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On the existence of solution of stage equations in implicit Runge–Kutta methods

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Abstract

This paper is concerned with the unique solvability of stage equations which arise when implicit Runge–Kutta methods apply to nonlinear stiff systems of differential equations $y' = f(t, y)$. Denoting by A the matrix of coefficients of the Runge–Kutta method and by $\mu_2[J]$ the logarithmic norm of the matrix J associated with the ℓ_2-norm, several authors (Crouzeix et al., BIT 23 (1983) 84–91; Hundsdorfer and Spijker, SIAM J. Numer. Anal. 24 (1987) 583–594; Kraaijevanger and Schneid, Numer. Math. 59 (1991) 129–157; Liu and Kraaijevanger, BIT 28(4) (1988) 825–838) have obtained conditions on A that ensure, for a given λ, the unique solvability of stage equations for all stepsize h and stiff system with $h\mu_2[f'(t, y)] < \lambda$, where $f'(t, y)$ is the jacobian matrix of f with respect to y. The aim of this paper is to study the unique solvability of stage equations in the frame of the ℓ_∞- and ℓ_1-norms. For a given real λ it will be proved that the condition $\mu_\infty[(\lambda I - A^{-1})D] < 0$, for some positive-definite diagonal matrix D, implies that the stage equations are uniquely solvable for all stepsize h and function f such that $h\mu_\infty[f'(t, y)] \leqslant \lambda$. Further, it is shown that these conditions also imply the BSI-stability i.e. the stability of stage equations under non uniform perturbations. Applications to some well-known families of Runge–Kutta methods are included.

MSC: 65L05

Keywords: Initial value problem; Implicit Runge–Kutta method; Solvability of stage equation

1. Introduction

We consider the numerical solution of stiff initial value problems

$$y'(t) = f(t, y(t)), \quad t \geqslant t_0, \tag{1.1}$$

$$y(t_0) = y_0 \in \mathbb{R}^m, \tag{1.2}$$

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by means of an s-stage Runge–Kutta method which advances the numerical solution from $(t_n, y_n) \to (t_{n+1} = t_n + h, y_{n+1})$, computing y_{n+1} from the formulas

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_{n,i}), \tag{1.3}$$

$$Y_{n,i} = y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_{n,j}) \quad (i = 1, \dots, s), \tag{1.4}$$

where c_i, b_i and a_{ij} $(1 \leqslant i, j \leqslant s)$ are real parameters defining the method with $c_i = a_{i1} + \cdots + a_{is}$, $(i = 1, \dots, s)$. Thus the method can be specified in terms of the $s \times s$ matrix $A = (a_{ij})$ and the column vector $b = (b_i)$. In the following, to simplify the presentation, it will be assumed that the function $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and its jacobian matrix $f'(t, y)$ are continuous functions on $\mathbb{R} \times \mathbb{R}^m$.

Eq. (1.4) that define the stage vectors $Y_{n,j}$ $(j = 1, \dots, s)$ will be called the stage equations and due to the implicitness of the methods a natural requirement is the existence of a unique solution of these equations for the class of problems and the range of stepsizes under consideration.

Conditions on h and f that guarantee the unique solvability of (1.4) have been given by several authors [2,7,10,11]. However, most of these studies have been carried out under the assumption that the function $f(t, y)$ satisfies a one-sided Lipschitz condition

$$\langle f(t,u) - f(t,v), u - v \rangle \leqslant \beta |u - v|^2, \quad t \in \mathbb{R}, \quad u, v \in \mathbb{R}^m, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^m$, $|\cdot|$ its induced norm and β a real constant. Denoting by $\mu[J]$ the logarithmic norm of a matrix J associated to a matrix norm $\| \cdot \|$ (see e.g. [3])

$$\mu[J] = \lim_{\Delta \to 0^+} \frac{\|(I + \Delta J)\| - 1}{\Delta}, \tag{1.6}$$

assumption (1.5) can be presented in the following equivalent forms:

$H_1(\beta)$. $\mu[f'(t,u)] \leqslant \beta$ for all $(t, y) \in \mathbb{R} \times \mathbb{R}^m$ where $\| \cdot \|$ is the matrix norm subordinated to $|x| = \sqrt{\langle x, x \rangle}$.

$H_2(\beta)$. For any two solutions $y(t)$ and $\tilde{y}(t)$ of (1.1) the function $\delta(t) := e^{(-\beta t)} |\tilde{y}(t) - y(t)|$ is a nondecreasing function of t in the interval of definition common to both solutions.

