A sequentially computable function that is not effectively continuous at any point

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Abstract

P. Hertling [Lecture Notes in Computer Science, vol. 2380, Springer, Berlin, 2002, pp. 962–972; Ann. Pure Appl. Logic 132 (2005) 227–246] showed that there exists a sequentially computable function mapping all computable real numbers to computable real numbers that is not effectively continuous. Here, that result is strengthened: a sequentially computable function on the computable real numbers is constructed that is not effectively continuous at any point.

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1. Introduction

In this paper we are concerned with functions that are defined on all computable real numbers and map each computable real number to a computable real number. One of the earliest computability notions for such functions is sequential computability, considered first by Banach and Mazur [1,10]. We call a function mapping all computable real numbers to computable real numbers sequentially computable if it maps every computable sequence of real numbers to a computable sequence of real numbers. Consider the following known facts:

- Every sequentially computable function is continuous. This result goes back to Mazur [10] who proved it for functions defined on all computable real numbers contained in some interval. See also [4].
- Every function that is computable in the sense of Markov, i.e., computable with respect to a standard Gödel numbering of the computable real numbers, is effectively continuous. A
function $F$ is called effectively continuous if one can enumerate a list of pairs of open rational intervals $(I, J)$ such that $F(\text{closure}(I)) \subseteq J$ for all such pairs and such that for any computable real number $x$ and any $\varepsilon > 0$ there is some pair $(I, J)$ in the list with $x \in I$ and $\text{length}(J) < \varepsilon$.

This result is due to Tseitin [13,14], who proved it more generally for functions on certain computable metric spaces; see also Kušner [8]. A short proof for the real number case can be found in [6].

It is clear that any effectively continuous function is computable in the sense of Markov. Thus, computability in the sense of Markov and effective continuity are equivalent for functions mapping all computable real numbers to computable real numbers. It is also clear that any function computable in the sense of Markov is sequentially computable. Kušner [7,9] posed the question whether sequential computability is also equivalent to Markov computability and, thus, to effective continuity. In [5,6] it was shown that this is not the case: there exists a sequentially computable function mapping all computable real numbers to computable real numbers that is not effectively continuous and, hence, not computable in the sense of Markov. Bauer and Simpson [2] gave a different proof, based on an analogous result by Friedberg [3] for the case of functions mapping all total recursive functions from $\mathbb{N}$ to $\mathbb{N}$ to natural numbers.

In view of the facts listed above, the question arises whether one can weaken the notion of effective continuity and perhaps show that a sequentially computable function is necessarily effectively continuous in some weaker sense. This question looks interesting also because Pour-El and Richards [11] base their development of computable analysis on the notion of a sequentially computable function that satisfies additionally a continuity condition.

We argue that the answer to this question seems to be no, because, as we will see, a sequentially computable function can even fail to be “effectively continuous at some point”. Let us fix a computable real number $x_0$ and a function $F$ mapping all computable real numbers to computable real numbers. We call $F$ effectively continuous at $x_0$ if there is a total computable function $g$ mapping natural numbers to natural numbers such that for all $n$ and all computable real numbers $x$

$$|x - x_0| \leq 2^{-g(n)} \Rightarrow |F(x) - F(x_0)| \leq 2^{-n}.$$  

Clearly, every effectively continuous function is effectively continuous at every computable real number. Thus, the main result of [5,6] (the existence of a sequentially computable function that is not effectively continuous) would already be strengthened if one could construct a sequentially computable function that is not effectively continuous at some point. In this paper, we show that there exists an even more pathological function.

**Theorem.** There exists a sequentially computable function mapping all computable real numbers to computable real numbers that is not effectively continuous at any computable real number.

It is interesting to note that nevertheless any sequentially computable function $F$ mapping all computable real numbers to computable real numbers enjoys the following property: if a computable sequence $(x_n)_n$ of real numbers converges computably to some (automatically computable) real number $x_\infty$ (that means that there exists a total computable function $g$ mapping natural numbers to natural numbers such that for all $n$ and all $m \geq g(n)$, $|x_m - x_\infty| \leq 2^{-n}$) then the image sequence $(F(x_n))_n$ converges computably to $F(x_\infty)$; see Hertling [4].

It is also interesting to note that there exists a function mapping all computable real numbers to computable real numbers that is effectively continuous at every computable real number and sequentially computable but not effectively continuous. Indeed, the function constructed in [5,6]...
has these properties. It is constructed so that it is sequentially computable but not effectively continuous. Furthermore, it is a linear spline function which is zero everywhere except that for some numbers \( i \in \mathbb{N} \), the interval \([i + \frac{1}{3}, i + \frac{2}{3}]\) contains a rational subinterval on which the function has a triangular shape with value 2 at the top. It is clear that such a function is effectively continuous at every computable real number.

In the following section we will introduce the notions precisely that are needed for formulating the theorem above and state it again. Then, in preparation of the proof, we introduce some more technical notions. Finally, we give the proof.

The proof uses some of the ideas in the proof of the main result of [5,6], but also differs in several points. In both proofs a function \( F \) with the desired properties is constructed by adding rational peak functions with compact support to the constant zero function. There, all added peaks had the same height and pairwise disjoint support. Here, the height of the peaks tends to zero and the supports of the peaks do not need to be disjoint. There, for each computable sequence of real numbers \((x_k)_k\), one could obtain an algorithm for computing the sequence \((F(x_k))_k\) by essentially simply following the construction. Here, for each computable sequence \((x_k)_k\) we explicitly construct a computably enumerable set which gives an algorithm for computing \((F(x_k))_k\). Finally, there no priority was needed, while here we use a finite injury priority argument.

2. Basic notions and the result

By \( \mathbb{N} \) we denote the set of natural numbers, i.e., nonnegative integers, and by \( \mathbb{Q} \) the set of rational numbers. By \( f : X \rightarrow Y \) we denote a total function from a set \( X \) to a set \( Y \). A sequence \( x_0, x_1, x_2, \ldots \) over a set \( X \) is nothing but a function \( x : \mathbb{N} \rightarrow X \) and will often be denoted by \((x_n)_n\) or \((x_i)_i\), etc. We use the standard bijection \( \langle \cdot, \cdot \rangle : \mathbb{N}^2 \rightarrow \mathbb{N} \) defined by \( \langle i, j \rangle = (i + j)(i + j + 1)/2 + j \), for all \( i, j \in \mathbb{N} \). Inductively, we define bijections between \( \mathbb{N}^{k+1} \) and \( \mathbb{N} \) by \( \langle i_1, \ldots, i_k, i_{k+1} \rangle := \langle \langle i_1, \ldots, i_k \rangle, i_{k+1} \rangle \), for \( k \geq 2 \). We use the notion of a possibly partial computable function \( f \) mapping natural numbers to natural numbers in the usual sense (of recursion theory; compare Soare [12]).

We define a total numbering \( v_{\mathbb{Q}} : \mathbb{N} \rightarrow \mathbb{Q} \) by \( v_{\mathbb{Q}}(\langle i, j, k \rangle) := (i - j)/(k + 1) \). A sequence \((q_n)_n\) of rational numbers is computable if there exists a computable function \( f : \mathbb{N} \rightarrow \mathbb{N} \) with \( q_n = v_{\mathbb{Q}}(f(n)) \) for all \( n \). A real number \( x \) is computable if there exists a computable sequence \((q_k)_k\) of rational numbers with \( |x - q_k| \leq 2^{-k} \) for all \( k \). Let \( \mathbb{R}_c \) be the set of all computable real numbers. A sequence \((x_n)_n\) of real numbers is called computable if there exists a computable sequence \((q_i)_i\) of rational numbers with \( |x_n - q_{(n,k)}| \leq 2^{-k} \) for all \( n \) and all \( k \). Note that every member of a computable sequence of real numbers is a computable real number. A function \( F : \mathbb{R}_c \rightarrow \mathbb{R}_c \) is sequentially computable if for every computable sequence \((x_n)_n\) of real numbers also the sequence \((F(x_n))_n\) is a computable sequence of real numbers. A function \( F : \mathbb{R}_c \rightarrow \mathbb{R}_c \) is effectively continuous at a point \( x_0 \in \mathbb{R}_c \) if there exists a computable function \( g : \mathbb{N} \rightarrow \mathbb{N} \) such that

\[
(\forall n)(\forall x \in \mathbb{R}_c) \ (|x - x_0| \leq 2^{-g(n)} \Rightarrow |F(x) - F(x_0)| \leq 2^{-n}).
\]

Now all notions needed for the formulation of the result of this paper are defined. We restate the result.

