) \quad \big(\text{by (2.25)}\big) \\ &= 0. \end{aligned} \tag{2.33}$$

It follows from the inequalities (2.31)–(2.33) that $U(\hat q)=\bar u$. Therefore, replacing $q^\dagger$ in (2.25) by $\hat q$, we also get

$$\limsup_n\bigg(\frac{1}{2}\big\|q^{\delta_n}_{\rho_n}\big\|^2_{L^2(\Omega)} + \int_\Omega\big|\nabla q^{\delta_n}_{\rho_n}\big|\bigg) \leqslant \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} + \int_\Omega|\nabla\hat q|. \tag{2.34}$$

Now, we have

$$\begin{aligned} \limsup_n\int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big| &\leqslant \limsup_n\bigg(\frac{1}{2}\big\|q^{\delta_{1n}}_{\rho_{1n}}-\hat q\big\|^2_{L^2(\Omega)} + \int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big|\bigg) \\ &= \limsup_n\bigg(\frac{1}{2}\big\|q^{\delta_{1n}}_{\rho_{1n}}\big\|^2_{L^2(\Omega)} + \int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big| + \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} - \big\langle q^{\delta_{1n}}_{\rho_{1n}},\hat q\big\rangle_{L^2(\Omega)}\bigg) \\ &\leqslant \limsup_n\bigg(\frac{1}{2}\big\|q^{\delta_{1n}}_{\rho_{1n}}\big\|^2_{L^2(\Omega)} + \int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big|\bigg) + \limsup_n\bigg(\frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} - \big\langle q^{\delta_{1n}}_{\rho_{1n}},\hat q\big\rangle_{L^2(\Omega)}\bigg). \end{aligned}$$

It follows from the last inequality, (2.34) and (2.28) that

$$\limsup_n\int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big| \leqslant \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} + \int_\Omega|\nabla\hat q| + \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} - \langle\hat q,\hat q\rangle_{L^2(\Omega)} = \int_\Omega|\nabla\hat q|.$$

This and (2.29) yield

$$\lim_n\int_\Omega\big|\nabla q^{\delta_{1n}}_{\rho_{1n}}\big| = \int_\Omega|\nabla\hat q|. \tag{2.35}$$

It follows from the last inequality and (2.34) that

$$\begin{aligned}\limsup_n \frac{1}{2}\|q^{\delta_{1n}}_{\rho_{1n}}-\hat q\|^2_{L^2(\Omega)} &= \limsup_n \frac{1}{2}\|q^{\delta_{1n}}_{\rho_{1n}}-\hat q\|^2_{L^2(\Omega)} + \lim_n\Big(\int_\Omega |\nabla q^{\delta_{1n}}_{\rho_{1n}}| - \int_\Omega |\nabla \hat q|\Big)\\ &= \limsup_n\Big(\frac{1}{2}\|q^{\delta_{1n}}_{\rho_{1n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1n}}_{\rho_{1n}}|\Big) + \lim_n\Big(\frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} - \int_\Omega |\nabla \hat q| - \langle q^{\delta_{1n}}_{\rho_{1n}}, \hat q\rangle_{L^2(\Omega)}\Big)\\ &= 0. \end{aligned} \tag{2.36}$$

Now, by the definition of $q^\dagger$ and (2.35)–(2.36), we obtain that

$$\begin{aligned}\frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| &\leqslant \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \hat q|\\ &= \lim_n\Big(\frac{1}{2}\|q^{\delta_{1n}}_{\rho_{1n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1n}}_{\rho_{1n}}|\Big)\\ &\leqslant \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| \quad \big(\text{by (2.26)}\big).\end{aligned}$$

Hence $\frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| = \frac{1}{2}\|\hat q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \hat q|$ or $q^\dagger = \hat q$, by the uniqueness of $q^\dagger$.

Finally, again using (2.35)–(2.36), we see that the sequence $(q^{\delta_{1n}}_{\rho_{1n}})$ weakly converges to $q^\dagger$ in $BV(\Omega)$ (see [1], Proposition 10.1.2, p. 374). Thus, for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$ we conclude that

$$\lim_n D^\ell_{TV}\big(q^{\delta_{1n}}_{\rho_{1n}}, q^\dagger\big) = \lim_n\Big(\int_\Omega |\nabla q^{\delta_{1n}}_{\rho_{1n}}| - \int_\Omega |\nabla q^\dagger| - \langle \ell, q^{\delta_{1n}}_{\rho_{1n}} - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}\Big) = 0.$$

The theorem is proved. □

2.2. Convergence rates

For any fixed $q \in Q$ the mapping

$$U'(q): L^\infty(\Omega) \to H^1_\diamond(\Omega)$$

is a continuous linear operator (see Lemma 2.3) with the dual operator

$$U'(q)^*: H^1_\diamond(\Omega)^* \to L^\infty(\Omega)^*.$$

Then

$$\langle w^*, U'(q)h\rangle_{(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega))} = \langle U'(q)^* w^*, h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle U'(q)^* w^*, h\rangle_{(\mathfrak{X}^*, \mathfrak{X})} \tag{2.37}$$

for all $w^* \in H^1_\diamond(\Omega)^*$ and $h \in \mathfrak{X}$.

In the following, for the simplicity of notation, we denote by

$$R(q) := \frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|, \quad q \in Q_{ad}$$

and note that

$$\partial R\big(q^\dagger\big) = q^\dagger + \partial\Big(\int_\Omega |\nabla(\cdot)|\Big)\big(q^\dagger\big) \subset \mathfrak{X}^*.$$

Now we state the main result of this section.

Theorem 2.11. *Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$U'\big(q^\dagger\big)^* w^* = q^\dagger + \ell \in \partial R\big(q^\dagger\big) \tag{2.38}$$

for some element ℓ *in* $\partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$. *Then,*

$$\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ *and* $\rho \sim \delta$. *Moreover, if* $\ell \in \mathfrak{X}^*$ *can be identified with an element of* $L^2(\Omega)$, *then the following convergence rate is obtained*

$$\left|\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right| = \mathcal{O}(\sqrt{\delta}) \quad \text{as } \delta \to 0 \text{ and } \rho \sim \delta. \tag{2.39}$$

To proving this result we need the following auxiliary results, the proofs of which are based on the convexity of the functional $J_{z^\delta}(\cdot)$.

Lemma 2.12. *(See [19, Lemma 2.10].) The estimate*

$$C_\Omega \|U(q) - z^\delta\|^2_{H^1(\Omega)} \leqslant \frac{2}{\underline{q}}(1 + C_\Omega) J_{z^\delta}(q)$$

holds for all q *belonging to* Q.

Lemma 2.13. *The estimate*

$$-\rho\langle \ell, q^\dagger - q_\rho^\delta\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \leqslant \frac{\bar{q}}{2}\delta^2 + \bar{q}\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}\rho \tag{2.40}$$

holds for q_ρ^δ *being the solution of the problem* $(P^q_{\rho,\delta})$ *and all* $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q_\rho^\delta)$.