The study of the unique solvability of (1.4) in the frame of stiff systems (1.5) has been justified by the fact that there exist a well-established theory that analyzes the stability and convergence properties of Runge–Kutta methods for such a class of stiff systems [1,3,6]. A remarkable fact is that a simple algebraic condition on the coefficients of a Runge–Kutta method, the so-called algebraic stability, which plays an essential role to ensure their stability and convergence does not imply the unique solvability of (1.4) for $\beta = 0$ and therefore some additional conditions on matrix A have been given by several authors to ensure the unique solvability. Thus, Crouzeix et al. [2] proved that for $\beta = 0$ a sufficient condition on A is the so-called Lyapunov diagonal stability which means that there exists a diagonal matrix D such that D and $DA + A^{\mathrm{T}} D$ are positive definite. This condition has been weakened in [7,11].

Furthermore, these theoretical studies have been applied to many families of Runge–Kutta methods of practical interest (see, e.g., [3,6]), in order to determine for a given β in (1.5) the range of stepsizes which allow the unique solvability of (1.4). More recently, Kraaijevanger and Schneid [10] in a detailed study presented necessary and sufficient conditions for the unique solvability of some classes of implicit equations which include Eq. (1.4) with f satisfying (1.5).

While the analysis of the stability and convergence properties of Runge–Kutta methods in the frame of inner product norms developed rapidly after the early papers of Burrage, Butcher and Crouzeix, for other norms like the maximum norm it has been rather lengthy. It must be remarked that the use of the ℓ_∞- and ℓ_1-norms may have some advantages over the inner product norms. First in view of the definitions of μ_∞, μ_1, μ_2 (see, e.g., [3, Section 1.5]) in many practical problems the conditions on $\mu_\infty[f'(t,y)]$ or else $\mu_1[f'(t,y)]$ can be checked more easily than those of the euclidean norm $\mu_2[f'(t,y)]$. Recall that

$$\mu_\infty[Q] = \mu_1[Q^{\mathrm{T}}] = \max_j \left\{ q_{jj} + \sum_{l\neq j} |q_{jl}| \right\} \quad \text{and} \quad \mu_2[Q] = \lambda_{\max}((Q^{\mathrm{T}} + Q)/2).$$

On the other hand, there are problems (1.1), (1.2) in which the Jacobian $J = f'(t,y)$ can be bounded more appropriately in the ℓ_∞- or ℓ_1- than in the ℓ_2-norm. To illustrate this fact consider the simple matrix

$$J = \begin{pmatrix} 0 & 0 \\ \alpha & -\alpha \end{pmatrix} \quad \text{where} \quad \alpha \gg 1.$$

An elementary calculation shows that $\mu_\infty[J] = 0$ while $\mu_2[J] = ((\sqrt{2}-1)/2)\alpha$. Thus for α large $\mu_2[J]$ will be large in spite of the well-conditioned global behavior of the solutions of $y' = Jy$ and therefore bounds based in the ℓ_2-norm can be useless. More generally, when J is almost symmetric, $\mu_2[J]$ is close to the spectral abscissa which is, in a sense, an optimal measure of the asymptotic behavior of the flow. However for problems where J has a moderately sized spectral abscissa and is very unsymmetric the quantity $\mu_2[J]$ can be very large. For these problems the ℓ_∞- or ℓ_1-norms may provide a more appropriate measure of the asymptotic measure of their solutions. This kind of problems appear, e.g., in singular perturbations and in some discretizations of partial differential equations (see, e.g., [3, pp. 275–277]). Moreover, in these problems the maximum norm leads to error bounds that hold uniformly in all space variables.

For problems in which $\mu_2[J]$ is very large but $\mu_1[J] \leqslant 0$ or $\mu_\infty[J] \leqslant 0$ the classical theory of B-stability and B-convergence based on the ℓ_2-norm can not be applied and a more appropriate theory, based on the ℓ_1- or ℓ_∞-norms should be developed. Among the contributions on the stability and convergence properties of Runge–Kutta methods on arbitrary norms we may mention those of Spijker [12] that deal with the contractivity of Runge–Kutta methods and more recently those of Kraaijevanger [8,9] who analyze not only the contractivity and convergence properties of these methods but also the unique solvability of stage equations. Let us recall that method (1.3),(1.4) is said to be (unconditionally) contractive for a given norm $|\cdot|$ if it preserves the dissipativity with respect to this norm for all stepsizes, i.e., if for all $f \in \mathrm{H}_1(0)$ the result of two parallel steps with the same method: $(t_n, y_n) \to (t_{n+1}, y_{n+1})$, $(t_n, \tilde{y}_n) \to (t_{n+1}, \tilde{y}_{n+1})$, satisfy $|\tilde{y}_{n+1} - y_{n+1}| \leqslant |\tilde{y}_n - y_n|$. Note that as shown by van Dorsselaer and Spijker [4], $H_1(\beta) \Leftrightarrow H_2(\beta)$ for any norm and therefore the dissipativity with respect to any norm can be characterized either by $H_1(0)$ or else by $H_2(0)$.