Theorem. There exists a sequentially computable function \( F : \mathbb{R}_c \rightarrow \mathbb{R}_c \) such that for any \( x_0 \in \mathbb{R}_c \) the function \( F \) is not effectively continuous at \( x_0 \).
3. Some more notions and preparations for the proof

For two sets $X$ and $Y$, by $f : \subseteq X \rightarrow Y$ we denote a possibly partial function whose domain of definition is a subset of $X$, and whose range is a subset of $Y$. We denote the domain of definition of $f$ by $\text{dom } f$ and the range of $f$ by $\text{range } f$. If $\text{dom } f = X$, we call the function $f$ total and may indicate this by writing $f : X \rightarrow Y$ instead of $f : \subseteq X \rightarrow Y$. For a function $f : \subseteq X \rightarrow Y$ and a point $x \in X$, $f(x) \downarrow$ means that $x \in \text{dom } f$, and $f(x) \uparrow$ means that $x \notin \text{dom } f$. For a function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ and a number $n \in \mathbb{N}$, by $f \upharpoonright n$ we mean the restriction of $f$ to the set $\{0, \ldots, n-1\} \cap \text{dom } f$, and $f \downharpoonright n$ means that $\{0, \ldots, n-1\} \subseteq \text{dom } f$.

For $x \in \mathbb{R}_c$ and $\varepsilon > 0$, set $B(x, \varepsilon) := \{y \in \mathbb{R}_c \mid |x-y| < \varepsilon\}$ and let $\overline{B}(x, \varepsilon) := \{y \in \mathbb{R}_c \mid |x-y| \leq \varepsilon\}$ be its closure in $\mathbb{R}_c$. For any subset $A \subseteq \mathbb{R}_c$, by interior($A$) we mean its interior in $\mathbb{R}_c$. The distance between two real numbers $x$ and $y$ is written either $|x-y|$ or $d(x, y)$. For $x \in \mathbb{R}$ and a subset $A \subseteq \mathbb{R}$, we define $d(x, A) := \inf_{y \in A} d(x, y)$.

Let $F : \mathbb{R}_c \rightarrow \mathbb{R}_c$ be some total function, let $x_0 \in \mathbb{R}_c$ be a computable real number, and let $g : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ be some possibly partial function. We say that $g$ is a modulus of continuity for $F$ at $x_0$ if $g$ is total and for all $n$, $F(\overline{B}(x_0, 2^{-g(n)})) \subseteq \overline{B}(F(x_0), 2^{-n})$. Thus, $F$ is effectively continuous at some point $x_0 \in \mathbb{R}_c$ if, and only if, there exists a computable modulus of continuity for $F$ at $x_0$.

A rational interval is an interval in $\mathbb{R}_c$ with rational endpoints. For rational $a, b, \delta$ with $a < b$ and $\delta > 0$ we define a function peak $([a, b], \delta) : \mathbb{R}_c \rightarrow \mathbb{R}_c$ by

$$\text{peak}([a, b], \delta)(x) := \begin{cases} 0 & \text{if } x \leq a \text{ or } x \geq b, \\ \delta \cdot \frac{x-a}{a+b} - \frac{a}{2} & \text{if } a \leq x \leq \frac{a+b}{2}, \\ \delta \cdot \frac{b-x}{b-a+b} - \frac{a+b}{2} & \text{if } \frac{a+b}{2} \leq x \leq b. \end{cases}$$

Such a function is called a (rational) peak function or simply a (rational) peak. Then $\delta$ is called the height of the peak, $(a+b)/2$ is the midpoint of the peak, and $[a, b]$ is the support of the peak function.

In the following we will often state that, given some data, one can compute some rational number or some object described by rational numbers. When we write something like that, we always mean that, given the data, one can compute $v_Q$-indices for these rational numbers.

For an integer $k \geq 1$, we denote by $P^{(k)}$ the set of all computable—in the sense of recursion theory; compare Soare [12]—functions $f : \subseteq \mathbb{N}^k \rightarrow \mathbb{N}$. Note that a function $f : \subseteq \mathbb{N}^k \rightarrow \mathbb{N}$ is computable if, and only if, the function $g : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ defined by $g((x_1, \ldots, x_k)) := f(x_1, \ldots, x_k)$ is computable. As usual, a set $A \subseteq \mathbb{N}^k$ is called computably enumerable if, and only if, there is a computable function $f : \subseteq \mathbb{N}^k \rightarrow \mathbb{N}$ with $\text{dom } f = A$. It is decidable if its characteristic function $\chi_A : \mathbb{N} \rightarrow \mathbb{N}$ (defined by $\chi_A(n) := 1$ if $n \in A$, and $\chi_A(n) := 0$ if $n \notin A$) is computable.

We fix a total standard numbering $\varphi$ of all computable natural number functions, i.e., a total surjective function $\varphi : \mathbb{N} \rightarrow P^{(1)}$ satisfying the following two conditions: (1) (universality) the function $\mu : \subseteq \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by $\mu(e, j) := \varphi(e)(j)$, for all $e, j \in \mathbb{N}$, is computable; (2) (smn-property) for any computable function $f : \subseteq \mathbb{N}^2 \rightarrow \mathbb{N}$ there exists a total computable function $r : \mathbb{N} \rightarrow \mathbb{N}$ with $f(e, j) = \varphi(r(e))(j)$, for all $e, j \in \mathbb{N$. Often we write $\varphi_e$ instead of
\( \varphi(e) \), and \( \varphi_e(j) \) instead of \( \varphi(e)(j) \). We wish to evaluate functions \( \varphi_e \) in stages. In order to do this, note that due to the properties of \( \varphi \), there exists a total computable function \( h : \mathbb{N} \to \mathbb{N} \) with range \( h = \{(e, x) \mid x \in \text{dom}(\varphi_e)\} \). Let \( \varphi_{e,s} \) be the restriction of \( \varphi_e \) to the set \( \{x \in \mathbb{N} \mid (e, x) \in \{h(0), \ldots, h(s - 1)\}\} \). It is clear that the set \( \{(x, e, s) \in \mathbb{N}^3 \mid x \in \text{dom} \varphi_{e,s}\} \) is decidable.

We say that a function \( g : \mathbb{N} \to \mathbb{N} \) describes a real number \( x \) if \( g \) is a total function and the sequence \( (q_n)_n \) defined by \( q_n := vQg(n) \) satisfies \( |q_n - q_m| \leq 2^{-\min[n,m]} \) for all \( n, m \) and \( \lim_{n \to \infty} q_n = x \). Note that if some \( \varphi_e \) describes a real number then this real number is computable. For any \( e, s \), we define \( I_{e,s} \) as follows. Let \( l \) be the largest number such that \( \varphi_{e,s}(l) \downarrow \) and, for all \( n, m < l, |vQ\varphi_e(n) - vQ\varphi_e(m)| \leq 2^{-\min[n,m]} \). If \( l > 0 \) then

\[
I_{e,s} := \bigcap_{n < l} \overline{B}(vQ\varphi_e(n), 2^{-n}).
\]

Otherwise, \( I_{e,s} \uparrow \).

**Lemma 1.** If \( I_{e,s} \) is defined, then it is a nonempty, closed interval with rational endpoints, with \( vQ\varphi_e(l - 1) \in I_{e,s} \), and with \( 2^{-l-1} \leq \text{length}(I_{e,s}) \leq 2 \cdot 2^{-l-1} \).