Proof. By (2.13), for $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q_\rho^\delta)$, we get

$$\begin{aligned} -\rho\langle \ell, q^\dagger - q_\rho^\delta\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} &\leqslant J'_{z^\delta}(q_\rho^\delta)(q^\dagger - q_\rho^\delta) + \rho\langle q_\rho^\delta, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} \\ &\leqslant J'_{z^\delta}(q_\rho^\delta)(q^\dagger - q_\rho^\delta) + \bar{q}\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}\rho. \end{aligned} \tag{2.41}$$

Since the function $J_{z^\delta}(\cdot)$ is convex, we obtain that

$$J'_{z^\delta}(q_\rho^\delta)(q^\dagger - q_\rho^\delta) \leqslant J_{z^\delta}(q^\dagger) - J_{z^\delta}(q_\rho^\delta) \leqslant J_{z^\delta}(q^\dagger) \leqslant \frac{\bar{q}}{2}\delta^2. \tag{2.42}$$

From the inequalities (2.41) and (2.42) we arrive at (2.40). □

Proof of Theorem 2.11. By the definition of q_ρ^δ, we have

$$J_{z^\delta}(q_\rho^\delta) + \rho\left(\frac{1}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q_\rho^\delta|\right) \leqslant J_{z^\delta}(q^\dagger) + \rho\left(\frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger|\right). \tag{2.43}$$

Then,

$$\begin{aligned} J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} &\leqslant J_{z^\delta}(q^\dagger) + \frac{\rho}{2}\left(\|q^\dagger\|^2_{L^2(\Omega)} - \|q_\rho^\delta\|^2_{L^2(\Omega)} + \|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)}\right) + \rho\left(\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right) \\ &= J_{z^\delta}(q^\dagger) + \rho\left(\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right). \end{aligned}$$

By (2.5), for any $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$ the last inequality leads to

$$\begin{aligned} &J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + \rho D^\ell_{TV}(q_\rho^\delta, q^\dagger) \\ &\quad \leqslant \frac{1}{2}\bar{q}\delta^2 + \rho\left(\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right) + \rho\left(\int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| - \langle \ell, q_\rho^\delta - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}\right) \\ &\quad = \frac{1}{2}\bar{q}\delta^2 + \rho\left(\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \langle \ell, q^\dagger - q_\rho^\delta\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}\right). \end{aligned} \tag{2.44}$$

By (2.11) and (2.12), we get that

$$\langle q^\dagger, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} = \langle q^\dagger, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*,\mathfrak{X})}$$

and

$$\langle \ell, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)})} = \langle \ell, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*,\mathfrak{X})}.$$

Hence, by the source condition (2.38), we have

$$\langle q^\dagger, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} + \langle \ell, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)})} = \langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*,\mathfrak{X})}.$$

It follows from the last equality and (2.37) that

$$\begin{aligned}\langle q^\dagger, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} + \langle \ell, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)})} &= \langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}\\ &= \langle w^*, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\rangle_{(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega))}.\end{aligned} \tag{2.45}$$

By the Riesz representation theorem, there exists an element $w \in H^1_\diamond(\Omega)$ such that

$$\langle w^*, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\rangle_{(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega))} = \langle w, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\rangle_{H^1_\diamond(\Omega)}. \tag{2.46}$$

By the similar reasonings as in the proof of Theorem 2.9 in [19], we get the following estimate

$$\langle w, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\rangle_{H^1_\diamond(\Omega)} \leqslant \bar{q}\delta\left(\int_\Omega |\nabla \hat{w}|^2\right)^{1/2} + \bar{q}\rho\int_\Omega |\nabla \hat{w}|^2 + \frac{1}{2\rho} J_{z^\delta}(q^\delta_\rho) \tag{2.47}$$

for some $\hat{w} \in H^1_\diamond(\Omega)$. It follows from (2.44)–(2.47) that

$$\frac{1}{2} J_{z^\delta}(q^\delta_\rho) + \frac{\rho}{2}\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} + \rho D^\ell_{TV}(q^\delta_\rho, q^\dagger) \leqslant \frac{1}{2}\bar{q}\delta^2 + \bar{q}\delta\rho\left(\int_\Omega |\nabla \hat{w}|^2\right)^{1/2} + \bar{q}\rho^2\int_\Omega |\nabla \hat{w}|^2. \tag{2.48}$$

By Lemma 2.12, the last inequality leads to the following convergence rates

$$\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q^\delta_\rho, q^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta) \tag{2.49}$$

as $\delta \to 0$ and $\rho \sim \delta$.

It remains to establish the convergence rate (2.39). Take $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\delta_\rho)$, from Lemma 2.13, we get

$$\int_\Omega |\nabla q^\delta_\rho| - \int_\Omega |\nabla q^\dagger| \leqslant -\langle \ell, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)})} \leqslant \frac{\bar{q}\delta^2}{2\rho} + \bar{q}\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)}. \tag{2.50}$$

On the other hand, for $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$, we have

$$\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho| \leqslant -\langle \ell, q^\delta_\rho - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)})} = -\langle \ell, q^\delta_\rho - q^\dagger\rangle_{(\mathfrak{X}^*,\mathfrak{X})}. \tag{2.51}$$

By the assumption that $\ell \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$, there exists an element of $L^2(\Omega)$ denoted by the same symbol such that

$$\langle \ell, q^\delta_\rho - q^\dagger\rangle_{(\mathfrak{X}^*,\mathfrak{X})} = \langle \ell, q^\delta_\rho - q^\dagger\rangle_{L^2(\Omega)}.$$

The last equality and (2.51) yield

$$\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho| \leqslant -\langle \ell, q^\delta_\rho - q^\dagger\rangle_{L^2(\Omega)} \leqslant \|\ell\|_{L^2(\Omega)}\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)}. \tag{2.52}$$

Since (2.49), it follows that $\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ as $\delta \to 0$ and $\rho \sim \delta$. Hence the inequalities (2.50) and (2.52) yield (2.39). The theorem is proved. □

2.3. Discussion of the source condition

Now we discuss the source condition (2.38), which ensures the convergence rate

$$\left\|q_\rho^\delta - q^\dagger\right\|_{L^2(\Omega)}^2 + D_{TV}^\ell\left(q_\rho^\delta, q^\dagger\right) = \mathcal{O}(\delta) \tag{2.53}$$

of the regularized solutions q_ρ^δ to the R-minimizing norm solution $q^\dagger$ of our inverse problem, where $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$. We remark that this source condition does not require any the regularity on $q^\dagger$ and the smallness of the source functions which is hard to check in the general convergence theory for nonlinear ill-posed problems [11,12,29]. Further, condition (2.38) is fulfilled if and only if there exists a function $w^* \in H_\diamond^1(\Omega)^*$ such that

$$\frac{1}{2}\|q\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla q| - \frac{1}{2}\left\|q^\dagger\right\|_{L^2(\Omega)}^2 - \int_\Omega \left|\nabla q^\dagger\right| - \left\langle U'\left(q^\dagger\right)^* w^*, q - q^\dagger\right\rangle_{(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)})} \geqslant 0 \tag{2.54}$$

for all $q \in \mathfrak{X}$. To further analyze this condition we assume that the sought coefficient belongs to $H^1(\Omega)$. Therefore, the admissible set of sought coefficients is restricted to

$$\hat{Q_{ad}} = Q \cap H^1(\Omega) \subset Q \cap BV(\Omega).$$

Moreover, if ℓ can be identified with an element of $L^2(\Omega)$, i.e., there exists an element $\widetilde{\ell}$ in $L^2(\Omega)$ such that

$$\langle \ell, q\rangle_{(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)})} = \langle \widetilde{\ell}, q\rangle_{L^2(\Omega)} \tag{2.55}$$

for all q in $\mathfrak{X}$, then the convergence rate

$$\left|\int_\Omega \left|\nabla q_\rho^\delta\right| - \int_\Omega \left|\nabla q^\dagger\right|\right| = \mathcal{O}(\sqrt{\delta}) \tag{2.56}$$

is also established.