As a consequence of the above studies Kraaijevanger has shown in [8] that a Runge–Kutta method is (unconditionally) — contractive in the maximum norm if and only if the so called K-function of the method defined by

$$K(Z) = \det(I - AZ + ee^{\mathrm{T}}Z)/\det(I - AZ)$$

with $e^{\mathrm{T}}=(1,\ldots,1)$ and $Z=\mathrm{diag}(z_1,\ldots,z_s)$, is absolutely monotonic for all $(z_1,\ldots,z_s)\in(-\infty,0]^s$. Moreover, this condition is equivalent to the following algebraic condition:

$$A \text{ is nonsingular, } A^{-1} \text{ is an } M\text{-matrix} \quad\text{and}\quad A^{-1}e\geqslant 0,\quad b^{\mathrm{T}}A^{-1}\geqslant 0,\quad b^{\mathrm{T}}A^{-1}e\leqslant 0. \tag{1.7}$$

This condition plays, for the maximum norm, the same role of the algebraic stability condition for the inner product norms. In addition Kraaijevanger [8] has proved that this absolute monotonicity is sufficient to ensure the unique solvability of stage equations. However, the main drawback of (1.7) is that the order of an (unconditionally) — contractive Runge–Kutta method is $\leqslant 1$ and therefore (1.7) is a too strong requirement to the unique solvability of stage equations in the maximum norm.

The aim of this paper is to give sufficient conditions on the matrix A (weaker than (1.7)) that imply the existence of a unique solution of (1.4) for the class of dissipative problems with respect to the ℓ_∞- and ℓ_1-norms. In Section 2, for any real λ, we give a sufficient condition on A which ensures the unique solvability of (1.4) for the more general class of differential equations (1.1) that satisfy either $H_1(\beta)$ or else $H_2(\beta)$ with $h\beta\leqslant\lambda$. In Section 3 it will be shown that this sufficient condition implies also the BSI-stability of stage equations (1.4). Finally, in Section 4, the unique solvability behavior of stage equations of some well-known families of Runge–Kutta methods is established by means of the new sufficient condition.

2. The sufficient conditions to the unique solvability of stage equations

We start introducing some notations to write the stage equations (1.4) in a more compact form. First of all putting

$$X_i=Y_{n,i}-y_n,\qquad f_i(X)=f(t_n+c_ih,y_n+X)\quad (i=1,\ldots,s),$$

Eq. (1.4) can be written as

$$X_i=h\sum_{j=1}^{s}a_{ij}f_j(X_j)\quad (i=1,\ldots,s). \tag{2.1}$$

Next in the space of stages $(\mathbb{R}^m)^s$ we introduce the vectors and matrix given by

$$\boldsymbol{X}=\begin{pmatrix}X_1\\ \vdots\\ X_s\end{pmatrix},\qquad \boldsymbol{F}(\boldsymbol{X})=\begin{pmatrix}f_1(X_1)\\ \vdots\\ f_s(X_s)\end{pmatrix},\quad \boldsymbol{A}=A\otimes I, \tag{2.2}$$

where $\otimes$ stands for the Kronecker product. With these notations we may rewrite (2.1) in the form

$$\boldsymbol{X}=h\boldsymbol{A}\,\boldsymbol{F}(\boldsymbol{X}). \tag{2.3}$$

Note that throughout the rest of this paper vectors and matrices in the space of stages will be denoted by boldface.

Further, we will assume in the remainder of the paper that A is a nonsingular matrix. Observe that we are interested in the existence of a unique solution to Eq. (1.4), thus in case that there exist explicit stages (e.g. in Lobatto IIIA or IIIB methods) they can be eliminated and the implicit equations can be written again in the form (2.3) with A the submatrix of implicit stages. More generally, if a A is a reducible matrix [5, pp. 50], as proved in [10, Theorem 2.1], the unique solvability of (2.3) reduces to lower-dimensional problems with the same type and irreducible matrices. This implies that, without loss of generality, we could assume in the following that A is irreducible. However, we do not introduce this assumption because most of our results hold true without it.

Now following the ideas of suitability of Kraaijevanger and Schneid [10] we introduce the following.

Definition. We say that (2.3) is *Uni*quely *Sol*vable (UniSolv) for a given norm $|\cdot|$ in $\mathbb{R}^m$ at some $\gamma \in \mathbb{R}$ if Eq. (2.3) has a unique solution for all set of continuously differentiable functions $f_j : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\mu[f_j'(X)] \leqslant \beta \quad (j = 1, \ldots, s) \tag{2.4}$$

with $h\beta = \gamma$, and μ defined by (1.6).