**Proof.** The sets \( \overline{B}(vQ\varphi_e(n), 2^{-n}) \) are closed intervals, and the intersection of such intervals is again a closed rational interval. The claim \( vQ\varphi_e(l - 1) \in I_{e,s} \) is clear. The estimate \( \text{length}(I_{e,s}) \leq 2 \cdot 2^{-l-1} \) is clear from \( I_{e,s} \subseteq \overline{B}(vQ\varphi_e(l - 1), 2^{-l-1}) \). The other estimate, \( 2^{-l-1} \leq \text{length}(\bigcap_{n < l} \overline{B}(vQ\varphi_e(n), 2^{-n})) \), follows by induction over \( l \), taking \( vQ\varphi_e(l - 1) \in \bigcap_{n < l} \overline{B}(vQ\varphi_e(n), 2^{-n}) \) into account. \( \square \)

If \( I_{e,s} \) is defined, then also \( I_{e,t} \) is defined for any \( t \geq s \). The sets \( I_{e,s} \) are nonincreasing for increasing \( s \), i.e., \( I_{e,s} \supseteq I_{e,s+1} \). Furthermore, given \( e \) and \( s \) one can decide whether \( I_{e,s} \uparrow \) or not, and, if not, one can compute it. If \( \varphi_e \) does not describe a real number \( x \), then for sufficiently large \( s \), the number \( l \) and the interval \( I_{e,s} \) will not change anymore. If \( \varphi_e \) describes a real number \( x \), then there exists a smallest number \( S \) such that \( I_{e,s} \downarrow \). Then, for all \( s \geq S, I_{e,s} \downarrow, x \in I_{e,s}, I_{e,s} \supseteq I_{e,s+1}, \) and, finally, \( \lim_{s \to \infty} \text{length}(I_{e,s}) = 0 \). For every \( e, n, s \) with \( e = (e_1, e_2), \) with \( \varphi_{e_2}(n) \downarrow, \) with \( I_{e_1,s} \downarrow, \) and with length \( (I_{e_1,s}) < 2^{-\varphi_{e_2}(n)} \), we define \( [a_{e_1,s}, b_{e_1,s}] := I_{e_1,s} \) and

\[
K_{e,n,s} := \left[ b_{e_1,s} - 2^{-\varphi_{e_2}(n)}, a_{e_1,s} + 2^{-\varphi_{e_2}(n)} \right] \setminus \text{interior}(I_{e_1,s}) = \left[ b_{e_1,s} - 2^{-\varphi_{e_2}(n)}, a_{e_1,s} \right] \cup \left[ b_{e_1,s}, a_{e_1,s} + 2^{-\varphi_{e_2}(n)} \right].
\]

This set is the union of two closed rational intervals, both of the same positive length. It contains exactly those \( x \in \mathbb{R} \setminus \text{interior}(I_{e_1,s}) \) with \( \max_{y \in I_{e_1,s}} |x - y| \leq 2^{-\varphi_{e_2}(n)} \). If \( K_{e,n,s} \) is defined, then also \( K_{e,n,t} \) is defined for any \( t \geq s \). The sets \( K_{e,n,s} \) are nondecreasing for increasing \( s \), i.e., \( K_{e,n,s} \subseteq K_{e,n,s+1} \).

Similarly, we say that a function \( g : \mathbb{N} \to \mathbb{N} \) describes a sequence \( (x_k)_k \) of real numbers if \( g \) is a total function and for each \( k \) the function \( j \mapsto g(k, j) \) describes \( x_k \). Note that if some \( \varphi_e \) describes a sequence of real numbers then this sequence is a computable sequence of real numbers. For any \( e, s \), let \( I_{e,s} \) be the largest number such that \( \varphi_{e,s}(k, n) \downarrow \) for all \( k < l_{e,s} \) and
all \( n < l_{e,s} \) and such that for all \( k, n, m < l_{e,s} \), \(|v_Q \varphi_e(k, n) - v_Q \varphi_e(k, m)| \leq 2^{-\min\{n, m\}}\). For any \( e, k, s \), we define \( I_{e,k,s} \) as follows. If \( l_{e,s} > k \) then

\[
I_{e,k,s} := \bigcap_{n < l_{e,s}} \overline{B}(v_Q \varphi_e(k, n), 2^{-n}).
\]

Otherwise, \( I_{e,k,s} \uparrow \). Thus, if \( I_{e,k,s} \) is defined, then it is a nonempty, closed interval with rational endpoints and \( 2^{-(l_{e,s}-1)} \leq \text{length}(I_{e,k,s}) \leq 2 \cdot 2^{-(l_{e,s}-1)} \) (proof as the proof of the lemma above).

Furthermore, given \( e, k, s \), one can decide whether \( I_{e,k,s} \uparrow \) or not, and, if not, one can compute it. If \( \varphi_e \) does not describe a sequence \((x_k)_k\) of real numbers, then for sufficiently large \( s \), the number \( l_{e,s} \), and the intervals \( I_{e,k,s} \) will not change anymore. If \( \varphi_e \) describes a sequence \((x_k)_k\) of real numbers, then \((I_{e,s})_s\) is a nondecreasing sequence with \( \lim_{s \to \infty} l_{e,s} = \infty \). Hence, then for each \( k \) there exists a smallest number \( S_k \) such that \( I_{e,k,S_k} \downarrow \). Then, for each \( k \) and for all \( s \geq S_k \), \( I_{e,k,s} \downarrow \), \( x_k \in I_{e,k,s} \), \( I_{e,k,s} \supseteq I_{e,k,s+1} \), and, finally, \( \lim_{s \to \infty} \sum_{j < l_{e,s}} \text{length}(I_{e,j,s}) = 0 \).

4. The proof

In this section we prove the result.

4.1. Strategy

It is our goal to construct a sequentially computable function \( F : \mathbb{R} \to \mathbb{R} \) that is not effectively continuous at any point. Thus, we have to take care that \( F \) satisfies the following “negative” and “positive” requirements for all \( e \in \mathbb{N} \):

Requirement \( \mathcal{N}_e \): If \( e = (e_1, e_2) \), if \( \varphi_{e_1} \) describes a real number \( x_{e_1} \), and if \( \varphi_{e_2} \) is total, then \( \varphi_{e_2} \) must not be a modulus of continuity of \( F \) at \( x_{e_1} \), i.e., it is not allowed that for all \( n \in \mathbb{N} \), \( F(\overline{B}(x_{e_1}, 2^{-\varphi_{e_2}(n)})) \subseteq \overline{B}(F(x_{e_1}), 2^{-n}) \).

Requirement \( \mathcal{P}_e \): If \( \varphi_e \) describes a sequence \((x_{e,n})_n \) of real numbers, then the sequence \((F(x_{e,n}))_n \) is computable as well.

The desired function \( F \) will be constructed in stages. During the stages we will take care of more and more of the above requirements. At the end of each stage \( t \) we will have a candidate \( F_t : \mathbb{R} \to \mathbb{R} \) for \( F \). During each stage \( t \) we may add one rational peak to the previous candidate. This ensures that for every \( t \), the set of all \( x \) with \( F_t(x) \neq 0 \) is bounded and that \( F_t \) is piecewise linear with finitely many breakpoints and such that both coordinates of any breakpoint are rational. Especially, each \( F_t \) is continuous. We will also make sure that the height of any added peak is of the form \( 2^{-m} \) for some \( m \in \mathbb{N} \), and that any such number can be the height of at most one added peak. This implies that the sequence \((F_t)_t \) converges uniformly to a function \( F \), which, therefore, must be continuous. This function \( F \) will have the desired properties.

The basic strategy for satisfying a negative requirement \( \mathcal{N}_e \) is to choose a number \( n \in \mathbb{N} \) such that \( \varphi_{e_2} \uparrow (n+1) \), to wait until the interval \( I_{e_1,t} \) (representing our present knowledge of any real number \( x_{e_1} \) that might be described by \( \varphi_{e_1} \)) is sufficiently small, and to add a rational peak of height either zero or \( 4 \cdot 2^{-n} \) to \( F_{t-1} \) such that the midpoint \( q \) of the peak has the two properties \( \max_{x \in I_{e_1,t}} |q - x| \leq 2^{-\varphi_{e_2}(n)} \) and \( \min_{x \in I_{e_1,t}} |F_t(q) - F_t(x)| \geq \frac{3}{2} \cdot 2^{-n} \). If we succeed to do that, we say that the number \( n \) is helping us to satisfy \( \mathcal{N}_e \).