We remark that, since $H^1(\Omega) \subset BV(\Omega)$, any ℓ in the dual space of $BV(\Omega)$ can be considered as an element of $H^1(\Omega)$ in the sense that there is a unique element in $H^1(\Omega)$, denoted by the same symbol such that

$$\langle \ell, q\rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \ell, q\rangle_{H^1(\Omega)}, \quad \forall q \in H^1(\Omega).$$

In fact, since $\ell \in BV(\Omega)^*$, there exists a positive constant C such that for all $q \in H^1(\Omega)$,

$$\begin{aligned}\left|\langle \ell, q\rangle_{(BV(\Omega)^*, BV(\Omega))}\right| &\leqslant C\|q\|_{BV(\Omega)} \\ &\leqslant C\sqrt{2\mathrm{mes}(\Omega)}\|q\|_{H^1(\Omega)}.\end{aligned}$$

This means that ℓ belongs to $H^1(\Omega)^*$. Hence, by the Riesz representation theorem, there is a unique element $\widetilde{\ell} \in H^1(\Omega)$ such that $\langle \ell, q\rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \widetilde{\ell}, q\rangle_{H^1(\Omega)}$ for all $q \in H^1(\Omega)$.

Lemma 2.14. *(See [19, Lemma 2.12].) Denote by*

$$\mathcal{B} = \left\{\ell \in BV(\Omega)^* \;\middle|\; \exists \hat{\ell} \in H^1(\Omega)\colon \langle \ell, q\rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \hat{\ell}, q\rangle_{L^2(\Omega)},\ \forall q \in H^1(\Omega)\right\}.$$

If the dimension $d \leqslant 4$ and the boundary $\partial\Omega$ is of class C^1, then

$$\bar{\mathcal{B}} = H^1(\Omega),$$

where the bar denotes the closure in $H^1(\Omega)$.

Theorem 2.15. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \leqslant 4$. Assume that $q^\dagger$ has the property that there is an element $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$ such that $\langle \ell, q\rangle_{(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)})} = \langle \hat{\ell}, q\rangle_{L^2(\Omega)}$ for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\hat{\ell}$ is some element of $H^1(\Omega)$. Further, suppose that the exact $\bar{u} = U(q^\dagger) \in W^{2,\infty}(\Omega)$, $|\nabla \bar{u}| \geqslant \varepsilon$ a.e. on Ω with ε being a positive constant. Then, the convergence rates* (2.53) *and* (2.56) *are obtained.*

Note that as the dimension $d \leqslant 4$, the requirement on $q^\dagger$ of the theorem is fulfilled at least on a set which is everywhere dense on $H^1(\Omega)$ (see Lemma 2.14).

We need the following auxiliary result, which is generalization of that in [26] and [33].

Lemma 2.16. *(See [19, Lemma 2.15].) Assume that the boundary $\partial\Omega$ is of class C^1, $u \in W^{2,\infty}(\Omega)$ and $|\nabla u| \geqslant \gamma$ a.e. on Ω, where γ is a positive constant. Then, for any element $p \in H^1(\Omega)$ there exists $v \in H^1(\Omega)$ satisfying*

$$\nabla u \cdot \nabla v = p. \tag{2.57}$$

Proof of Theorem 2.15. Due to Lemma 2.16, there exists $\psi \in H^1(\Omega)$ satisfying

$$\nabla U(q^\dagger) \cdot \nabla \psi = q^\dagger + \hat{\ell}.$$

Set

$$\hat{\psi} := \frac{\int_\Omega v}{\operatorname{mes}(\Omega)} - \psi.$$

Then,

$$-\nabla U(q^\dagger) \cdot \nabla \hat{\psi} = q^\dagger + \hat{\ell} \quad \text{and} \quad \hat{\psi} \in H^1_\diamond(\Omega).$$

By (2.12), for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$ we have

$$\begin{aligned}
\langle \ell + q^\dagger, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} &= \langle \ell, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} + \langle q^\dagger, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\
&= \langle \hat{\ell}, q\rangle_{L^2(\Omega)} + \langle q^\dagger, q\rangle_{L^2(\Omega)} \\
&= \langle q^\dagger + \hat{\ell}, q\rangle_{L^2(\Omega)} \\
&= -\int_\Omega q \nabla U(q^\dagger) \nabla \hat{\psi}.
\end{aligned}$$

In virtue of (2.8), the last equality leads to

$$\langle \ell + q^\dagger, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \int_\Omega q^\dagger \nabla U'(q^\dagger)(q) \nabla \hat{\psi}, \quad \forall q \in L^\infty(\Omega) \cap H^1(\Omega).$$

Using the equivalent scalar product on $H^1_\diamond(\Omega)$, we obtain that there exist an element $\hat{w} \in H^1_\diamond(\Omega)$ independent of $q \in L^\infty(\Omega) \cap H^1(\Omega)$ such that

$$\langle \ell + q^\dagger, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle \hat{w}, U'(q^\dagger)(q)\rangle_{H^1_\diamond(\Omega)}, \quad \forall q \in L^\infty(\Omega) \cap H^1(\Omega).$$

Thus, there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\langle \ell + q^\dagger, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle w^*, U'(q^\dagger)(q)\rangle_{(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega))} = \langle U'(q^\dagger)^* w^*, q\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}. \tag{2.58}$$

Since

$$\langle U'(q^\dagger)^* w^*, q\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle q^\dagger + \hat{\ell}, q\rangle_{L^2(\Omega)}$$

with $q^\dagger, \hat{\ell} \in H^1(\Omega)$, the boundary $\partial\Omega$ being of class C^1 and the dimension $d \leqslant 4$ and by the Sobolev embedding theorem, it follows that $U'(q^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega) \cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm and

$$\langle U'(q^\dagger)^* w^*, q\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle U'(q^\dagger)^* w^*, q\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}, \quad \forall q \in L^\infty(\Omega) \cap H^1(\Omega)$$

(see the proof of Theorem 2.13 in [19]). Since $q^\dagger + \ell \in \partial(\frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|)(q^\dagger)$, it follows from the last equality and (2.58) that

$$\begin{aligned}
&\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\
&\quad = \frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla q^\dagger| - \langle q^\dagger + \ell, q - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\
&\quad \geqslant 0
\end{aligned}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. The theorem is proved. □

3. The reaction coefficient identification problem

In this section we investigate the following coefficient identification problem.

Find the coefficient a in the problem (1.3)–(1.4) *subject to the constraints*

$$a \in A := \{a \in L^{\infty}(\Omega) \mid 0 < \underline{a} \leqslant a(x) \leqslant \bar{a} \text{ a.e. on } \Omega\} \tag{3.1}$$

with $\underline{a}$ and $\bar{a}$ being given positive constants, when the solution u is imprecisely given in Ω.