Remark. (1) Since in the context of Runge–Kutta methods $f_j(X) = f(t_n + c_j h, y_n + X)$, condition (1.5) on f implies (2.4). This means that the UniSolv of (2.3) introduced in the above definition is a stronger requirement than necessary for the unique solvability of stage equations in (confluent) Runge–Kutta methods (see comments on the definition of suitability given in [10]). However, the conditions that will be derived here to the existence of a unique solution of (2.3) hold in this general setting.

(2) If (2.3) is UniSolv at some γ and $\gamma' < \gamma$ then it is clear from the above definition that (2.3) is UniSolv at γ'. This implies that (2.3) is UniSolv at all values $(-\infty, \gamma]$ and motivates (see, e.g., [10]) the definition of the abscissa of UniSolv $s(A)$ for a given norm $|\cdot|$ as the $\sup\{\gamma\}$ for which (2.3) is UniSolv at γ in this norm.

(3) Following the ideas used in Theorem 2.10 of [10] it can be proved that (2.3) is UniSolv at λ if and only if $(I - \lambda A)$ is nonsingular and

$$\boldsymbol{X} = h\boldsymbol{A}(\lambda)\boldsymbol{F}(\boldsymbol{X}) \tag{2.5}$$

with

$$\boldsymbol{A}(\lambda) = A(\lambda) \otimes I \quad \text{and} \quad A(\lambda) = A(I - \lambda A)^{-1},$$

is UniSolv at γ=0. This follows from the fact that UniSolv of (2.3) at λ=$h\beta$ for f satisfying (2.4) is equivalent to the UniSolv of (2.3) at γ=0 for $\hat{f}_j(X)$=$f_j(X)-\beta X$, since $\mu[\hat{f}_j']$=$\mu[f_j'-\beta I]$=$\mu[f_j']-\beta$. Clearly this result allows us to reduce the study of the UniSolv of (2.3) at any real λ to the UniSolv of (2.5) at the fixed value $\gamma = 0$.

Theorem 2.1. *Let λ be a real number. If there exists a positive-definite diagonal matrix D such that*

$$\mu_\infty[(\lambda I - A^{-1})D] < 0, \tag{2.6}$$

then (2.3) *is UniSolv at λ for the maximum norm.*

Proof. In view of Remark 3 and the fact that when $I - \lambda A$ is nonsingular $-A(\lambda)^{-1}D$=$(\lambda I - A^{-1})D$ it will be enough to prove the theorem for $\lambda = 0$.

Let D be a positive-definite diagonal matrix and $\boldsymbol{D}=D\otimes I$. Putting $\boldsymbol{X}=\boldsymbol{DZ}$ the UniSolv of (2.3) at λ is equivalent to the existence of unique solution of $\boldsymbol{G}(\boldsymbol{Z})=\boldsymbol{0}$ with the function $\boldsymbol{G}:(\mathbb{R}^m)^s\to(\mathbb{R}^m)^s$ given by

$$\boldsymbol{G}(\boldsymbol{Z}):=-\boldsymbol{A}^{-1}\boldsymbol{DZ}+h\boldsymbol{F}(\boldsymbol{DZ}). \tag{2.7}$$

To prove the existence of a unique solution of (2.7) we apply Theorem 3.6 of van Dorsselaer and Spijker [4]. We take $\boldsymbol{Z}^{(0)}=\boldsymbol{0}$ the zero vector of $(\mathbb{R}^m)^s$, and in this space we consider the norm

$$\|\boldsymbol{X}\|=\max_{1\leqslant i\leqslant s}|X_i|=\max_{1\leqslant i\leqslant s}\ \max_{1\leqslant j\leqslant m}|X_{i,j}|, \tag{2.8}$$

i.e., the maximum norm in the space of stages.

From (2.7) it follows that the Jacobian matrix of $\boldsymbol{G}$ can be written as

$$\boldsymbol{G}'(\boldsymbol{Z})=-\boldsymbol{A}^{-1}\boldsymbol{D}+h\boldsymbol{F}'(\boldsymbol{DZ})\boldsymbol{D}.$$

Denoting by $\boldsymbol{\mu}$ the logarithmic norm of matrices in $(\mathbb{R}^m)^s$ associated to the norm (2.8) we have

$$\boldsymbol{\mu}[\boldsymbol{G}'(\boldsymbol{Z})]\leqslant\boldsymbol{\mu}[-\boldsymbol{A}^{-1}\boldsymbol{D}]+\boldsymbol{\mu}[h\boldsymbol{F}'(\boldsymbol{DZ})\boldsymbol{D}]. \tag{2.9}$$

Since for all $Q\in\mathbb{R}^{s\times s}$ and $I=I_m$, we have $\boldsymbol{\mu}[Q\otimes I]=\mu_\infty[Q]$ then

$$\boldsymbol{\mu}[-\boldsymbol{A}^{-1}\boldsymbol{D}]=\boldsymbol{\mu}[(-A^{-1}D)\otimes I]=\mu_\infty[-A^{-1}D]=-r_0<0. \tag{2.10}$$