The basic strategy for satisfying \( \mathcal{P}_e \) is to enumerate a set \( C_e \) of pairs \((k, J)\) with \( k \in \mathbb{N} \) and with \( J \) being a closed rational interval such that \( F(x_{e,k}) \in J \) and such that for each \( k \)
and each \( \varepsilon > 0 \), the set \( C_\varepsilon \) will contain a pair \((k, J)\) with \( \text{length}(J) \leq \varepsilon \). Thus, using \( C_\varepsilon \) one can compute \( F(x_{e,k}) \) with any desired precision, uniformly in \( k \). That means the sequence \( (F(x_{e,k}))_k \) is computable.

Since satisfying the negative requirements might force us to add some unexpectedly high peak quite late, while our strategy for satisfying the positive requirements asks us to fix the value of \( f \) up to some precision already in certain points (and, thus, preliminarily in certain intervals, since after finitely many steps, usually we know these points only with some finite precision), it is clear that these two strategies are in conflict with each other. Therefore, we use a priority argument where we order the requirements as follows (from left to right the priority decreases):

\[ N_0, P_0, N_1, P_1, N_2, P_2, \ldots. \]

We describe in an informal way how the construction steps for requirements of different priority interact with each other. We already explained that the basic strategy for fulfilling a negative requirement \( N_e \) is to add a peak to the current approximation of the function. When we do that, we do not care about lower priority requirements, positive or negative, and simply initialize them. But we have to be careful not to violate higher priority requirements. For negative higher priority requirements \( N_d \) we can ensure this by adding only small peaks after having taken care of \( N_d \). This works, since we took care that the difference \( \min_{x \in I_{d,t}} |F_t(x) - F_t(x)| \geq \frac{3}{2} \cdot 2^{-n} \) was so large that later addition of peaks whose heights sum up to some number \( \leq \frac{1}{4} \cdot 2^{-n} \) still guarantees \( |F(q) - F(x_d)| \geq \frac{5}{2} \cdot 2^{-n} \). For positive higher priority requirements \( P_d \) things are not so easy since we try all the time to improve the set \( C_d \) and to add more pairs \((k, J)\) with \( J \)'s of decreasing height. Once the process of computing \( \varphi_{e_1} \) and \( \varphi_{e_2} \) would finally allow us to add a peak of a certain height \( 4 \cdot 2^{-n} \) in a certain area to the function in order to satisfy \( N_e \), it can happen that some pairs \((k, J)\) in \( P_d \) for some \( d < e \) do not allow us to add this peak. This can happen if the current intervals \( I_{d,k,t} \) (the interval \( I_{d,k,t} \) represents our present knowledge about the number \( x_k \) if \( \varphi_d \) describes a sequence \((x_i)\) of real numbers) cover the area in which we would like to add the peak and if the corresponding intervals \( J \) have already too small height. In this case the idea is to try a new, much larger candidate for helping us to satisfy \( N_e \), i.e., a new, much larger number \( n \) so that the height \( 4 \cdot 2^{-n} \) of the new peak to be added would be much smaller. We will explain below why this works. In order to satisfy a positive requirement \( P_d \), where \((x_k)_k\) is a sequence of real numbers described by \( \varphi_d \), we try to add more and more pairs \((k, J)\) to \( C_d \) with \( J = \left[ \min F_t(I_{d,k,t}), \max F_t(I_{d,k,t}) + 2^{-t} \right] \). Note the buffer \( 2^{-t} \). Its purpose is to allow us to add later small peaks in order to satisfy negative requirements of lower priority. We will add pairs \((k, J)\) to \( C_d \) only at stages \( t \) when the number \( I_{d,t} \) is larger than \( I_{d,t-1} \), i.e., when the quality of the description of \((x_k)_k\) given by \( \varphi_d \) has increased. This has the effect that for irrelevant \( \varphi_d \)'s that do not describe any sequence of real numbers and for which therefore \( I_{d,t} \) does not change anymore for sufficiently large \( t \), the set \( C_{d,t} \) will also not change anymore for sufficiently large \( t \). Thus, such irrelevant \( \varphi_d \)'s will not impose ever new restrictions on our wish to add peaks to the function. Then, for any negative requirement \( N_e \), if a sufficiently large candidate \( n \) is considered, the wish to add a peak of height \( 4 \cdot 2^{-n} \) will not be in conflict anymore with any irrelevant \( P_d \) of higher priority. And the conflict with relevant higher priority requirements \( P_d \), i.e., with \( \varphi_d \)'s that describe a sequence of real numbers, will be resolved eventually since the area blocked by such a requirement becomes smaller and smaller, while the area in which we can add a peak of height \( 4 \cdot 2^{-n} \) for this specific candidate \( n \) will not shrink. Thus, eventually, we can add a peak as desired and satisfy \( N_e \). In fact, for each negative
requirement $\mathcal{N}_e$ we will have a finite, strictly increasing list of candidates. At stage $t$, this list is empty if $e \geq t$ or if $\mathcal{N}_e$ has already been satisfied. Otherwise it is nonempty. Whenever a higher priority negative requirement initializes $\mathcal{N}_e$, the list for $\mathcal{N}_e$ is initialized to just one new candidate, larger than all candidates that have been used so far. If at some stage we try to take care of $\mathcal{N}_e$ but do not succeed (“none of the current candidates for helping us to satisfy $\mathcal{N}_e$ is helpful at this stage”) then we add a new, larger candidate to the list. The relevant positive requirements $\mathcal{P}_d$ will also be satisfied, since we add infinitely often pairs $(k, J)$ with the $J$’s in general decreasing in height.

In the construction of the function $F$ we will use a number of variables of various types. Usually, they have some indices, and the last index—often $t$ or $s$—will refer to the value of the variable at the end of stage $t$ (resp. $s$). We conclude this subsection by explaining some of these variables. The precise assignment of values to the variables at the different stages will be given in the description of the construction. First of all,

- $F_t : \mathbb{R}^c \rightarrow \mathbb{R}^c$ is the candidate and approximation for the function $F$ at the end of stage $t$.

The following variables are mostly related to the negative requirements.

- $\text{SATISFIED}_t \subseteq \mathbb{N}$ is the set of all indices $e$ of requirements $\mathcal{N}_e$ that have been satisfied already. But note whenever the algorithm has taken care of some $\mathcal{N}_d$, in the subsequent initialization of all lower priority requirements, all indices $e > d$ that are already in $\text{SATISFIED}$ are removed from $\text{SATISFIED}$ again.
- $m_{e,t} \in \mathbb{N}$ is the number of natural numbers that are the current candidates for helping us to satisfy requirement $\mathcal{N}_e$. Once $\mathcal{N}_e$ is satisfied, $m_{e,t}$ will be set to zero.
- And $n_{e,0,t}, \ldots, n_{e,m_{e,t}-1,t} \in \mathbb{N}$ are these candidates. They will always form a strictly increasing sequence of multiples of 5.

The following variables are mostly related to the positive requirements.

- $C_{e,t} \subseteq \{(k, J) \mid k \in \mathbb{N}, J \text{ a closed, positive length interval with rational ends}\}$ is the current approximation to the final set $C_e$ as described above. It is a finite set. We will always try to add more elements to this set in order to ensure that, in case $\varphi_e$ describes a sequence $(x_{e,k})_k$ of real numbers, the sequence $(F(x_{e,k}))_k$ will be computable. Sometimes we may empty this set, but only finitely often for each $e$.

4.2. Construction

Now we give the formal description of the construction. As explained above, we proceed in stages.

Stage 0: In stage 0 we will only initialize some variables. The others will be computed at every stage without referring to their values at earlier stages. We define $F_0 : \mathbb{R}^c \rightarrow \mathbb{R}^c$ by $F_0(x) := 0$ for all $x$. We set $\text{SATISFIED}_0 := \emptyset$. We define $m_{0,0} := 1$ and $n_{0,0,0} := 0$. We set $C_{d,0} := \emptyset$ for all $d \in \mathbb{N}$.