3.1. Problem setting and regularization

A function $u \in H^1(\Omega)$ is said to be a weak solution of (1.3)–(1.4), if it satisfies the equality

$$\int_{\Omega} \nabla u \nabla v + \int_{\Omega} auv = \int_{\Omega} fv + \int_{\partial\Omega} gv, \quad \forall v \in H^1(\Omega). \tag{3.2}$$

For all $u \in H^1(\Omega)$ and $a \in A$ the following coercivity condition

$$\int_{\Omega} |\nabla u|^2 + \int_{\Omega} au^2 \geqslant \beta \|u\|^2_{H^1(\Omega)} \tag{3.3}$$

holds. Here,

$$\beta := \min\{1, \underline{a}\} > 0. \tag{3.4}$$

In virtue of the Lax–Milgram lemma, for each $a \in A$ there exists a unique weak solution of (1.3)–(1.4) which satisfies inequality

$$\|u\|_{H^1(\Omega)} \leqslant \Lambda_{\beta}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big), \tag{3.5}$$

where Λ_{β} is a positive constant depending only on β.

Therefore, we can define the nonlinear coefficient-to-solution mapping $U : A \subset L^{\infty}(\Omega) \to H^1(\Omega)$ which maps each $a \in A$ to the unique solution $U(a) \in H^1(\Omega)$ of (1.3)–(1.4). Thus, our inverse problem in this section is that of the form: given $\bar{u} = U(a) \in H^1(\Omega)$ find $a \in A$.

Now we suppose that $\bar{u}$ is the exact solution of (1.3)–(1.4), i.e., there exists some $a \in A$ such that $\bar{u} = U(a)$, where the set A is defined by (3.1) and $U(a)$ is the coefficient-to-solution mapping. We assume that instead of the exact $\bar{u}$ we have only its observations $z^{\delta} \in H^1(\Omega)$ such that

$$\big\|\bar{u} - z^{\delta}\big\|_{H^1(\Omega)} \leqslant \delta, \tag{3.6}$$

where $\delta > 0$. Our problem is to reconstruct a from z^{δ}. For solving this ill-posed problem we minimize the *strictly convex* functional (see Lemma 3.2 below)

$$\min_{a \in A_{ad}} G_{z^{\delta}}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_{\Omega} |\nabla a|\right), \tag{$P^a_{\rho,\delta}$}$$

where $A_{ad} := A \cap BV(\Omega)$ is the admissible set and $\rho > 0$ is the regularization parameter and

$$G_{z^{\delta}}(a) := \frac{1}{2}\int_{\Omega} \big|\nabla\big(U(a) - z^{\delta}\big)\big|^2 + \frac{1}{2}\int_{\Omega} a\big(U(a) - z^{\delta}\big)^2, \quad a \in A. \tag{3.7}$$

We remark that the problem ($P^a_{\rho,\delta}$) has a *unique solution* a^{δ}_{ρ} on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm set A_{ad}, which is called regularized solution to our inverse problem. On the other hand, due to the nonempty convexity, closedness and boundedness in the $L^2(\Omega)$-norm of the set

$$\Pi_{A_{ad}}(\bar{u}) := \{a \in A_{ad} \mid U(a) = \bar{u}\}$$

we can conclude that there is a unique solution $a^{\dagger}$ of the problem

$$\min_{a \in \Pi_{A_{ad}}(\bar{u})} \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_{\Omega} |\nabla a|, \tag{Π^a}$$

which we call *T-minimizing norm solution* to our inverse problem, where

$$T(\cdot) := \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|.$$

In this section we investigate the convergence rates of a^δ_ρ to the solution $a^\dagger$ of the equation $U(a) = \bar{u}$.

We note that the functional $G_{z^\delta}(\cdot)$ is convex and Fréchet differentiable. The Fréchet differential of it is defined by

$$G'_{z^\delta}(a)h = -\frac{1}{2}\int_\Omega h\big(U(a) - z^\delta\big)\big(U(a) + z^\delta\big)$$

for $a \in A$ and $h \in L^\infty(\Omega)$.

Lemma 3.1. *(See [19, Lemma 3.1].) The mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable with the derivative $U'(a)$. For each h in $L^\infty(\Omega)$, the differential $\eta := U'(a)h \in H^1(\Omega)$ is the unique solution of the problem*

$$-\Delta\eta + a\eta = -hU(a) \quad \text{in } \Omega, \qquad \frac{\partial\eta}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

in the sense that it satisfies the equation

$$\int_\Omega \nabla\eta\nabla v + \int_\Omega a\eta v = -\int_\Omega hU(a)v \tag{3.8}$$

for all $v \in H^1(\Omega)$. Furthermore, the estimate

$$\|\eta\|_{H^1(\Omega)} \leqslant \frac{\Lambda_\beta}{\beta}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big)\|h\|_{L^\infty(\Omega)}$$

holds for all $h \in L^\infty(\Omega)$.

Lemma 3.2. *(See [19, Lemma 3.2].) The functional $G_{z^\delta}(\cdot)$ defined by (3.7) is convex on the convex set A.*

Similar to the previous section we can prove the following results.

Theorem 3.3. (i) *There exists a unique solution of the problem $(P^a_{\rho,\delta})$. Further, an element a^δ_ρ in A_{ad} is a solution to $(P^a_{\rho,\delta})$ if and only if for any $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\delta_\rho)$, the inequality*

$$G'_{z^\delta}\big(a^\delta_\rho\big)\big(a - a^\delta_\rho\big) + \rho\big\langle a^\delta_\rho, a - a^\delta_\rho\big\rangle_{L^2(\Omega)} + \rho\big\langle\lambda, a - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \geqslant 0 \tag{3.9}$$

holds for all a in A_{ad}.

(ii) *There exists a unique solution of the problem (Π^a). Further, an element $a^\dagger$ in $\Pi_{A_{ad}}(\bar{u})$ is a solution of the problem (Π^a) if and only if for any $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$, the inequality*

$$\big\langle a^\dagger, a - a^\dagger\big\rangle_{L^2(\Omega)} + \big\langle\lambda, a - a^\dagger\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \geqslant 0$$

holds for all a in $\Pi_{A_{ad}}(\bar{u})$.

Theorem 3.4. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence in $H^1(\Omega)$ which converges to z^δ in $H^1(\Omega)$ and $(a^{\delta_n}_\rho)$ be the unique minimizers of the problems*

$$\min_{a\in A_{ad}} \frac{1}{2}\int_\Omega \big|\nabla\big(U(a) - z^{\delta_n}\big)\big|^2 + \frac{1}{2}\int_\Omega a\big(U(a) - z^{\delta_n}\big)^2 + \rho\bigg(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\bigg).$$

Then, $(a^{\delta_n}_\rho)$ converges to the unique solution a^δ_ρ of $(P^a_{\rho,\delta})$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega \big|\nabla a^{\delta_n}_\rho\big| = \int_\Omega \big|\nabla a^\delta_\rho\big|.$$

Theorem 3.5. *For any positive sequence* $(\delta_n) \to 0$, *let* $\rho_n := \rho(\delta_n)$ *be such that*

$$\rho_n \to 0 \quad \text{and} \quad \frac{\delta_n^2}{\rho_n} \to 0 \quad \text{as } n \to \infty.$$

Moreover, let (z^{δ_n}) *be a sequence in* $H^1(\Omega)$ *satisfying* $\|\bar{u} - z^{\delta_n}\|_{H^1(\Omega)} \leqslant \delta_n$ *and* $(a_{\rho_n}^{\delta_n})$ *be the unique minimizers of the problems*

$$\min_{a \in A_{ad}} \frac{1}{2} \int_\Omega \left|\nabla\left(U(a) - z^{\delta_n}\right)\right|^2 + \frac{1}{2} \int_\Omega a\left(U(a) - z^{\delta_n}\right)^2 + \rho_n \left(\frac{1}{2}\|a\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla a|\right).$$

Then, $(a_{\rho_n}^{\delta_n})$ *converges to the unique solution* $a^\dagger$ *of the problem* (Π^a) *in the* $L^2(\Omega)$*-norm. Further,*

$$\lim_n \int_\Omega \left|\nabla a_{\rho_n}^{\delta_n}\right| = \int_\Omega \left|\nabla a^\dagger\right| \quad \text{and} \quad \lim_n D_{TV}^\ell\left(a_{\rho_n}^{\delta_n}, a^\dagger\right) = 0$$

for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$.