On the other hand, $h\boldsymbol{F}'(\boldsymbol{DZ})\boldsymbol{D}$ is a block diagonal matrix with the form

$$\mathrm{diag}(Q_1,\ldots,Q_s)\quad\text{with}\quad Q_j=hd_jf'(d_jZ_j),$$

where $D=\mathrm{diag}(d_1,\ldots,d_s)$. Hence,

$$\boldsymbol{\mu}[h\boldsymbol{DF}'(\boldsymbol{DZ})]=\max_{1\leqslant j\leqslant s}\mu_\infty[Q_j]\leqslant 0. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9) we get

$$\boldsymbol{\mu}[\boldsymbol{G}'(\boldsymbol{Z})]\leqslant -r_0$$

with $r_0>0$ given by (2.10). Now by the above mentioned Theorem 3.6 of [4] the equation $\boldsymbol{G}(\boldsymbol{Z})=0$ has a unique solution.

Remark. (1) It can be seen that Theorem 2.3 can be easily modified to show that the unique solvability of (2.3) at λ for the maximum norm also holds if there exist two positive-definite diagonal matrices D_1 and D_2 such that

$$\mu_\infty[D_1(\lambda I-A^{-1})D_2]<0. \tag{2.12}$$

Although this condition is apparently more general than (2.6), both conditions turn out to be equivalent and therefore we have formulated Theorem 2.3 with the more simple condition (2.6).

(2) It follows from Theorem 2.1 that the existence of a positive-definite diagonal matrix D such that $\mu_\infty[-A^{-1}D]<0$ ensures the UniSolv of (2.3) at $\lambda=0$ in the ℓ_∞-norm, i.e., the UniSolv of (2.3) for all dissipative systems and all stepsizes in the uniform norm. Furthermore, for $\lambda>0$, since

$$\begin{aligned}\mu_\infty[(\lambda I-A^{-1})D]&\leqslant\mu_\infty[\lambda D]+\mu_\infty[-A^{-1}D]\\&=\lambda\max_{1\leqslant j\leqslant s}d_j+\mu_\infty[-A^{-1}D],\end{aligned}$$

the same theorem implies the unique solvability for

$$h\beta = \lambda < \frac{-\mu_\infty[-A^{-1}D]}{\max_{1\leqslant j\leqslant s} d_j}.$$

Notice that the above Theorem 2.1 provides a set of sufficient conditions for the UniSolv of (2.3) at a single value of λ. Thus for a given matrix A will be important to get insight into the set

$$\Lambda_A := \{\lambda;\, \mu_\infty[(\lambda I - A^{-1})D] < 0,\ \text{for some } D > 0\}, \tag{2.13}$$

where $D > 0$ denotes a positive-definite diagonal matrix.

First of all since $\mu_\infty[(\lambda I - A^{-1})D] \to -\infty$, when $\lambda \to -\infty$, the set Λ_A is $\neq \emptyset$. Moreover if $\lambda \in \Lambda_A$ there exist $D > 0$ such that $\mu_\infty[(\lambda I - A^{-1})D] < 0$. Hence for all $\lambda' < \lambda$, utilizing the properties of the logarithmic norm, we have

$$\mu_\infty[(\lambda' I - A^{-1})D] \leqslant \mu_\infty[(\lambda I - A^{-1})D] + \mu_\infty[(\lambda' - \lambda)D] < 0,$$

and therefore $\lambda' \in \Lambda_A$.

On the other hand, since $(\lambda, D) \to \mu_\infty[(\lambda I - A^{-1})D]$ is a continuous mapping, Λ_A is an open subset of $\mathbb{R}$. Then the above remarks show that Λ_A is an interval of type $(-\infty, \lambda_\infty(A))$ with $\lambda_\infty(A) = \sup \Lambda_A < +\infty$. In fact, by the definition of μ_∞ for all $\lambda \in \Lambda_A$,we have $\lambda < \min_{1\leqslant i\leqslant s}\{(A^{-1})_{ii}\}$, and therefore

$$\lambda_\infty(A) < \min_{1\leqslant i\leqslant s}\{\alpha_{ii} = (A^{-1})_{ii}\}. \tag{2.14}$$

In order to determine $\lambda_\infty(A)$, we consider the matrix $T \in \mathbb{R}^{s\times s}$ defined by

$$T = \begin{cases} t_{ii} = (A^{-1})_{ii}, \\ t_{ij} = -|(A^{-1})_{ij}|\ (i \neq j). \end{cases} \tag{2.15}$$