Stage $t$ for $t > 0$: Stage $t$ consists of two parts, a negative part and a positive part. They will be executed after each other in this order. In the first, negative, part we will try to take care of at least one negative requirement $\mathcal{N}_e$ with $e < t$ and perhaps modify the current function $F_{t-1}$ by adding a peak. In the second, positive, part we will try to add elements to as many sets $C_{e,t-1}$ with $e \leq t$ as possible. Thus, we will try to take care of the positive requirements $\mathcal{P}_e$ with $e \leq t$.
4.2.1. Negative part

At first, we define a number $n'_t$ that we will need in the construction:

$$n'_t := \max\{5 \cdot t, 5 + \max\{n_{d,j,s} \mid d < t, s < t, j < m_{d,s}\}\}.$$ 

That means $n'_t$ is equal to $5 \cdot t$ or equal to $5 +$ the largest number that has so far been a candidate for helping us to satisfy a negative requirement. By induction, all these numbers are multiples of 5. Thus, also $n'_t$ is a multiple of 5. Furthermore, $n'_t \geq 5 \cdot t$.

In the following we write $e = (e_1, e_2)$. We say that a negative requirement $N_e$ needs action, if $e < t$, if $e \notin SATISFIED_{t-1}$, if $\varphi_{e_2,t}(n_e,m_{e,t-1-1},t-1) + 1 < 1$, if $I_{e_1,t} \downarrow$, if length($I_{e_1,t}$) < $2^{-\varphi_{e_2}(n_e,m_{e,t-1-1},t-1)}$, and if length($F_{t-1}(I_{e_1,t})$) $\leq 2^{-\varphi_{e_2}(n_e,m_{e,t-1-1},t-1)}$.

Case I: There is no negative requirement that needs action. In that case we simply set $F_t := F_{t-1}$, SATISFIED$_t := SATISFIED_{t-1}$.

$$m_{d,t} := \begin{cases} m_{d,t-1} & \text{if } d < t, \\ 1 & \text{if } d = t, \end{cases}$$

$$n_{d,j,t} := \begin{cases} n_{d,j,t-1} & \text{if } d < t \text{ and } j < m_{d,t-1}, \\ n'_t & \text{if } d = t \text{ and } j = 0. \end{cases}$$

That means that we keep all the already chosen candidates for helping us to satisfy the requirements $N_d$ for $d < t$, and we choose a first candidate for helping us to satisfy the requirement $N_e$. Then we continue with the positive part.

Case II: There is a negative requirement that needs action. Let $e$ be the smallest number such that $N_e$ needs action. Then we say that this negative requirement $N_e$ is active at stage $t$. We define $e_1$ and $e_2$ by $(e_1, e_2) := e$. There are now three possible subcases.

- (First subcase) Either the function $F_{t-1}$ already has the property that $\varphi_{e_2}$ cannot be a modulus of continuity of this function at any point $x_{e_1}$ that might be described by $\varphi_{e_1}$.
- (Second subcase) The function $F_{t-1}$ does not have this property. But by adding a rational peak of a certain height to $F_{t-1}$ we can obtain a function $F_t$ that has this property. In this case, we have to be cautious with adding a peak for two reasons: (I) if the function already satisfies a negative requirement of higher priority, this should still be valid after adding a peak; (II) the resulting function $F_t$ after adding a peak should respect the conditions defined by the sets $C_{d,t-1}$ for $d < e$.
- (Third subcase) Our algorithm for adding a peak would not produce a function $F_t$ with the desired property. In that case we give up trying to satisfy $N_e$ at this stage.

In the rest of the treatment of Case II, we will consider numbers $n$ in the set

$$N_{e,t} := \{n_{e,j,t-1} \mid j < m_{e,t-1}\}.$$ 

Note that, due to the fact that currently $N_e$ needs action, we have $I_{e_1,t} \downarrow$ and

$$\varphi_{e_2}(n) \downarrow \text{ and } \text{length}(I_{e_1,t}) < 2^{-\varphi_{e_2}(n)} \tag{1}$$

for $n = n_{e,m_{e,t-1-1},t-1}$. In fact, (1) is true for any $n \in N_{e,t}$, due to the fact that for every number $n \in N_{e,t}$ there is some $s \leq t$ such that $n = n_{e,m_{e,t-1-1},s-1}$ and $N_e$ was/is active at stage $s$ and, hence, needed/needs action at stage $s$. We will see that indeed a number $n$ can have entered the set $N_{e,t}$ only in such a situation. Hence, for any $n \in N_{e,t}$, the triple $e, n, t$ satisfies all the assumptions in the definition of $K_{e,n,t}$ in Section 3.
First we check whether there exists a number \( n \in N_{e,t} \) such that

\[
d \left( \max_{x \in K_{e,n,t}} F_{t-1}(x), F_{t-1}(I_{e_1,t}) \right) \geq \frac{3}{2} \cdot 2^{-n}
\]  

(2)

or

\[
d \left( \min_{x \in K_{e,n,t}} F_{t-1}(x), F_{t-1}(I_{e_1,t}) \right) \geq \frac{3}{2} \cdot 2^{-n}.
\]  

(3)

(First subcase) If that is the case then there exists a rational number \( y \in K_{e,n,t} \), hence with

\[
\max_{x \in I_{e_1,t}} |y - x| \leq 2^{-\varphi_2(n)}
\]

that satisfies \( d(F_{t-1}(y), F_{t-1}(I_{e_1,t})) \geq \frac{3}{2} \cdot 2^{-n} \), hence, \( \min_{x \in I_{e_1,t}} |F_{t-1}(y) - F_{t-1}(x)| \geq \frac{3}{2} \cdot 2^{-n} \). In this case, we set \( F_t := F_{t-1}, SATISFIED_t := \{e\} \cup (SATISFIED_{t-1} \cap \{0, \ldots, e-1\}) \),

\[
m_{d,t} := \begin{cases} 
m_{d,t-1} & \text{if } d < e, \\
0 & \text{if } d = e, \\
1 & \text{if } e < d \leq t,
\end{cases}
\]

and

\[
n_{d,j,t} := \begin{cases} 
n_{d,j,t-1} & \text{if } d < e \text{ and } j < m_{d,t-1}, \\
n' e + 5 \cdot (d - e - 1) & \text{if } e < d \leq t \text{ and } j = 0.
\end{cases}
\]

That means that we keep all candidates for satisfying the requirements \( N_d \) with \( d < e \), that we do not need any candidates for satisfying the requirement \( N_e \) anymore (because it is satisfied already at this moment), that for each \( d \) with \( e < d < t \) we throw away all the candidates that had been chosen already, and that for each \( d \) with \( e < d \leq t \) we choose one new candidate that is larger than all previous candidates. We remark that by induction all these candidates \( n_{d,j,t} \) are multiples of 5. And we keep in mind that in this case there is a rational number \( y \) with

\[
\max_{x \in I_{e_1,t}} |y - x| \leq 2^{-\varphi_2(n)} \quad \text{and} \quad \min_{x \in I_{e_1,t}} |F_t(y) - F_t(x)| \geq \frac{3}{2} \cdot 2^{-n}.
\]  

(4)

(Second subcase) If there does not exist a number \( n \in N_{e,t} \) satisfying (2) or (3), then we try to satisfy \( N_e \) by adding a peak of height \( 4 \cdot 2^{-n} \) to \( F_{t-1} \). But we must be careful with that. For \( d < e \) and \( n \in \mathbb{N} \), we define

\[
B_{d,n,t} := \bigcup_{(k,J) \in C_{d,t-1}} \{x \in I_{d,k,t} \mid F_{t-1}(x) + 2^{-t} + 4 \cdot 2^{-n} \notin \text{interior}(J) \}.
\]

This set contains the numbers that currently “are blocked” by \( P_d \) for negative lower priority requirements and possible peaks of height \( 4 \cdot 2^{-n} \). Since each \( C_{d,t-1} \) is a finite set (this is clear for \( t = 1 \), and for larger \( t \) it will be clear from the construction in the second part of any stage), each set \( B_{d,n,t} \) is a union of finitely many closed rational intervals. For each \( n \in N_{e,t} \), we define the set \( A_n \) of “allowed” points by

\[
A_n := \text{interior}(K_{e,n,t}) \cap \left( \mathbb{R}_c \backslash \bigcup_{d < e} B_{d,n,t} \right).
\]