3.2. Convergence rates

We now state and briefly prove the result on the convergence rates of regularization solutions a_ρ^δ to the solution $a^\dagger$ of the equation $U(a) = \bar{u}$.

Theorem 3.6. *Assume that there exists a function* $w^* \in H^1(\Omega)^*$ *such that*

$$U'\left(a^\dagger\right)^* w^* = a^\dagger + \lambda \in \partial T\left(a^\dagger\right) \tag{3.10}$$

for some element λ *in* $\partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$. *Then,*

$$\left\|a_\rho^\delta - a^\dagger\right\|_{L^2(\Omega)}^2 + D_{TV}^\lambda\left(a_\rho^\delta, a^\dagger\right) = \mathcal{O}(\delta) \quad \text{and} \quad \left\|U\left(a_\rho^\delta\right) - z^\delta\right\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ *and* $\rho \sim \delta$. *Further, if* $\lambda \in \mathfrak{X}^*$ *can be identified with an element of* $L^2(\Omega)$, *then the convergence rate*

$$\left|\int_\Omega \left|\nabla a^\dagger\right| - \int_\Omega \left|\nabla a_\rho^\delta\right|\right| = \mathcal{O}(\sqrt{\delta}) \quad \text{as } \delta \to 0 \text{ and } \rho \sim \delta, \tag{3.11}$$

is also established.

We need the following lemmas.

Lemma 3.7. *(See [19, Lemma 3.7].) The estimate*

$$\left\|U(a) - z^\delta\right\|_{H^1(\Omega)}^2 \leqslant \frac{2}{\beta} G_{z^\delta}(a)$$

holds for all a belonging to A.

Lemma 3.8. *The following estimate*

$$-\rho\left\langle\lambda, a^\dagger - a_\rho^\delta\right\rangle_{(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)})} \leqslant \frac{1}{2}\max\{\bar{a}, 1\}\delta^2 + \bar{a}\left\|a_\rho^\delta - a^\dagger\right\|_{L^2(\Omega)}\rho \tag{3.12}$$

holds for all a_ρ^δ *being the solutions of the problems* $(P_{\rho,\delta}^a)$ *and* $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(a_\rho^\delta)$.

Proof. Using the convexity of the function $G_{z^\delta}(\cdot)$ and the inequality (3.9) we obtain that

$$\begin{aligned}
-\rho\left\langle\lambda, a^\dagger - a_\rho^\delta\right\rangle_{(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)})} &\leqslant G'_{z^\delta}\left(a_\rho^\delta\right)\left(a^\dagger - a_\rho^\delta\right) + \rho\left\langle a_\rho^\delta, a^\dagger - a_\rho^\delta\right\rangle_{L^2(\Omega)} \\
&\leqslant G_{z^\delta}\left(a^\dagger\right) - G_{z^\delta}\left(a_\rho^\delta\right) + \rho\left\langle a_\rho^\delta, a^\dagger - a_\rho^\delta\right\rangle_{L^2(\Omega)} \\
&\leqslant G_{z^\delta}\left(a^\dagger\right) + \bar{a}\left\|a_\rho^\delta - a^\dagger\right\|_{L^2(\Omega)}\rho.
\end{aligned}$$

Since $G_{z^\delta}(a^\dagger) \leqslant \frac{1}{2}\max\{\bar{a}, 1\}\delta^2$, the last inequality yields (3.12). The lemma is proved. □

Proof of Theorem 3.6. Using the definition of a^δ_ρ we have

$$G_{z^\delta}\big(a^\delta_\rho\big) + \rho\bigg(\frac{1}{2}\big\|a^\delta_\rho\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla a^\delta_\rho\big|\bigg) \leqslant G_{z^\delta}\big(a^\dagger\big) + \rho\bigg(\frac{1}{2}\big\|a^\dagger\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla a^\dagger\big|\bigg)$$
$$\leqslant \frac{1}{2}\max\{\bar a, 1\}\delta^2 + \rho\bigg(\frac{1}{2}\big\|a^\dagger\big\|^2_{L^2(\Omega)} + \int_\Omega \big|\nabla a^\dagger\big|\bigg). \tag{3.13}$$

Take $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$ such that $U'(a^\dagger)^* w^* = a^\dagger + \lambda$. Then,

$$G_{z^\delta}\big(a^\delta_\rho\big) + \frac{\rho}{2}\big\|a^\delta_\rho - a^\dagger\big\|^2_{L^2(\Omega)} + \rho D^\lambda_{TV}\big(a^\delta_\rho, a^\dagger\big)$$
$$\leqslant \frac{1}{2}\max\{\bar a, 1\}\delta^2 + \rho\big(\big\langle a^\dagger, a^\dagger - a^\delta_\rho\big\rangle_{L^2(\Omega)} + \big\langle \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}\big). \tag{3.14}$$

Now we have

$$\begin{aligned}
\big\langle a^\dagger, a^\dagger - a^\delta_\rho\big\rangle_{L^2(\Omega)} + \big\langle \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} &= \big\langle a^\dagger, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*, \mathfrak{X})} + \big\langle \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*, \mathfrak{X})} \\
&= \big\langle a^\dagger + \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*, \mathfrak{X})} \\
&= \big\langle U'\big(a^\dagger\big)^* w^*, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*, \mathfrak{X})} \\
&= \big\langle U'\big(a^\dagger\big)^* w^*, a^\dagger - a^\delta_\rho\big\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\
&= \big\langle w^*, U'\big(a^\dagger\big)\big(a^\dagger - a^\delta_\rho\big)\big\rangle_{(H^1(\Omega)^*, H^1(\Omega))}.
\end{aligned}$$

By the Riesz representation theorem, the last equation follows that there exists an element $w \in H^1(\Omega)$ such that

$$\big\langle a^\dagger, a^\dagger - a^\delta_\rho\big\rangle_{L^2(\Omega)} + \big\langle \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \big\langle w, U'\big(a^\dagger\big)\big(a^\dagger - a^\delta_\rho\big)\big\rangle_{H^1(\Omega)}. \tag{3.15}$$