Then $\mu_\infty[(\lambda I - A^{-1})D] < 0$ holds for $D = \operatorname{diag}(d_i) > 0$ if and only if $Td > \lambda d$ for $d = (d_1, \ldots, d_s)^{\mathrm{T}}$ (here $u > v$ stands for $u_i > v_i$, $i = 1, \ldots, s$). Hence an alternative definition of $\lambda_\infty(A)$ is

$$\lambda_\infty(A) = \sup \Lambda_A = \sup\{\lambda \in \mathbb{R};\, Td > \lambda d,\ \text{for some } d > 0\}. \tag{2.16}$$

Now suppose that A is an irreducible matrix [5, Chapter XIII]. Then A^{-1} and T are also irreducible and according to Frobenius theorem [5, pp. 53] for some $\delta > 0$ sufficiently large $-T + \delta I$ is a nonnegative irreducible matrix that has an eigenvalue $r > 0$ with a positive eigenvector u. Hence $Tu = (-r + \delta)u$ and we may ensure that T has a real eigenvalue λ_0 with a positive eigenvector d_0. In view of this, for all $\lambda < \lambda_0$ we may write $\lambda d_0 = (\lambda - \lambda_0)d_0 + Td_0 < Td_0$ which implies that $\lambda \in \Lambda_A$. This shows that such a real eigenvalue λ_0 corresponding to a positive eigenvector is a possible candidate to the value of $\lambda_\infty(A)$. In fact, with a similar argument to the one used in the proof of Frobenius theorem [5, pp. 56] it can be seen that $\lambda_\infty(A)$ must be an eigenvalue of T associated to a positive eigenvector.

In conclusion we have proved:

Theorem 2.2. *If A is an irreducible matrix then $\lambda_\infty(A)$ is an eigenvalue of T corresponding to a positive eigenvector.*

For the ℓ_1-norm proceeding in a similar way to the above theorem we can prove:

Theorem 2.3. *Let $\lambda \in \mathbb{R}$ and D a positive-definite diagonal matrix such that*

$$\mu_1[D(\lambda I - A^{-1})] < 0. \tag{2.17}$$

Then (2.3) *is UniSolv at λ for the ℓ_1 norm.*

Observe that the remarks and consequences derived from the sufficient condition for the UniSolv for the ℓ_∞-norm can be easily translated to the ℓ_1-norm.

To end this section let us remark that the approach used in Theorems 2.1 and 2.3 to derive sufficient conditions on the UniSolv. of (2.3) at λ for the ℓ_∞- and ℓ_1-norms can be applied also to the euclidean norm. Thus, it can be proved that the existence of two positive definite diagonal matrices D_1 and D_2 such that

$$\mu_2[D_1(\lambda I - A^{-1})D_2] < 0, \tag{2.18}$$

implies the UniSolv of (2.3) at λ with respect to the euclidean norm. Furthermore (2.18) implies that

$$\lambda < \psi_0(A^{-1}) = \sup_{D>0} \psi_D(A^{-1}), \tag{2.19}$$

where $\psi_D(A) = \min_{\xi \neq 0} \langle A\xi, \xi\rangle_D / \langle \xi, \xi\rangle_D$ is the function defined by Dekker and Verwer [3, Chapter V] in their study of the unique solvability with respect to inner product norms. Note that for appropriate D_1 and D_2 condition (2.19) implies (2.18).

3. The stability of stage equations

In this section we study the effect of nonuniform perturbations of the stage equations (1.4) on their solutions for the classes of functions f satisfying $\mu_\infty[f'(t,y)] \leqslant \beta$ or else $\mu_1[f'(t,y)] \leqslant \beta$. The Runge–Kutta methods (A,b) whose matrices A show a stable behavior are usually called BSI-stable for the corresponding class of functions and stepsizes.

Together with (1.4) we consider the perturbed equations

$$\tilde{Y}_{n,i} = y_n + h\sum_{j=1}^{s} a_{ij} f(t_n + c_j h, \tilde{Y}_{n,j}) + \Delta_i \quad (i = 1,\dots,s), \tag{3.1}$$

where Δ_i are arbitrary perturbations introduced in the ith stage. Furthermore, we introduce the stage vectors

$$\boldsymbol{U} := (U_i = \tilde{Y}_{n,i} - Y_{n,i}), \qquad \boldsymbol{\Delta} := (\Delta_i).$$

Definition. The matrix A is said to be BSI-stable at some $\lambda \in \mathbb{R}$ for the ℓ_∞-norm if for all continuously differentiable function f with $\mu_\infty[f'(t,y)] \leqslant \beta$ and all stepsize $h \geq 0$, with $h\beta \leqslant \lambda$ Eqs. (1.4) and (3.1) are uniquely solvable and their solutions satisfy

$$\|\boldsymbol{U}\| \leqslant C\|\boldsymbol{\Delta}\| \tag{3.2}$$

with some constant C independent of the stiffness.

