Journal of Pure and Applied Algebra 39 (1986) 275–281
North-Holland

ON THE STEINBERG MODULE, REPRESENTATIONS OF THE SYMMETRIC GROUPS, AND THE STEENROD ALGEBRA

Stephen MITCHELL*
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Communicated by J. Stasheff
Received 13 March 1984
Revised 8 September 1984

Let B denote one of the subalgebras $A(n)$ of the mod p Steenrod algebra A, or one of the subalgebras $P(n)$ of $A/A\beta A$. In [4] it was shown that B admits a left A-module structure extending its left B-module structure. The proof made use of invariant theory, and the Steinberg representation St of $GL_n\mathbb{F}_p$, to construct a certain A-module; this module was then shown to be free on one generator as a B-module. Later, Jeff Smith [6] gave another proof of this result, using certain 'triangular' representations of the symmetric groups (our terminology is derived from the associated partitions). The question then arises whether these two A-module structures are the same. We show here that if $B=P(n)$ (or $p=2$ and $B=A(n)$), there is in fact a rather natural isomorphism between the two structures. At first glance this is somewhat surprising. However, we also show (Theorem 1.4) that St is precisely the 'Weyl module' associated to a suitable triangular representation via the classical theory described in [8]. (This fact is certainly well known, but we provide an elementary proof.) Furthermore – and this is really the main point of the paper – we show that Smith's theorem is in fact a *corollary* of the Weyl module result by making use of an unpublished result of Steward Priddy (Theorem 1.5 below). Hence this paper can be read independently of [6].

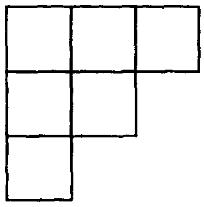
It is a pleasure to thank Jeff Smith for several conversations about his work [6], and for an introduction to the representation theory of the symmetric groups.

Notation. $P(n)$ is the subalgebra of $A/A\beta A$ generated by $P^1,\dots,P^{p^n}$. ($P^i=\mathrm{Sq}^{2i}$ if $p=2$).) Our results will be stated in terms of $P(n)$, but if $p=2$, then $P(n)$ can be replaced throughout by $A(n)$ – the subalgebra of A generated by $\mathrm{Sq}^1,\dots,\mathrm{Sq}^{2^n}$.

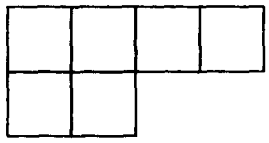
* Supported by NSF Grant No. MCS 83-11676.

1. Smith's Theorem, and the Steinberg module as a Weyl module

We begin by summarizing what we need from the representation theory of the symmetric groups; for further details we refer to [2]. The irreducible representations of Σ_k over $\mathbb{Q}$ correspond to partitions of k (any field is a splitting field for Σ_k). Such a partition λ is usually represented by a 'Young diagram': for example, the diagrams



and



represent the partitions (3, 2, 1) and (4, 2) of $k=6$. The irreducible S_λ corresponding to λ is constructed explicitly by giving a primitive idempotent f_λ, called a Young symmetrizer; then $S_\lambda = f_\lambda \mathbb{Q}\Sigma_k$. To define f_λ, let R_λ, C_λ denote respectively the row and column stabilizer groups of the Young diagram. For example, if $\lambda=(4,2)$, then $R_\lambda = \Sigma_4 \times \Sigma_2$ and $C_\lambda = \Sigma_2 \times \Sigma_2 \times \Sigma_1 \times \Sigma_1$. Here we have labelled the diagram as follows:

1	2	3	4
5	6		

so that the orbits of R_λ are $\{1,2,3,4\}$ and $\{5,6\}$, while those of C_λ are $\{1,5\}$, $\{2,6\}$, $\{3\}$, $\{4\}$. One then defines

$$f_\lambda = \frac{1}{c_\lambda} \bar{R}_\lambda \tilde{C}_\lambda,$$

where c_λ is a certain constant and a bar (resp. tilda) over a set of elements in Σ_k denotes the sum (resp. signed sum) of those elements in the group ring. There is a beautiful combinatorial formula for c_λ – it is the product of the 'hook lengths' associated with the Young diagram [2, 20.1]. In particular f_λ is defined over $\mathbb{Z}_{(p)}$ if and only if all hook lengths are prime to p. For example, if $k=(p-1)\binom{n}{2}$, and $\lambda \equiv \Delta_n$ is the 'triangular' partition $((p-1)(n-1), (p-1)(n-2), \dots, p-1)$, it is easy to show by induction on n that $p \nmid c_\lambda$ [6]. In fact for $p=2$ an amusing exercise is to show that $p \nmid c_\lambda$ if and only if $k=\binom{n}{2}$ and $\lambda = \Delta_n$. (We remark here that by general theory, the projective irreducible representations of $\mathbb{F}_p\Sigma_k$ are precisely the reductions mod p of the S_λ with $c_\lambda^{-1} \in \mathbb{Z}_{(p)}$.) Let $f_n = f_{\Delta_n}$.

Smith makes the following use of the idempotents f_n: Let $\mathbb{F}_p[y]$ denote the unstable A-algebra with $|y|=2$ (i.e., $H^*\mathbb{C}P^\infty$). Let $Z_n = A \cdot y \subseteq \mathbb{F}_p[y]/y^{p^n}$. Note that $Z_n = P(n-2) \cdot y$, and that Z_n has basis $y, y^p, \dots, y^{p^{n-1}}$. For any k, $\bigotimes^k Z_n$ is then an $A[\Sigma_k]$-module.

1.1. Theorem [6]. *As a* $P(n-2)$*-module,* $(\bigotimes^{(p-1)\binom{n}{2}} Z_n)f_n$ *is free on one generator.*

The first step in the proof is to observe that the dimensions are right. Here one uses a classical formula [2, 26..19], again involving hook lengths, that expresses $\dim((\bigotimes^k W)f_\lambda)$ in terms of λ and $\dim W$ (for any vector space W). In particular one has:

1.2. Lemma (Smith). *If* $\dim W = n$, *then* $\dim((\bigotimes^{(p-1)\binom{n}{2}} W)f_n) = p^{\binom{n}{2}}$.

Since $\dim P(n-2) = p^{\binom{n}{2}}$, the remaining (and hardest) part of the argument is to show the given A-module is $P(n-2)$-free; this is accomplished in [6] by analyzing the appropriate Margolis homology groups.

In attempting to understand how Smith's construction is related to [4], we were naturally led to ask how the Steinberg module is related to the representations of Σ_n. The result is exactly what one would expect, but there is a surprise: The answer to the question in fact *implies* Smith's theorem. To explain what one would expect, we first recall the classical construction of irreducible representations of $\mathrm{GL}_n\mathbb{C}$ [8]. Let V denote the standard n-dimensional representation of $\mathrm{GL}_n\mathbb{C}$. Given a partition λ of k, $k \geq 0$, the corresponding Weyl module $W_\lambda = (\bigotimes^k V)f_\lambda$. Weyl shows [8, Chap. IV]:

1.3. W_λ *is either zero or irreducible.*

The dimension of W_λ is given by the formula alluded to earlier. In characteristic p the situation is much more complicated [1]. However, if f_λ is defined over $\mathbb{Z}_{(p)}$, the definition of W_λ makes sense over $\mathbb{F}_p$. In Section 2 we prove:

1.4. Theorem. *The Weyl module* W_{Δ_n} *is isomorphic to the Steinberg representation of* $\mathrm{GL}_n\mathbb{F}_p$.

Remark. If $n=1$, then $\binom{1}{2}=0$ and by convention $\Sigma_0=\{1\}$, and W_{Δ_1} is the trivial representation of $\mathrm{GL}_1(\mathbb{F}_p)$, i.e., the Steinberg module. If $n=2$, S_{Δ_2} is again the trivial representation (of Σ_{p-1}), and we recover the familiar fact that St is the $(p-1)$-st symmetric power of V. We also note that 1.4 remains true if we replace V by its contragredient $V^\#$ in the definition of W_λ. This is because St is self-dual, i.e., $\mathrm{St}^\# \cong \mathrm{St}$.