By the similar reasonings as in the proof of Theorem 3.6 in [19] we get the following estimate

$$\big\langle w, U'\big(a^\dagger\big)\big(a^\dagger - a^\delta_\rho\big)\big\rangle_{H^1(\Omega)} \leqslant \delta\bar a\bigg(\int_\Omega \hat w^2\bigg)^{1/2} + \rho\bar a\int_\Omega \hat w^2 + \frac{1}{4\rho}\int_\Omega a^\delta_\rho\big(z^\delta - U\big(a^\delta_\rho\big)\big)^2$$
$$+ \delta\bigg(\int_\Omega |\nabla\hat w|^2\bigg)^{1/2} + \rho\int_\Omega |\nabla\hat w|^2 + \frac{1}{4\rho}\int_\Omega \big|\nabla\big(z^\delta - U\big(a^\delta_\rho\big)\big)\big|^2$$

for some $\hat w \in H^1(\Omega)$. Thus,

$$\rho\big\langle w, U'\big(a^\dagger\big)\big(a^\dagger - a^\delta_\rho\big)\big\rangle_{H^1(\Omega)} \leqslant \delta\rho\bar a\bigg(\int_\Omega \hat w^2\bigg)^{1/2} + \rho^2\bar a\int_\Omega \hat w^2 + \delta\rho\bigg(\int_\Omega |\nabla\hat w|^2\bigg)^{1/2} + \rho^2\int_\Omega |\nabla\hat w|^2 + \frac{1}{2}G_{z^\delta}\big(a^\delta_\rho\big). \tag{3.16}$$

It follows from the inequalities (3.14), (3.15) and (3.16) that

$$\frac{1}{2}G_{z^\delta}\big(a^\delta_\rho\big) + \frac{\rho}{2}\big\|a^\delta_\rho - a^\dagger\big\|^2_{L^2(\Omega)} + \rho D^\lambda_{TV}\big(a^\delta_\rho, a^\dagger\big)$$
$$\leqslant \frac{1}{2}\max\{\bar a, 1\}\delta^2 + \delta\rho\bar a\bigg(\int_\Omega \hat w^2\bigg)^{1/2} + \rho^2\bar a\int_\Omega \hat w^2 + \delta\rho\bigg(\int_\Omega |\nabla\hat w|^2\bigg)^{1/2} + \rho^2\int_\Omega |\nabla\hat w|^2.$$

Using Lemma 3.7, we obtain the following convergence rates

$$\big\|a^\delta_\rho - a^\dagger\big\|^2_{L^2(\Omega)} + D^\lambda_{TV}\big(a^\delta_\rho, a^\dagger\big) = \mathcal{O}(\delta) \quad\text{and}\quad \big\|U\big(a^\delta_\rho\big) - z^\delta\big\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

It remains to prove the convergence rate (3.11). Take $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\delta_\rho)$, we get from Lemma 3.8 that

$$\int_\Omega \big|\nabla a^\delta_\rho\big| - \int_\Omega \big|\nabla a^\dagger\big| \leqslant -\big\langle \lambda, a^\dagger - a^\delta_\rho\big\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \leqslant \frac{1}{2}\max\{\bar a, 1\}\frac{\delta^2}{\rho} + \bar a\big\|a^\delta_\rho - a^\dagger\big\|_{L^2(\Omega)}. \tag{3.17}$$

On the other hand, since $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$, by the assumption that $\lambda \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$ which is denoted by the same symbol, we get

$$\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| \leqslant -\langle \lambda, a^\delta_\rho - a^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = -\langle \lambda, a^\delta_\rho - a^\dagger\rangle_{(\mathfrak{X}^*, \mathfrak{X})} = -\langle \lambda, a^\delta_\rho - a^\dagger\rangle_{L^2(\Omega)}.$$

The last inequality yields

$$\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| \leqslant \|\lambda\|_{L^2(\Omega)} \|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)}. \tag{3.18}$$

From the inequalities (3.17) and (3.18), and the fact that $\|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ we arrive at (3.11). The theorem is proved. □

3.3. Discussion of the source condition

Now we discuss the source condition (3.10) which is equivalent to the following one: there exists a function $w^* \in H^1(\Omega)^*$ such that

$$\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \geqslant 0 \tag{3.19}$$

for all $a \in \mathfrak{X}$. To further analyze this condition we assume that the admissible set of coefficients is restricted to

$$\hat{A_{ad}} = A \cap H^1(\Omega) \subset A \cap BV(\Omega).$$

Theorem 3.9. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \leqslant 4$. Suppose that $a^\dagger$ has the property that there is an element $\lambda \in \partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$ such that $\langle \lambda, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle \hat{\lambda}, a\rangle_{L^2(\Omega)}$ for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\hat{\lambda}$ is some element of $H^1(\Omega)$. Further, assume that there exists a constant $\varepsilon > 0$ such that $|U(a^\dagger)| \geqslant \varepsilon$ a.e. on Ω. Then, the condition* (3.19) *is fulfilled and hence convergence rates*

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \left|\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho|\right| = \mathcal{O}(\sqrt{\delta})$$

are obtained.

Proof. For any $a \in L^\infty(\Omega) \cap H^1(\Omega)$ we have

$$\begin{aligned}\langle a^\dagger + \lambda, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} &= \langle a^\dagger, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} + \langle \lambda, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\ &= \langle a^\dagger, a\rangle_{L^2(\Omega)} + \langle \lambda, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\ &= \langle a^\dagger + \hat{\lambda}, a\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.20}$$

Since $a^\dagger + \hat{\lambda} \in H^1(\Omega)$ and $|U(a^\dagger)| \geqslant \varepsilon > 0$, we have $\psi := -\frac{a^\dagger + \hat{\lambda}}{U(a^\dagger)} \in H^1(\Omega)$. Hence

$$-\int_\Omega aU(a^\dagger)\psi = \langle a^\dagger + \hat{\lambda}, a\rangle_{L^2(\Omega)} \tag{3.21}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. It follows from (3.20), (3.21) and (3.8) that

$$\langle a^\dagger + \lambda, a\rangle_{(\hat{\mathfrak{X}}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \int_\Omega \nabla U'(a^\dagger)(a)\nabla\psi + \int_\Omega a^\dagger U'(a^\dagger)(a)\psi = \langle \hat{w}, U'(a^\dagger)(a)\rangle_{H^1(\Omega)}$$

for some $\hat{w} \in H^1(\Omega)$ independent of $a \in L^\infty(\Omega) \cap H^1(\Omega)$. Therefore, there exists an element $w^* \in H^1(\Omega)^*$ such that

$$\langle a^\dagger + \lambda, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = \langle w^*, U'(a^\dagger)(a)\rangle_{(H^1(\Omega)^*, H^1(\Omega))} = \langle U'(a^\dagger)^* w^*, a\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}.$$

Since $\langle U'(a^\dagger)^* w^*, a\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle a^\dagger + \hat{\lambda}, a\rangle_{L^2(\Omega)}$ with $a^\dagger + \hat{\lambda} \in H^1(\Omega)$, we obtain that $U'(a^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega) \cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm and

$$\langle U'(a^\dagger)^* w^*, a\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle U'(a^\dagger)^* w^*, a\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}, \quad \forall a \in L^\infty(\Omega) \cap H^1(\Omega).$$

Therefore, since $a^\dagger + \lambda \in \partial(\frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|)(a^\dagger)$, we conclude that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$\begin{aligned}
&\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\
&\quad = \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle a^\dagger + \lambda, a - a^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \\
&\quad \geqslant 0
\end{aligned}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. The theorem is proved. □

4. Related inverse problems

Let Γ be an open piece of $\partial\Omega$, $\Gamma \subsetneq \partial\Omega$. In this section, we extend the above results to problems with Dirichlet or mixed boundary conditions

$$-\operatorname{div}(q\nabla u) + au = f \quad \text{in } \Omega, \tag{4.1}$$

$$q\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \tag{4.2}$$

$$u = 0 \quad \text{on } \partial\Omega \setminus \Gamma \tag{4.3}$$

from imprecise values of u in the domain Ω. Here, the functions

$$a \in A := \{a \in L^\infty(\Omega) \mid 0 \leqslant a(x) \leqslant \bar{a} \text{ a.e. on } \Omega\}$$

with $\bar{a}$ being a given positive constant, $f \in L^2(\Omega)$ and g in $L^2(\Gamma)$ are given.