Note that in the above definition the same ℓ_∞-norm is used in the spaces $\mathbb{R}^m$ and $(\mathbb{R}^m)^s$, however, more general norms could be considered in the space of stage vectors $(\mathbb{R}^m)^s$. Further, a similar definition can be given for the ℓ_1-norm replacing $\mu_\infty[f'(t,y)]\leqslant\beta$ by $\mu_1[f'(t,y)]\leqslant\beta$ and the norms (2.8) of (3.2) by their corresponding ℓ_1-norm, i.e.,

$$\|\boldsymbol{X}\| = \sum_{i=1}^{s} |X_i| = \sum_{i=1}^{s}\sum_{j=1}^{m} |X_{i,j}|.$$

As a final remark note that in [10] a similar concept has been called stable suitability at λ.

Substracting (3.1) from (1.4) we have

$$\boldsymbol{U} = h\boldsymbol{A}\,[\boldsymbol{F}(\boldsymbol{X}+\boldsymbol{U}) - \boldsymbol{F}(\boldsymbol{X})] + \boldsymbol{\Delta}. \tag{3.3}$$

Next, we will show that the sufficient conditions (2.6) or else (2.12) for the unique solvability also imply the BSI-stability.

Theorem 3.1. *Let $\lambda\in\mathbb{R}$ and $D_1, D_2 \in \mathbb{R}^{s\times s}$ positive-definite diagonal matrices such that* (2.12) *holds, then the matrix A is BSI-stable at λ for the ℓ_∞-norm.*

Proof. Each component of the bracket in (3.3) $f_i(X_i+U_i) - f_i(X_i)$ can be written in the form

$$f_i(X_i+U_i) - f_i(X_i) = \left(\int_0^1 f'(X_i+\theta U_i)\,\mathrm{d}\theta\right) U_i \equiv J_i U_i.$$

Moreover as shown in [3, pp. 29] the assumption $\mu_\infty[f'(t,y)]\leqslant\beta$ implies that $\mu_\infty[J_i]\leqslant\beta$.

Putting $\boldsymbol{J}=$ diag $(J_1,\dots,J_s)$, (3.3) can be written equivalently in the form

$$[\boldsymbol{I} - h\boldsymbol{A}\boldsymbol{J}]\boldsymbol{U} = \boldsymbol{\Delta},$$

or else

$$[\boldsymbol{A}^{-1} - h\boldsymbol{J}]\boldsymbol{U} = \boldsymbol{A}^{-1}\boldsymbol{\Delta}.$$

Introducing the matrices $\boldsymbol{D}_1 = D_1\otimes I$ and $\boldsymbol{D}_2 = D_2\otimes I$ where D_1 and D_2 are positive diagonal matrices we have

$$[\boldsymbol{D}_1^{-1}(\boldsymbol{A}^{-1} - h\boldsymbol{J})\boldsymbol{D}_2]\boldsymbol{V} = -\boldsymbol{D}_1^{-1}\boldsymbol{A}^{-1}\boldsymbol{\Delta}$$

with $\boldsymbol{V} = \boldsymbol{D}_2^{-1}\boldsymbol{U}$.

Now the matrix $\boldsymbol{Q} = \boldsymbol{D}_1^{-1}(\boldsymbol{A}^{-1} - h\boldsymbol{J})\boldsymbol{D}_2$ with $\mu_\infty[J_i]\leqslant\beta$ $(i=1,\dots,s)$ and $h\beta\leqslant\lambda$ satisfies

$$\mu_\infty[\boldsymbol{Q}]\leqslant\mu_\infty[D_1(\lambda I - A^{-1})D_2] = -r_0$$

with some constant $r_0>0$. Therefore, the matrix $\boldsymbol{Q}$ is nonsingular and $\|\boldsymbol{Q}^{-1}\|\leqslant 1/r_0$ and this implies that

$$\|\boldsymbol{U}\| = \|\boldsymbol{D}_2\boldsymbol{V}\| = \|\boldsymbol{D}_2\boldsymbol{Q}^{-1}\boldsymbol{D}_1^{-1}\boldsymbol{A}^{-1}\boldsymbol{\Delta}\|$$

and then

$$\|\boldsymbol{U}\| \leqslant \frac{|D_2|_\infty \, |D_1^{-1}A^{-1}|_\infty}{r_0} \|\boldsymbol{\Delta}\| \tag{3.4}$$

which proves the BSI-stability.

Remark. (1) The above theorem can be stated with the simpler condition (2.6) ($D_1 = I, D_2 = D$) instead of (2.12), however the stability constant derived from (3.4)

$$C = \inf_{D_1>0, D_2>0} \frac{|D_2|_\infty |D_1 A^{-1}|_\infty}{\mu_\infty[D_1(\lambda I - A^{-1})D_2]}$$

may be smaller than that corresponding to $D_1 = I,\ D_2 = D$.