Remark. Theorem 1.4 can be deduced from the standard correspondence between dominant weights and partitions; cf. Remark 3.2 below. However, there is also an elementary proof based on the flag complex (Tits building) description of St Theorem (Section 3).

Comparing Smith's theorem and Theorem 1.4, we see that they are strikingly similar. In fact, we claim the two theorems are almost equivalent. The necessary link is provided by a remarkable observation of Stewart Priddy: Let U_n denote the unipotent subgroup of $\mathrm{GL}_n \mathbb{F}_p$, consisting of upper triangular matrices with ones on the diagonal. If A is an augmented algebra, E^0A is the graded algebra associated with the augmentation ideal filtration.

1.5. Theorem (Priddy). *$E^0(\mathbb{F}_p U_n)$ and $E^0P(n-2)$ are isomorphic as Hopf algebras. (If $p=2$, $P(n-2)$ can be replaced by $A(n-2)$.)*

Proof. By a theorem of Quillen [5], $E^0(\mathbb{F}_p U_n) = V(\mathscr{L}_n)$, the restricted universal enveloping algebra of the restricted Lie algebra associated with the mod p lower central series of U_n. Similarly, by the Milnor–Moore theorem, $E^0(P(n-2)) \cong V(\mathscr{L}_n')$, where $\mathscr{L}_n'$ is the (restricted) Lie algebra of primitives in $E^0(P(n-2))$. But in fact $\mathscr{L}_n \cong \mathscr{L}_n'$ as (ungraded) restricted Lie algebras: On the one hand, it is well known that $\mathscr{L}_n$ is just the usual Lie algebra of upper triangular nilpotent matrices (with trivial restriction). It has a basis $\{x_{ij} : i<j\}$, where x_{ij} has a one in the (ij)-position and zeros elsewhere. On the other hand in his thesis P. May [3] showed that $\mathscr{L}_n'$ has basis $\{P_t^s : s+t<n-1,\ t>0\}$. Moreover, Theorem II.2.9, (3) and (4), of [3] states precisely that the map $\mathscr{L}_n \to \mathscr{L}_n'$ defined by $x_{ij} \mapsto P_{j-i}^{i-1}$ is an isomorphism of restricted Lie algebras. □

Remark. The correspondence is easily remembered in terms of the display

$$\begin{array}{ccccc}
0 & P_1^0 & P_2^0 & \cdots & P_{n-1}^0 \\
 & 0 & \ddots & & \vdots \\
 & & \ddots & \ddots & P_2^{n-3} \\
 & & & 0 & P_1^{n-2} \\
 & & & & 0
\end{array}$$

Proof of Theorem 1.1. If R is a Hopf algebra over $\mathbb{F}_p$, and N is an R-module, let $T(N)$ denote the R-module $(\bigotimes^{(p-1)\binom{n}{2}} N) f_n$. Let $E = E^0\mathbb{F}_p U_n = E^0P(n-2)$. Then the point of the proof is simply that $(V^n)^{\#}$ and Z_n can both be regarded as E-modules, and in fact are identical as E-modules. More precisely, $E^0(V^n)^{\#}$ is an E-module (by restricting the $\mathrm{GL}_n\mathbb{F}_p$ action to U_n), with basis $e_1, \dots, e_n$ and E acting by $(x_{ij})e_k = \delta_{ik}e_j$. On the other hand E^0Z_n has basis $y, y^p, \dots, y^{p^{n-1}}$, with $P_t^s y^{p^u} = \delta_{su} y^{p^{t+u}}$. Hence $e_k \to y^{p^{k-1}}$ is an isomorphism of E-modules. Now, by 1.4, $T(V^{\#})$ is free as an $\mathbb{F}_p U_n$-module. Hence $T(E^0V^{\#})) = T(E^0Z) = E^0T(Z)$ is free as an E-module, and hence $T(Z)$ is free on one generator as a $P(n-2)$-module. □