Problem: *Find the coefficient $q \in Q$ defined by* (2.1) *in the problem* (4.1)–(4.3).

We see that problem (4.1)–(4.3) is of the mixed type, if neither Γ nor $\partial\Omega \setminus \Gamma$ is empty; of the Dirichlet type, if $\Gamma = \emptyset$; of the Neumann type if $\partial\Omega \setminus \Gamma = \emptyset$. We note that the solution space of the Neumann problem (4.1)–(4.3) with $a \geqslant \underline{a} > 0$ is $H^1(\Omega)$, while that of this problem with $a = 0$ is $H^1_\diamond(\Omega)$. On the other hand, the solution space of the Dirichlet and mixed problem (4.1)–(4.3) are $H^1_0(\Omega)$ and $H^1_0(\Omega \cup \Gamma)$, respectively, indifferently of a. This is a reason why we choose the identification problem in the Neumann problem to present in detail. Indeed, all results that stating for the inverse problem in the Neumann problem remain valid for that in the Dirichlet and mixed problems. The definition of the space $H^1_0(\Omega \cup \Gamma)$ can be found in [31], p. 67. We also note that if $\Gamma = \emptyset$, then $H^1_0(\Omega \cup \Gamma) = H^1_0(\Omega)$. Therefore, in the following we only state results for the inverse problem of identifying the coefficient q in the mixed boundary value problem for elliptic equations (4.1)–(4.3), in fact, these results are valid also for the Dirichlet problem as $\Gamma = \emptyset$.

We recall that a function $u \in H^1_0(\Omega \cup \Gamma)$ is said to be a weak solution of (4.1)–(4.3) if

$$\int_\Omega q\nabla u \nabla v + \int_\Omega auv = \int_\Omega fv + \int_\Gamma gv, \quad \forall v \in H^1_0(\Omega \cup \Gamma).$$

Since the Poincaré–Friedrichs inequality remains valid on the $H^1_0(\Omega \cup \Gamma)$ space (see, [31], p. 69 and p. 81), there exists a positive constant κ depending only on $\underline{q}$ and Ω such that the coercivity condition $\int_\Omega q|\nabla u|^2 + \int_\Omega au^2 \geqslant \kappa\|u\|^2_{H^1(\Omega)}$ holds for all u in $H^1_0(\Omega \cup \Gamma)$. Then, by the Lax–Milgram lemma, we conclude that there exists a unique solution u of (4.1)–(4.3) satisfying the inequality $\|u\|_{H^1(\Omega)} \leqslant \Lambda(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)})$, where Λ is a positive constant depending only on $\underline{q}$ and Ω. Thus, we can define the nonlinear coefficient-to-solution mapping

$$\widetilde{U} : Q \subset L^\infty(\Omega) \to H^1_0(\Omega \cup \Gamma)$$

which maps each $q \in Q$ to the unique solution $\widetilde{U}(q)$ of (4.1)–(4.3). The inverse problem is then set as follows: given $\bar{u} = \widetilde{U}(q) \in H^1_0(\Omega \cup \Gamma)$ find $q \in Q$.

We remark that the mapping $\widetilde{U} : Q \subset L^\infty(\Omega) \to H^1_0(\Omega \cup \Gamma)$ is continuously Fréchet differentiable on the set Q. For each $q \in Q$, the Fréchet derivative $\widetilde{U}'(q)$ of $\widetilde{U}(q)$ has the property that the differential $\eta := \widetilde{U}'(q)h$ with $h \in L^\infty(\Omega)$ is the (unique) solution in $H^1_0(\Omega \cup \Gamma)$ of the mixed boundary value problem

$$-\operatorname{div}(q\nabla\eta) + a\eta = \operatorname{div}(h\nabla\widetilde{U}(q)) \quad \text{in } \Omega, \qquad q\frac{\partial\eta}{\partial n} = -h\frac{\partial\widetilde{U}(q)}{\partial n} \quad \text{on } \Gamma, \qquad \eta = 0 \quad \text{on } \partial\Omega \setminus \Gamma,$$

in the sense that $\widetilde{U}'(q)h$ in $H_0^1(\Omega\cup\Gamma)$ solves the variational equation

$$\int_\Omega q\nabla\eta\nabla v+\int_\Omega a\eta v=-\int_\Omega h\nabla\widetilde{U}(q)\nabla v$$

for all v in $H_0^1(\Omega\cup\Gamma)$ and satisfies the estimate $\|\eta\|_{H^1(\Omega)}\leqslant\frac{\Lambda}{\kappa}(\|f\|_{L^2(\Omega)}+\|g\|_{L^2(\Gamma)})\|h\|_{L^\infty(\Omega)}$ (the proof of this fact is similar to that of Lemma 2.2 in [18]).

Now we assume that instead of the exact $\bar u\in H_0^1(\Omega\cup\Gamma)$ we have only its observations $z^\delta\in H_0^1(\Omega\cup\Gamma)$ such that $\|\bar u-z^\delta\|_{H^1(\Omega)}\leqslant\delta$, where $\delta>0$. Our problem is to reconstruct $q\in Q$ from z^δ.

4.1. Pure total variation regularization and convergence rates

For regularizing the identification problem we solve the *convex* minimization problem

$$\min_{q\in Q_{ad}}\widetilde{J}_{z^\delta}(q)+\rho\int_\Omega|\nabla q|, \qquad (\widetilde{P}^q_{\rho,\delta})$$

where $Q_{ad}:=Q\cap BV(\Omega)$ is the admissible set, $\rho>0$ is the regularization parameter and

$$\widetilde{J}_{z^\delta}(q):=\frac12\int_\Omega q\big|\nabla\big(\widetilde{U}(q)-z^\delta\big)\big|^2+\frac12\int_\Omega a\big(\widetilde{U}(q)-z^\delta\big)^2,\quad q\in Q \tag{4.4}$$

is convex functional (the proof of this fact is similar to that of Lemma 2.3 in [18]).

Similar to the proof of Lemma 2.1 in [9] we conclude that the problem $(\widetilde{P}^q_{\rho,\delta})$ has a solution q^δ_ρ. Further, the problem

$$\min_{q\in\widetilde{\Pi}_{Q_{ad}}(\bar u)}\int_\Omega|\nabla q| \qquad (\widetilde{\Pi}^q)$$

also has a solution (see Theorem 2.5 in [19]) which is called the total variation-minimizing solution of the equation $\widetilde{U}(q)=\bar u$, where

$$\widetilde{\Pi}_{Q_{ad}}(\bar u):=\big\{q\in Q_{ad}\;\big|\;\widetilde{U}(q)=\bar u\big\}. \tag{4.5}$$

Our aim is to investigate the convergence rates of regularized solutions q^δ_ρ to the total variation-minimizing solution $q^\dagger$ of the equation $\widetilde{U}(q)=\bar u$.

Similar to Theorems 2.7–2.9 in [19] we can prove the following results.