(2) A similar statement holds for the ℓ_1-norm.

(3) For the ℓ_2-norm we have observed that (2.18) implies also the unique solvability of stage equations with $h\mu_2[f'(t,y)] \leqslant \lambda$. Assumption (2.18) implies again the BSI-stability for the ℓ_2-norm.

4. Applications

In this section we calculate $\lambda_\infty(A)$ for some well-known implicit Runge–Kutta methods. First of all let us recall that in the case of the ℓ_2-norm (and also for all inner product norms) we have the upper bound $\lambda_2(A) = \psi_0(A^{-1}) \leqslant \min_{1\leqslant i\leqslant s}(A^{-1})_{ii}$ (see [3, Chapter 5]). Furthermore, it has been found [3] that by choosing an appropriately positive diagonal matrix D, $\lambda_2(A)$ may attain this upper bound for some well-known families of implicit methods (Gauss–Legendre, Radau IA and Radau IIA). Since we have the same bound for the ℓ_∞-norm the question arises whether a suitable choice of D allows for $\lambda_\infty(A)$ to attain again this upper bound. It will be seen next that unfortunately this question has a negative answer.

Theorem 4.1. *For the two-stage Gauss–Legendre, Radau IA, Radau IIA and Lobatto IIIC Runge–Kutta methods* $\lambda_\infty(A) < \min\{\alpha_{ii},\ i = 1,2\}$.

Proof. For the two stage Gauss–Legendre method we have

$$A^{-1} = \begin{pmatrix} 3 & -3+2\sqrt{3} \\ 3+2\sqrt{3} & 3 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 3 & 3-2\sqrt{3} \\ -3-2\sqrt{3} & 3 \end{pmatrix}.$$

Taking into account Theorem 2.2 we calculate the eigenvalues of T which are $\lambda_\pm = 3 \pm \sqrt{3}$. Since $\lambda_+ > \min(A^{-1})_{ii} = 3$ it can be disregarded. For $\lambda_- = 3 - \sqrt{3}$ there is a positive eigenvector $(d_1, d_2)^{\mathrm{T}}$. Therefore,

$$\lambda_\infty(A) = 3 - \sqrt{3} < \min_{i=1,2} \alpha_{ii} = 3.$$

Similar calculations for the other method results are given in Tables 1 and 2.

For the four-stages Lobatto IIIA method we have $\lambda_\infty(A) = -1.6821 < 0$.

As can be seen from Table 1, the λ-values that give the bound $h\mu[f'(t,y)] < \lambda$ for the ℓ_∞-norm are smaller than that corresponding to the ℓ_2-norm in the classical two-stage implicit Runge–Kutta

Table 1
Two-stages Runge–Kutta methods

Method	$\lambda_\infty(A)$	$\min_{i=1,2} \alpha_{ii}$
Gauss–Legendre	$3-\sqrt{3}$	3
Radau IA	$2-\sqrt{10}/2$	3/2
Radau IIA	$2-\sqrt{10}/2$	3/2
Lobatto IIIC	0	1

Table 2
Three-stages Runge–Kutta methods

Method	$\lambda_\infty(A)$	$\min_{i=1,2} \alpha_{ii}$
Gauss–Legendre	−0.895631	2
Radau IA	−1.748719	$2-\sqrt{6}/2$
Radau IIA	−1.748719	$2-\sqrt{6}/2$
Lobatto IIIC	−2	0
Lobatto IIIA	$3-\sqrt{5}$	2

methods. However, as remarked above, there are stiff problems in which $\mu_2[f'(t,y)]$ can be positive and even very large while $\mu_\infty[f'(t,y)]\leqslant 0$. For these problems, since $h\mu_\infty[f'(t,y)]\leqslant 0<\lambda_\infty(A)$ for all $h>0$ our study allows to ensure the unique solvability for all stepsize while the study in the ℓ_2-norm only implies the unique solvability for $h\mu_2[f'(t,y)]<\min\alpha_{ii}$. In addition it is straightforward to estimate the logarithmic norm of the jacobian in the ℓ_∞-norm but the ℓ_2-norm requires the computation of the largest eigenvalue of the matrix $(f'+f'^T)/2$ and this clearly more complicated, particularly for higher-dimensional matrices.

In Table 2, it is found that, except in the Lobatto IIIA methods, the value of $\lambda_\infty(A)$ is negative and therefore the unique solvability, which is ensured for stepsizes and problems with $h\mu_\infty[f'(t,y)]<\lambda_\infty(A)$, does not have practical relevance. In fact for methods such as $\mu_\infty[-A^{-1}D]>0$ for all diagonal matrix D, our approach does not provide practical results on the solvability of stage equations in stiff problems.

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