Remark. If one tries to reverse the above argument, by assuming Smith's theorem, one arrives at the conclusion that W_{Δ_n} is a projective $\mathrm{GL}_n\mathbb{F}_p$-module of dimension $p^{\binom{n}{2}}$. However, this in itself does not imply that $W_{\Delta_n} \cong \mathrm{St}$, even when $p=2$.

2. Generators for Weyl modules

In this section we write down explicit generators (in fact highest weight vectors) for the Weyl modules W_λ. As usual, $n>0$ is fixed, $V=K^n$ (K is a field), $k\geq 0$ and $\lambda=(k_1,\dots,k_m)$ (with $k_1\geq k_2\geq\cdots\geq k_m$) is a partition of k into at most n terms (when Char $K=0$, this is precisely the range where $W_\lambda\neq 0$). We define elements x_λ, z_λ of $\bigotimes^k V$ as follows: First form a 'tableau' of elements of V by filling in the ith row of the Young diagram for λ with copies of the standard basis vector e_i. Then form the tensor product x_λ of all the elements in the tableau, tensoring first the columns from top to bottom and then the rows from left to right. For example, if $n=k=3$ and $\lambda=(2,1)$, $x_\lambda=e_1\otimes e_2\otimes e_1$. Next, let $(r_1,\dots,r_s)$ be the partition of k given by the columns of the Young diagram (ordered from left to right; note $s\leq k$, and $r_i\leq n$). In $\bigotimes_{i=1}^s \bigwedge^{r_i} V$ we have the standard basis vector

$$z_\lambda=(e_1\wedge\cdots\wedge e_{r_1})\otimes\cdots\otimes(e_1\wedge\cdots\wedge e_{r_s}).$$

The signed trace $\tilde{C}_\lambda$ induces an embedding $\bigotimes_{i=1}^s \bigwedge^{r_i} V\to\bigotimes^k V$; we use the same notation for the image of z_λ under this map. The following is immediate from the definitions:

Corollary. *If* Char $K\nmid c_\lambda$ *(so f_λ defined over K), then $z_\lambda f_\lambda=z_\lambda$.*

Let $B=$ Borel subgroup of upper triangular matrices in $\mathrm{GL}(V)$. The following is then obvious, using the exterior product version of z_λ:

2.1. Proposition. *If* Char $K=0$, *then* $(c_\lambda/|R_\lambda|)x_\lambda f_\lambda=z_\lambda$.

2.2. Proposition. *The line spanned by z_λ is B-stable. When $K=\mathbb{F}_p$ and $\lambda=\Delta_n$, $z_n\equiv z_{\Delta_n}$ is fixed by B.*

3. Proof of Theorem 1.4

Let K be a field, and let $V=K^n$, with standard basis $e_1,\dots,e_n$. The set of flags (resp. complete flags) in V is denoted $F(V)$ (resp. $F_c(V)$). (Here a flag is a chain of *proper* nontrivial subspaces of V.) If $K=\mathbb{F}_q$ and we regard $F(V)$ as a simplicial complex, then $\tilde{H}^{n-2}(F(V);\mathbb{F}_q)\cong\mathrm{St}$, the Steinberg module [7]. Since St is self-dual we can also identify St with H_{n-2}, and hence with the submodule Z_{n-2} of cycles in $\mathbb{F}_q\cdot(F_c(V))$.