The set \( A_n \) is either empty or a union of finitely many open, nonempty, rational intervals. We check whether for some \( n \in N_{e,t} \), the set \( A_n \) is nonempty. If that is the case then let \( n \in \mathbb{N} \).
be the smallest number with this property. Then \( A_n \) contains an open, nonempty, rational interval. We choose such an interval, call its closure \( L \), and define \( F_t := F_{t-1} + \text{peak}(L, 4 \cdot 2^{-n}) \), \( SATISFIED_t := \{ e \} \cup (SATISFIED_{t-1} \cap \{ 0, \ldots, e - 1 \}) \),

\[
m_{d,t} := \begin{cases} 
m_{d,t-1} & \text{if } d < e, \\
0 & \text{if } d = e, \\
1 & \text{if } e < d \leq t,
\end{cases}
\]

and

\[
n_{d,j,t} := \begin{cases} 
n_{d,j,t-1} & \text{if } d < e \text{ and } j < m_{d,t-1}, \\
n_t' + 5 \cdot (d - e - 1) & \text{if } e < d \leq t \text{ and } j = 0.
\end{cases}
\]

As in the first subcase, that means that we keep all candidates for satisfying the requirements \( N_d \) with \( d < e \), that we do not need any candidates for satisfying the requirement \( N_e \) anymore (because we are managing to satisfy it at this moment, as we will see now), that for each \( d \) with \( e < d < t \) we throw away all the candidates that had been chosen already, and that for each \( d \) with \( e < d \leq t \) we choose one new candidate that is larger than all previous candidates. Note that midpoint(\( L \)) \( \in K_{e,n,t} \). Hence, \( \max_{x \in I_{e,1,t}} |\text{midpoint}(L) - x| \leq 2^{-\varphi_{e2}(n)} \). On the other hand, (2) and (3) are not satisfied in this case, hence

\[
d(F_{t-1}(\text{midpoint}(L)), F_{t-1}(I_{e,1,t})) < \frac{3}{2} \cdot 2^{-n}.
\]

Together with \( \text{length}(F_{t-1}(I_{e,1,t})) \leq 2^{-n_{e,m_{e,t-1},t-1}} \leq 2^{-n} \) (it is also clear that the numbers \( n_{d,j,t-1} \) in \( N_{e,t} \) form an increasing finite sequence, for \( j = 0, \ldots, m_{e,t-1} - 1 \)) and \( F_t(\text{midpoint}(L)) = F_{t-1}(\text{midpoint}(L)) + 4 \cdot 2^{-n} \) we obtain

\[
d(F_t(\text{midpoint}(L)), F_{t-1}(I_{e,1,t})) > 4 \cdot 2^{-n} - 2^{-n} - \frac{3}{2} \cdot 2^{-n} = \frac{3}{2} \cdot 2^{-n},
\]

hence, due to \( F_t(x) = F_{t-1}(x) \) for all \( x \in I_{e,1,t} \),

\[
\min_{x \in I_{e,1,t}} |F_t(\text{midpoint}(L)) - F_t(x)| > \frac{3}{2} \cdot 2^{-n}.
\]

We summarize that also in this case there is a rational number \( y \) with (4), namely the number \( y = \text{midpoint}(L) \).

(Third subcase) If there does not exist a number \( n \in N_{e,t} \) with \( A_n \neq \emptyset \), then we give up trying to satisfy \( N_e \) at this stage. Instead, we set \( F_t := F_{t-1} \), \( SATISFIED_t := SATISFIED_{t-1} \cap \{ 0, \ldots, e - 1 \} \),

\[
m_{d,t} := \begin{cases} 
m_{d,t-1} & \text{if } d < e, \\
m_{d,t-1} + 1 & \text{if } d = e, \\
1 & \text{if } e < d \leq t
\end{cases}
\]

and

\[
n_{d,j,t} := \begin{cases} 
n_{d,j,t-1} & \text{if } d \leq e \text{ and } j < m_{d,t-1}, \\
n_t' & \text{if } d = e \text{ and } j = m_{d,t} - 1, \\
n_t' + 5 \cdot (d - e) & \text{if } e < d \leq t \text{ and } j = 0.
\end{cases}
\]

Note that in this case, we add \( n_t' \) as a new number to our list of candidates for helping us to satisfy \( N_e \). Note also that whenever some \( N_e \) is active, all negative lower priority requirements are initialized. In that case we will also initialize all positive lower priority requirements, as we will see now.
4.2.2. Positive part

For every \( d \leq t \) we define a set \( C'_{d,t} \) by

\[
C'_{d,t} := \begin{cases} 
C_{d,t-1} & \text{if in the negative part of stage } t \text{ (the current stage) we entered Case I (no negative requirement needed action)}, \\
C_{d,t-1} & \text{if in the negative part of stage } t \text{ we entered Case II,} \\
\emptyset & \text{if in the negative part of stage } t \text{ we entered Case II,} \\
& \text{if } N_e \text{ was the active negative requirement,} \\
& \text{and if } d < e, \\
& \emptyset & \text{if in the negative part of stage } t \text{ we entered Case II,} \\
& \text{if } N_e \text{ was the active negative requirement,} \\
& \text{and if } d \geq e.
\end{cases}
\]

The set \( C_{d,t} \) will be defined by adding pairs \((k, J)\) to \( C'_{d,t} \) where \( k \) is a number and \( J \) a closed, rational interval. Note that setting \( C'_{d,t} = \emptyset \) in the fourth case, i.e., if \( N_e \) was active and is of higher priority than \( P_d \), means that in this case we have to start from scratch our attempt to satisfy the positive requirements of lower priority than \( N_e \). For every \( d > t \) we set \( C_{d,t} := \emptyset \), and for every \( d \leq t \), we define the new set \( C_{d,t} \) by

\[
C_{d,t} := \begin{cases} 
C'_{d,t} & \text{if } l_{d,t} = l_{d,t-1}, \\
C'_{d,t} \cup \{(k, [\min F_t(Id,k,t), \max F_t(Id,k,t) + 2^{-t}] | k < l_{d,t}) \} & \text{if } l_{d,t} > l_{d,t-1}.
\end{cases}
\]

This ends the description of the construction.

4.3. Verification

In this subsection we show that the construction is correct.

At the end of each stage \( t \) we have an approximation \( F_t : \mathbb{R}_c \to \mathbb{R}_c \) for \( F \). During each stage \( t \) we either leave the previous approximation unchanged or we add one rational peak to it. This ensures that for every \( t \), the set of all \( x \) with \( F_t(x) \neq 0 \) is bounded and that \( F_t \) is piecewise linear with finitely many breakpoints and such that both coordinates of any breakpoint are rational. Thus, each \( F_t \) is continuous. The height of any added peak is of the form \( 4 \cdot 2^{-n} \) for some \( n \in \mathbb{N} \). Each \( n \) here is a number of the form \( ne,j,t \) for some \( e, j, t \) and, thus, a multiple of 5. Since we have taken care that all these numbers are pairwise different, for any \( m \in \mathbb{N} \) we add at most one peak of height \( 4 \cdot 2^{-5m} \) during the whole construction. This implies that the sequence \((F_t)_t\) converges uniformly to a function \( F \), which, therefore, must be continuous.