Theorem 4.1. *For a fixed regularization parameter $\rho>0$, let (z^{δ_n}) be a sequence in $H_0^1(\Omega\cup\Gamma)$ which converges to z^δ in the $H^1(\Omega)$-norm and $(q^{\delta_n}_\rho)$ be minimizers of the problems*

$$\min_{q\in Q_{ad}}\frac12\int_\Omega q\big|\nabla\big(\widetilde{U}(q)-z^{\delta_n}\big)\big|^2+\frac12\int_\Omega a\big(\widetilde{U}(q)-z^{\delta_n}\big)^2+\rho\int_\Omega|\nabla q|.$$

Then, there exists a subsequence $(q^{\delta_{k_n}}_\rho)$ of $(q^{\delta_n}_\rho)$ and $q^\delta_\rho\in Q_{ad}$ such that $(q^{\delta_{k_n}}_\rho)$ converges to q^δ_ρ in the $L^1(\Omega)$-norm and $\lim_n\int_\Omega|\nabla q^{\delta_{k_n}}_\rho|=\int_\Omega|\nabla q^\delta_\rho|$. Further, q^δ_ρ is a solution to $(\widetilde{P}^q_{\rho,\delta})$.

Theorem 4.2. *For any positive sequence $(\delta_n)\to0$, let $\rho_n:=\rho(\delta_n)$ be such that $\rho_n\to0$ and $\delta_n^2/\rho_n\to0$ as $n\to\infty$. Moreover, let (z^{δ_n}) be a sequence in $H_0^1(\Omega\cup\Gamma)$ satisfying $\|\bar u-z^{\delta_n}\|_{H^1(\Omega)}\leqslant\delta_n$ and $(q^{\delta_n}_{\rho_n})$ be minimizers of the problems*

$$\min_{q\in Q_{ad}}\frac12\int_\Omega q\big|\nabla\big(\widetilde{U}(q)-z^{\delta_n}\big)\big|^2+\frac12\int_\Omega a\big(\widetilde{U}(q)-z^{\delta_n}\big)^2+\rho_n\int_\Omega|\nabla q|.$$

Then, there exists a subsequence $(q^{\delta_{k_n}}_{\rho_{k_n}})$ of $(q^{\delta_n}_{\rho_n})$ and an element $q^\dagger\in Q_{ad}$ such that

$$\lim_n\big\|q^{\delta_{k_n}}_{\rho_{k_n}}-q^\dagger\big\|_{L^1(\Omega)}=0\quad\textit{and}\quad\lim_n\int_\Omega\big|\nabla q^{\delta_{k_n}}_{\rho_{k_n}}\big|=\int_\Omega\big|\nabla q^\dagger\big|.$$

Further, $q^\dagger$ is the solution to the problem $(\widetilde{\Pi}^q)$ and $\lim_n D^\ell_{TV}(q^{\delta_{k_n}}_{\rho_{k_n}},q^\dagger)=0$, for each element $\ell\in\partial(\int_\Omega|\nabla(\cdot)|)(q^\dagger)$.

Theorem 4.3. *Assume that there exists a function $w^* \in H_0^1(\Omega \cup \Gamma)^*$ such that*

$$\widetilde{U}'(q^\dagger)^* w^* \in \partial\bigg(\int_\Omega |\nabla(\cdot)|\bigg)(q^\dagger). \tag{4.6}$$

Then,

$$D_{TV}^{\widetilde{U}'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta), \quad \bigg|\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\bigg| = \mathcal{O}(\delta) \quad \text{and} \quad \|\widetilde{U}(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

4.2. Total variation plus L^2-norm regularization and convergence rates

For solving the problem of identifying the coefficient q in the problem (4.1)–(4.3) in this subsection we solve the *strictly convex* minimization problem

$$\min_{q \in Q_{ad}} \widetilde{J}_{z^\delta}(q) + \rho\bigg(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\bigg), \tag{$\hat{P}^q_{\rho,\delta}$}$$

where $Q_{ad} := Q \cap BV(\Omega)$ is the admissible set, $\rho > 0$ is the regularization parameter and $\widetilde{J}_{z^\delta}(\cdot)$ is the convex functional defined by (4.4).

We remark that the problem $(\hat{P}^q_{\rho,\delta})$ has a *unique* solution q_ρ^δ. Further, the problem

$$\min_{q \in \widetilde{\Pi}_{Q_{ad}}(\bar{u})} R(q) \quad \text{with } R(\cdot) := \frac{1}{2}\|\cdot\|_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)| \tag{$\hat{\Pi}^q$}$$

also has a solution (see Theorem 2.4), which is called the R-minimizing solution of equation $\widetilde{U}(q) = \bar{u}$, where the set $\widetilde{\Pi}_{Q_{ad}}(\bar{u})$ defined by (4.5).

Similar to Theorems 2.9–2.11 we can prove the following results.

Theorem 4.4. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence in $H_0^1(\Omega \cup \Gamma)$ which converges to z^δ in the $H^1(\Omega)$-norm and $(q_\rho^{\delta_n})$ be the unique minimizers of the problems*

$$\min_{q \in Q_{ad}} \frac{1}{2}\int_\Omega q\big|\nabla(\widetilde{U}(q) - z^{\delta_n})\big|^2 + \frac{1}{2}\int_\Omega a(\widetilde{U}(q) - z^{\delta_n})^2 + \rho\bigg(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\bigg).$$

Then, $(q_\rho^{\delta_n})$ converges to the unique solution q_ρ^δ of $(\hat{P}^q_{\rho,\delta})$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega |\nabla q_\rho^{\delta_n}| = \int_\Omega |\nabla q_\rho^\delta|.$$

Theorem 4.5. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that $\rho_n \to 0$ and $\delta_n^2/\rho_n \to 0$ as $n \to \infty$. Moreover, let (z^{δ_n}) be a sequence in $H_0^1(\Omega \cup \Gamma)$ satisfying $\|\bar{u} - z^{\delta_n}\|_{H^1(\Omega)} \leqslant \delta_n$ and $(q_{\rho_n}^{\delta_n})$ be unique minimizers of the problems*

$$\min_{q \in Q_{ad}} \frac{1}{2}\int_\Omega q\big|\nabla(\widetilde{U}(q) - z^{\delta_n})\big|^2 + \frac{1}{2}\int_\Omega a(\widetilde{U}(q) - z^{\delta_n})^2 + \rho_n\bigg(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\bigg).$$

Then, $(q_{\rho_n}^{\delta_n})$ converges to the unique solution $q^\dagger$ of the problem $(\hat{\Pi}^q)$ in the $L^2(\Omega)$-norm. Further, $\lim_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| = \int_\Omega |\nabla q^\dagger|$ and $\lim_n D_{TV}^\ell(q_{\rho_n}^{\delta_n}, q^\dagger) = 0$ for each element $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$.

Theorem 4.6. *Assume that there exists a function $w^* \in H_0^1(\Omega \cup \Gamma)^*$ such that*

$$\widetilde{U}'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger) \tag{4.7}$$

for some element ℓ in $\partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$. Then,

$$\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + D_{TV}^\ell(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \|\widetilde{U}(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. Moreover, if $\ell \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$, then the convergence rate

$$\left| \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho| \right| = \mathcal{O}(\sqrt{\delta}) \quad \text{as } \delta \to 0 \quad \text{and} \quad \rho \sim \delta$$

is also established.

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