In particular St contains an obvious (spherical)$(n-2)$-cycle a defined as follows: There is an obvious inclusion of the poset of proper nonempty subsets of $\{1,\dots,n\}$ into the poset of proper nonzero subspaces of V (using the standard basis). Passing to the associated complexes, we obtain an embedding of the barycentric subdivision of the standard $(n-2)$-sphere into $F(V)$; this defines the cycle a. Explicitly,

$a=\tilde{\Sigma}_n[\hat{n}]$, where $[\hat{n}]$ is the standard flag $\langle e_1\rangle \subsetneq \langle e_1, e_2\rangle \subsetneq \cdots \subsetneq \langle e_1, \dots, e_{n-1}\rangle$.

Now identify $\mathbb{F}_p \cdot F_c(V)$ with the induced representation $\mathbb{F}_p\mathrm{GL}(V) \otimes_{\mathbb{F}_pB} \mathbb{F}_p$, where B is the Borel subgroup stabilizing $[\hat{n}]$. Then by adjunction there is a unique map $\phi : \mathbb{F}_p \cdot F_c(V) \to \bigotimes^{(p-1)\binom{n}{2}} V$ such that $\phi([\hat{n}]) = z_n$ (using 2.2). Moreover it is clear on inspection that $\phi(a)$ is nonzero (use the exterior power description of z_n). Hence ϕ restricts to a nonzero map $\mathrm{St} \to W_{\Delta_n}$; this map is injective (since St is irreducible) and hence an isomorphism (since the dimensions agree).

3.1. Remark. We may regard ϕ as a map $\mathbb{F}_p \cdot F_c(V) \to \bigotimes_{i=1}^{n-1} (\Lambda^i V)^{p-1}$. In this form it is just an obvious modification of the usual 'Plücker–Segre' embedding of the flag variety in the projective space $\mathbb{P}(\bigotimes_{i=1}^{n-1} \Lambda^i V)$.

3.2. Remark. From the viewpoint of algebraic groups Theorem 1.4 is seen as follows: The irreducible polynomial representations W of $\mathrm{SL}_n\mathbb{C}$ are classified by their 'highest weights' – that is, by the character of the diagonal subgroup H acting on the unique B-stable line in W. Such a highest weight can be written uniquely in the form $w = \sum_{i=1}^{n-1} a_i w_i$ ($a_i \in \mathbb{Z}$, $a_i \geq 0$), where the fundamental dominant weights w_i correspond to the exterior powers $\Lambda^i V^n$. From Section 2 it is then obvious that the weight w corresponds to the partition $(a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \dots, a_n)$. Since St can be obtained (sometimes by definition) as the 'mod p reduction' of the module with highest weight $(p-1)\sum_{i=1}^{n-1} w_i$, it follows easily that $W_{\Delta_n} \cong \mathrm{St}$.

4. A-module structures on $P(n)$

In this section we show that the A-module structures on $P(n)$ (or $A(n)$, if $p=2$) obtained in [4] and [6] are isomorphic. In order to be consistent with the conventions of [4], we replace V^n by $W^n = (V^n)^{\#}$ – the contragredient module. W^n has basis $y_1, \dots, y_n$ dual to $e_1, \dots, e_n$. For $k \leq n$, we regard W^k as the subspace of W^n spanned by $y_{n-k+1}, \dots, y_n$. We set $\dim y_i = 2$, so that the symmetric algebra $S^n = S(W^n) = \mathbb{F}_p(y_1, \dots, y_n]$ becomes, as usual, an unstable A-algebra. Let $G_n = \mathrm{GL}_n\mathbb{F}_p$. Then S^n is in fact an $A[G_n]$-algebra. Finally, let M_n denote the 'covariant algebra' $S^n \otimes_{S^{G_n}} \mathbb{F}_p$ and let e_n denote the Steinberg idempotent in $\mathbb{F}_pG_n$. Then one of the main results of [4] is:

4.1. Theorem. *As a $P(n-2)$-module, $M_n e_n$ is free on one generator.*

This theorem exhibits an A-module structure on $P(n-2)$ extending its $P(n-2)$-module structure. Theorem 1.1 provides another such structure, but in fact the two coincide:

4.2. Theorem. *$(\bigotimes^{(p-1)\binom{n}{2}} Z_n) f_n$ and $M_n e_n$ are isomorphic as A-modules.*

Proof. Identify $W^{(p-1)\binom{n}{2}}$ with $(p-1)W^{n-1}\oplus(p-1)W^{n-1}\oplus\cdots\oplus(p-1)W^1$. The flag $W^1\subset W^2\subset\cdots\subset W^{n-1}$ then defines an obvious map $\phi: W^{(p-1)\binom{n}{2}}\to W^n$; the restriction of ϕ to each W^k is just inclusion. Hence we obtain a map $S(\phi): S^{(p-1)\binom{n}{2}}\to S^n$ of A-algebras.

4.3. Lemma. *If* $z\in S^n$, $\dim z>0$, *then* z^{p^n} *is zero in* M_n.

By the lemma, $S(\phi)$ induces a map of A-algebras $\alpha: Q_n\to M_n$, where $Q_n=\bigotimes^{(p-1)\binom{n}{2}}\mathbb{F}_p[y]/(y^{p^n})$. At this point it is convenient to replace e_n by the conjugate idempotent $\chi(e_n)$ [4, 2.2]. We then define ψ to be the composite

$$\bigotimes^{(p-1)\binom{n}{2}} Z_n f_n \to Q_n \xrightarrow{\alpha} M_n \to M_n\chi(e_n).$$

Now identify Z_n with V^n by the correspondence $e_k\leftrightarrow y^{p^{k-1}}$. We then obtain $z_n\in\bigotimes^{(p-1)\binom{n}{2}} Z_n f_n$ as in 2.2. Moreover it follows immediately from the definitions that $\psi(z_n)=L_{n-1}^{p-1}\cdots L_1^{p-1}$, where $L_k=(y_{n-k+1}\cdots y_n^{p^{k-1}})\bar{\Sigma}_k$ (here Σ_k permutes the last k coordinates in W^n). But this is precisely the generator of $M_n\chi(e_n)$ [4, §3]. Hence ψ is onto, and is then an isomorphism by 4.1 and 1.2.

Proof of 4.3. Let $N=S^n\otimes_T\mathbb{F}_p$, where $T=\mathbb{F}_p[y_1^{p^n},\ldots,y_n^{p^n}]$. By [4, (3.4)], there is a map of S-modules $\eta: M\to N$ defined by $\eta(x)=L_n x$. Recall that M and N are Poincaré duality algebras. If $a=y_1^{p^n-2}\cdots y_n^{p^n-p^{n-1}-1}$, then $\eta(a)=y_1^{p^n-1}\cdots y_n^{p^n-1}$ – the fundamental class of N. It follows that a must be a fundamental class for M (since it has the right dimension). Thus η maps fundamental class to fundamental class, and hence is injective. Now if $z\in M$, obviously $\eta(z^{p^n})=0$, and hence $z^{p^n}=0$.

References

[1] R. Carter and C. Lusztig, On the modular representations of the general linear and symmetric groups, Math. Z. 136 (1974) 193–242.

[2] G. James, The Representation Theory of the Symmetric Groups (Springer, New York, 1978).

[3] P. May, Thesis, Princeton University, 1964.

[4] S. Mitchell, Finite complexes with $A(n)$-free cohomology, Topology 24 (1985) 227–248.

[5] D. Quillen, On the associated graded ring of a group ring, J. Algebra 10 (1968) 411–418.

[6] J. Smith, in preparation.

[7] L. Solomon, The Steinberg character of a finite group with BN-pair, in: Theory of Finite Groups (Harvard Symposium), (Benjamin, New York, 1969) 213–221.

[8] H. Weyl, The Classical Groups (Princeton, 1949).