**Lemma 2.**

\[ (\forall t)(\forall d \leq t)(\forall (k, J) \in C_{d,t}) \left[ \min F_t(Id,k,t), \max F_t(Id,k,t) + 2^{-t} \right] \subseteq J. \]

**Proof.** This is clear by construction and by induction. Let us fix some \( t \) and some \( d \leq t \). If \( d = t \), then \( C'_{d,t} = C_{d,t-1} = \cdots = C_{d,0} = \emptyset \). Hence, then every \((k, J) \in C_{d,t}\) has entered \( C_d \) at stage \( t \). Then the assertion is clear by construction. Let us assume that \( d < t \), and let us fix some \((k, J) \in C_{d,t}\) that did not enter \( C_d \) at stage \( t \) but earlier. Then \((k, J) \in C'_{d,t} = C_{d,t-1}\). Then, by induction hypothesis

\[ \left[ \min F_{t-1}(Id,k,t-1), \max F_{t-1}(Id,k,t-1) + 2^{-(t-1)} \right] \subseteq J. \]
Remember that \( I_{d,k,t} \subseteq I_{d,k,t-1} \) and that \( F_{t-1}(x) \leq F_t(x) \) for all \( x \in \mathbb{R}_c \). Thus, \( \min F_{t-1}(I_{d,k,t-1}) \leq \min F_t(I_{d,k,t}) \). We need to show that also

\[
\max F_t(I_{d,k,t}) + 2^{-t} \in J.
\]

Either \( F_t \) is identical with \( F_{t-1} \) (then the assertion is clear) or a peak has been added by an active negative requirement in the negative part of stage \( t \). But, during stage \( t \) only a negative requirement \( \mathcal{N}_e \) with \( d < e \) can have been active. (Otherwise the positive requirement \( \mathcal{P}_d \) would have been initialized, i.e., the set \( C_d \) would have been emptied. Then \( (k,J) \) would have had to enter \( C_d \) again at stage \( t \).) An active requirement \( \mathcal{N}_e \) of lower priority than \( \mathcal{P}_d \), i.e., with \( d < e \), may add a peak of height \( 4 \cdot 2^{-n} \) to the function \( F_{t-1} \) only if the new function \( F_t \) still “respects” the pairs \( (k,J) \) already in \( C_{d,t-1} \); see the definition of the set \( A_n \) of “allowed” points in the negative part of the construction. That is just what we need. \( \square \)

**Lemma 3.** Assume that \( \mathcal{N}_e \) is active at some stage \( s \) and that no \( \mathcal{N}_d \) with \( d < e \) is active at any stage \( t > s \). Then, for all \( x \in \mathbb{R}_c \) and all \( t > s \),

\[
F_s(x) \leq F_{t-1}(x) < F_s(x) + 5 \cdot 2^{-5s}.
\]

**Proof.** The assertion \( F_s(x) \leq F_{t-1}(x) \) for \( t > s \) is clear since we only add peaks to the function. Since we assume that no higher priority negative requirement will be active after stage \( s \), only lower priority negative requirements can add peaks to \( F_s \). But, since \( \mathcal{N}_e \) is active at stage \( s \), all lower priority requirements are initialized at stage \( s \). Especially, all peaks that may be added later have height \( 4 \cdot 2^{-5r} \) for pairwise different integers \( r \) with \( r \geq s \). Hence, for all \( t > s \),

\[
F_{t-1}(x) \leq F_s(x) + \sum_{r=s}^{\infty} 4 \cdot 2^{-5r} < F_s(x) + 5 \cdot 2^{-5s}. \quad \square
\]

**Corollary 4.** Assume that \( \mathcal{N}_e \) is active at some stage \( s \geq 2 \) and that no \( \mathcal{N}_c \) with \( c < e \) is active at any stage \( t > s \). Assume further that \( d < e \) and \( l_{d,t} = l_{d,s} \) for all \( t > s \). Then \( B_{d,n,t} = \emptyset \) for all \( n \geq 5s \) and all \( t > s \).

**Proof.** Since no \( \mathcal{N}_c \) with \( c < e \) is active at any stage \( t > s \), no \( \mathcal{N}_c \) with \( c \leq d \) is active at any stage \( t > s \). Together with \( l_{d,t} = l_{d,s} \) for all \( t > s \) this implies \( C_{d,t} = C_{d,s}' = C_{d,s} \) for all \( t > s \). Hence, for any \( t > s \), for any \( (k,J) \in C_{d,t-1} = C_{d,s} \), and for any \( x \in I_{d,k,t} \subseteq I_{d,k,s} \), due to Lemma 2 we obtain

\[
[F_s(x), F_s(x) + 2^{-5}] \subseteq J.
\]

In the following we assume \( t > s \). If \( n \geq 5s \), using Lemma 3, we obtain for \( x \in \mathbb{R}_c \)

\[
F_s(x) \leq F_{t-1}(x) < F_{t-1}(x) + 2^{-t} + 4 \cdot 2^{-n} < F_s(x) + 2^{-t} + 9 \cdot 2^{-5s} < F_s(x) + 2^{-s}.
\]

The last estimate is due to \( s \geq 2 \). Thus, \( B_{d,n,t} = \emptyset \) for all \( t > s \). \( \square \)

We define four sets as follows:

\[
RELEVANT-NEG-REQ := \{ e \in \mathbb{N} \mid \varphi_{e_1} \text{ describes a real number } x_{e_1} \text{ and } \varphi_{e_2} \text{ is total, where } \langle e_1, e_2 \rangle := e \},
\]

\[
\mathcal{P}_d := \{ e \in \mathbb{N} \mid x_{e_1} \text{ and } \varphi_{e_2} \text{ are defined and } \varphi_{e_2} \text{ is total, where } \langle e_1, e_2 \rangle := e \},
\]

\[
\mathcal{N}_e := \{ e \in \mathbb{N} \mid x_{e_1} \text{ and } \varphi_{e_2} \text{ are defined, where } \langle e_1, e_2 \rangle := e \},
\]

\[
\mathcal{N}_c := \{ e \in \mathbb{N} \mid x_{e_1} \text{ and } \varphi_{e_2} \text{ are defined, where } \langle e_1, e_2 \rangle := e \}.
\]
IRRELEVANT-NEG-REQ := \( \mathbb{N} \setminus \text{RELEVANT-NEG-REQ} \),
RELEVANT-POS-REQ := \{ d \in \mathbb{N} \mid \varphi_d \) describes a sequence of real numbers\},

IRRELEVANT-POS-REQ := \( \mathbb{N} \setminus \text{RELEVANT-POS-REQ} \).

RELEVANT-NEG-REQ is the set of “relevant negative requirements”, and so on.

**Lemma 5.** (1) Each negative requirement is active only finitely often.

(2) For each negative requirement \( \mathcal{N}_e \) with \( e \in \text{RELEVANT-NEG-REQ} \) and all sufficiently large \( t \), we have \( e \in \text{SATISFIED}_t \).

**Proof.** Let us fix some \( e \). We wish to show both claims for the negative requirement \( \mathcal{N}_e \). By induction we can assume that each negative requirement \( \mathcal{N}_d \) with \( d < e \) is active only finitely often. Let \( S > 0 \) be a number such that no \( \mathcal{N}_d \) with \( d < e \) is active at any stage \( t \geq S \) and such that \( l_{d,t} = l_{d,S} \) for all \( t \geq S \) and all \( d \in \text{IRRELEVANT-NEG-REQ} \cap \{0, \ldots, e-1\} \).

If there is some \( s > S \) such that \( e \in \text{SATISFIED}_{s-1} \), then \( e \) will stay in this set forever, i.e., \( e \in \text{SATISFIED}_s \) for all \( t \geq s \), since no negative higher priority requirement will ever be active again. Furthermore, then \( \mathcal{N}_e \) will never need action again and, thus, will never be active again, and the lemma is proved. So, in the rest of the proof, we assume that \( e \notin \text{SATISFIED}_{s-1} \) for all \( s > S \).

For the sake of a contradiction, let us assume that \( \mathcal{N}_e \) is active infinitely often. Let \( s > S \) be a stage at which \( \mathcal{N}_e \) is active. Since we assume that \( e \) is not added to the set \( \text{SATISFIED}_{s-1} \), the number \( m_{e,s-1} \) is increased to \( m_{e,s} = m_{e,s-1} + 1 \) and a new number \( n'' := n_{e,m_{e,s-1},s} \) is added to the list of numbers \( n_{e,j,s-1} \) for \( j = 0, \ldots, m_{e,s-1} - 1 \). This number \( n'' \) satisfies \( n'' > S \). Note that \( s > 2 \) because of \( S > 0 \). Hence, Corollary 4 tells us that for all \( d \in \text{IRRELEVANT-POS-REQ} \cap \{0, \ldots, e-1\} \) and all \( t > s \). That means that no point is blocked by any irrelevant positive requirement \( \mathcal{P}_d \) with \( d < e \) for \( \mathcal{N}_e \) and for peaks of height \( 4 \cdot 2^{-n''} \) at any stage \( t > s \). For the following, remember that \( \mathcal{K}_{e,n'',t} \) has positive measure and cannot decrease with increasing \( t \). Since for each \( d \in \text{RELEVANT-POS-REQ} \cap \{0, \ldots, e-1\} \) the measure of \( B_{d,n'',s} \) tends to zero for \( s \) tending to infinity (for a proof note that \( B_{d,n'',s} \subseteq \bigcup_{k < l_{d,t}} I_{d,k,t} \) and that even the sum \( \sum_{k < l_{d,t}} \) length\( (I_{d,k,t}) \) tends to zero for \( t \) tending to infinity) the set \( \mathcal{K}_{e,n'',t} \) cannot be blocked forever by positive higher priority requirements. Thus, at some stage \( t > s \), the set \( A_{n''} \) will be nonempty and \( e \) will enter the set \( \text{SATISFIED}_{t-1} \) and be an element of \( \text{SATISFIED}_{t} \). Contradiction. This proves the first statement of the lemma: each negative requirement is active only finitely often.

Finally, we consider the case \( e \in \text{RELEVANT-NEG-REQ} \). Let \( s \geq S \) be a stage such that \( \mathcal{N}_e \) is not active at any stage \( t > s \). Since negative requirements of higher priority than \( \mathcal{N}_e \) are not active at any stage \( t > S \) anyway, this means that \( \mathcal{N}_e \) does not need action at any stage \( t > s \). Remember that we assume that \( e \notin \text{SATISFIED}_{t-1} \) for all \( t > s \). We will show that this assumption leads to a contradiction. For all \( t \geq s \), we have \( n_{e,m_{e,t-1},t-1} = n_{e,m_{e,t-1},t-1-1} \). Remember that \( e \in \text{RELEVANT-NEG-REQ} \) means that for \( \langle e_1, e_2 \rangle = e \) the function \( \varphi_{e_2} \) is total and the function \( \varphi_{e_1} \) describes a real number \( x_{e_1} \). This implies that for sufficiently large \( t \geq s \), we have \( e < t, \varphi_{e_2,t} (n_{e,m_{e,t-1},t-1-1} + 1) \downarrow, I_{e_1,t} \downarrow \), and length\( (I_{e_1,t}) < 2^{-\varphi_{e_2}(n_{e,m_{e,t-1-1},t-1})} \).

Since the functions \( F_{t-1} \) are continuous and converge uniformly to \( F \), and the intervals \( I_{e_1,t} \) are nonincreasing and converge to the real number \( x_{e_1} \), for sufficiently large \( t \) we also have length\( (F_{t-1}(I_{e_1,t})) \leq 2^{-n_{e,m_{e,t-1-1},t-1}} \). Thus, \( \mathcal{N}_e \) will need action for sufficiently large \( t > s \). Contradiction. □
Lemma 6. Each negative requirement $N_e$ is satisfied.

Proof. Nothing needs to be shown for $e \in IRRELEVANT-NEG-REQ$. Let us fix some $e \in RELEVANT-NEG-REQ$. According to the last lemma, there is a first stage $t$ such that $e \in SATISFIED_s$ for all $s \geq t$. Then $N_e$ is active at stage $t$, and no negative higher priority requirement is active at any stage $s \geq t$. Let $x_{e_1} \in I_{e_1}$ be the point described by $\phi_{e_1}$. Remember that $x_{e_1} \in I_{e_1}$ for all $s$ with $I_{e_1,s} \downarrow$, and $I_{e_1,s} \downarrow$ is true for $s \geq t$ because $N_e$ is active at stage $t$.

Since $N_e$ is active at stage $t$, we have $e \notin SATISFIED_{t-1}$. Remember that the two different conditions leading to $e \in SATISFIED_t$ in Case II of the negative part of Stage $t$ both imply that there exist an $n \in N_{e,t}$ and a rational number $y$ satisfying (4), i.e., satisfying

$$\max_{x \in I_{e_1,t}} |y - x| \leq 2^{-\phi_{e_2}(n)} \quad \text{and} \quad d(F_t(y), F_t(I_{e_1,t})) \geq \frac{3}{2} \cdot 2^{-n}.$$

Furthermore, at stage $t$, we have initialized all negative requirements $N_f$ with $f > e$. Especially, all numbers $n_{f,j,s}$ for any $f > e$ are pairwise different, are multiples of 5, and are at least as large as $5 + n$. In fact, all this is true even for all numbers $n_{f,j,s}$ for any $f > e$ and any $s \geq t$. Since during each of the stages $t + 1, t + 2, \ldots$, we may add at most one peak to the function $F_t$, since these peaks are of height $4 \cdot 2^{-5r}$ for pairwise different integers $r$ with $5r \geq 5 + n$, and since $I_{e_1,s} \subseteq I_{e_1,t}$ for $s > t$, we see that for any $s > t$,

$$d(F_s(y), F_s(I_{e_1,s})) \geq d(F_t(y), F_t(I_{e_1,t})) - \sum_{r=(5+n)/5}^{\infty} 4 \cdot 2^{-5r} > \frac{5}{4} \cdot 2^{-n}.$$

Hence, also

$$d(F(y), F(x_{e_1})) \geq \frac{5}{4} \cdot 2^{-n}.$$

Since $|y - x_{e_1}| \leq \max_{x \in I_{e_1,t}} |y - x| \leq 2^{-\phi_{e_2}(n)}$, the function $\phi_{e_2}$ cannot be a modulus of continuity of $F$ at $x_{e_1}$. Thus, $N_e$ is satisfied. \qed

We still have to show that every positive requirement is satisfied.

Lemma 7. Every positive requirement $P_d$ is satisfied.

Proof. For irrelevant positive requirements, i.e., for requirements $P_d$ with $d \in IRRELEVANT-POS-REQ$, nothing needs to be shown.

Let us fix some $d \in RELEVANT-POS-REQ$. Let $T$ be the last stage during which a negative higher priority requirement was active, i.e., some $N_c$ with $c \leq d$ (if no such stage exists, set $T := 0$). Then at stage $T$, $P_d$ is being initialized, i.e., all pairs that have possibly been added to $C_d$ already are thrown away, we start with $C_{d,T} = \emptyset$ again at this stage $T$, and we will never throw anything out of $C_d$ again. Hence, the set $C_d$ is computably enumerable. Let $(x_k)_k$ be the sequence of real numbers described by $\phi_k$. Remember that for each $k$, the interval $I_{d,k,t}$ contains $x_k$ and that the sequence $(I_{d,k,t})_t$ converges to the set $\{x_k\}$ containing only the real number $x_k$. Remember also that the continuous functions $F_t$ converge uniformly to $F$. For every $k$ we obtain

$$\lim_{t \to \infty} \min_{t \to \infty} F_t(I_{d,k,t}) = F(x_k) = \lim_{t \to \infty} (\max_{t \to \infty} F_t(I_{d,k,t}) + 2^{-t}).$$

(5)
Now let us consider some \( t \geq T \) and some \((k, J) \in C_{d,t}\). Since for \( s \geq t \) we have \( C_{d,t} \subseteq C_{d,s} \), hence, \((k, J) \in C_{d,s}\), Lemma 2 tells us \( \min F_t(I_{d,k,s}) \in J \) for all \( s \geq t \). Since \( J \) is closed, by (5), we obtain
\[
F(x_k) \in J.
\]
(6)

Finally, we remember that the sequence \((l_{d,t})_t\) is nondecreasing and unbounded. Hence, there are infinitely many stages \( t \geq T \) during which we add pairs
\[
(k, J) = (k, [\min F_t(I_{d,k,t}), \max F_t(I_{d,k,t}) + 2^{-t}])
\]
for all \( k < l_{d,t} \) to the set \( C_d \). Taking (5) and (6) into account, this shows that the enumeration of \( C_d \) from stage \( T \) on gives an algorithm for computing the sequence \((F(x_k))_k\). Thus, the positive requirement \( P_d \) is satisfied. We have shown that all positive requirements are satisfied. \( \square \)

This ends the verification.

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