PERGAMON

Applied Mathematics Letters 14 (2001) 783–788

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Conformal Symmetric Model of the Porous Media

G. A. KUZ'MIN
Institute of Thermophysics, 630090 Novosibirsk, Russia

O. N. SOBOLEVA
Institute of Computational Mathematics and Mathematical Geophysics
630090 Novosibirsk, Russia
olga@nmsf.sscc.ru

(Received December 1999; accepted January 2000)

Communicated by A. S. Alekseev

Abstract—We present a conformal theory for the random fractal fields. As an example, the density of the porous matter is considered. The equation that expresses density in terms of a nonfractal field is evaluated. Assuming the hypothesis of scale and conformal symmetry for the latter, we derive the correlation functions for density. The log-normal conformal model is studied. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Conformal symmetry, Scaling, Random fractals, Porous matter.

1. INTRODUCTION

Rocks have a developed porous structure that determines the fluid flows, waves, and other processes in them. It has been recognized that the porous structure may be approximated by the scale invariant models. This has led to use of various models and of the terminology of the physics of disordered phenomena. In particular, the percolation theory and the fractal models were recognized to be useful for modelling the flow and dispersion in the porous media (see, for example, the review in [1,2]). Some characteristics of the wave fields may also reflect the fractal structure of the media. We consider the conformal model of random fractal fields. For concreteness, we shall study the correlation functions of density in the porous media, keeping in mind that the same relations may be valid for the wave field and for other related characteristics.

Kolmogorov introduced the conception of scale invariance into the modern physics. The scale invariance lies in the fundament in his theory of turbulence [3]. The logarithmic normal distribution follows from his scale invariant model of the fine crushing [4]. Improved scaling [5] was developed to describe the scale dependent intermittency in turbulence. Those ideas were fruitfully explored in various regions of physics (in statistical physics, in the physics of disordered phenomena, and so on). In this paper, the improved scaling theory [5] is used as a base for the conformal symmetric model of the porous media.

The conformal symmetry is the simplest extension of the scale one. It gives more definite predictions for the statistical characteristics of the random media than the simple scaling hypothesis,

0893-9659/01/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(01)00044-1

see review in [6]. The main aim of the present paper is to incorporate the conformal symmetry into the scaling theory of fractal fields.

2. THE FRACTAL MODEL OF THE POROUS MEDIA

We define the fractal fields as a limit of the usual smooth functions. Consider the density field $\rho(\mathbf{x})$. The density that is smoothed over a sphere of radius l is

$$\rho^l(\mathbf{x}) = \frac{3}{4\pi l^3}\int_{r\le l}\rho(\mathbf{x}+\mathbf{r})\,d^3r.$$

At $l \to 0$, $\rho^l(\mathbf{x}) \to \rho(\mathbf{x})$. Let us consider the dimensionless field $\psi(\mathbf{x},l,l') = \rho^{l'}(\mathbf{x})/\rho^l(\mathbf{x})$ that is similar to the dimensionless fields of [5] and will be supposed to have the scale symmetry. We suppose that all correlation functions of $\psi(\mathbf{x},l,l') = \rho^{l'}(\mathbf{x})/\rho^l(\mathbf{x})$ are invariant to the scale transform: when the points $\mathbf{x}$ and the scales l are transformed as $\mathbf{x} \to K\mathbf{x}$, $l \to Kl$, where K is any numeric factor, the correlation functions of $\psi(\mathbf{x},l,l')$ remain the same.

The field $\psi(\mathbf{x},l,l')$ has too many arguments. We define a simpler field that has the same information. By the definition of $\psi(\mathbf{x},l,l')$,

$$\psi(\mathbf{x},l,l'') = \psi(\mathbf{x},l,l')\,\psi(\mathbf{x},l',l''). \tag{1}$$

Let us consider the case $l'' \to l'$. In the first order in $l''-l'$, we obtain the differential equation for $\psi(\mathbf{x},l,l')$:

$$\frac{\partial\psi(\mathbf{x},l,l')}{\partial l'} = \frac{1}{l'}\psi(\mathbf{x},l,l')\,\varphi(\mathbf{x},l'), \tag{2}$$

where $\varphi(\mathbf{x},l') = \left.\frac{\partial\psi(\mathbf{x},l',l'y)}{l'\partial y}\right|_{y=1}$ is a dimensionless field that has the scale symmetric correlation functions. From the definition of $\psi(\mathbf{x},l,l')$, we obtain the equation that expresses the fractal field ρ in terms of $\varphi(\mathbf{x},l)$,

$$\frac{\partial[\rho^l(\mathbf{x})]}{\partial\ln l} = \rho^l(\mathbf{x})\varphi(\mathbf{x},l). \tag{3}$$

In practice, the fluctuations may be observed in some finite range of scales $l_\eta < l < L$. Equation (3) has to be supplemented by the boundary condition on any end of the range (l_η, L). For definiteness, the boundary condition at $l = L$ will be assumed to be fixed $\rho(\mathbf{x},L) = \rho_0 = \mathbf{const}$. The solution to (3) is

$$\rho(\mathbf{x},l) = \rho_0\exp\left[-\int_l^L\varphi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right]. \tag{4}$$

All statistical properties of the fractal density $\rho(\mathbf{x}) = \rho(\mathbf{x},l/L\to 0)$ are determined by the field φ. Assuming some model for φ, we obtain full description of ρ. In the following sections, the correlation functions for ρ are derived, assuming that φ is the scale and conformal symmetric field in some subrange $l_0 < l < L_0$, where $l_0 > l_\eta$, $L_0 < L$.

3. SCALE AND CONFORMAL SYMMETRY

We consider the scale and conformal pair correlation (cumulant) function of $\varphi(\mathbf{x},l)$:

$$\Phi(\mathbf{x},\mathbf{y},l,l') = \langle\varphi(\mathbf{x},l)\,\varphi(\mathbf{y},l')\rangle_c.$$

The spatial homogeneity and isotropy implies

$$\Phi(\mathbf{x}-\mathbf{y},l,l') = \Phi\left((\mathbf{x}-\mathbf{y})^2,l,l'\right),$$

where the same letter Φ is used for the sake of simplicity in the right-hand side.

The scale symmetry says that the correlation remains unchanged if all spatial scales are extended in K times

$$\Phi\left((\mathbf{x}-\mathbf{y})^2,\ l,l'\right) = \Phi\left(K^2(\mathbf{x}-\mathbf{y})^2,\ Kl, Kl'\right),$$

where K is any positive factor. This equation implies that Φ depends on two arguments rather than on three:

$$\Phi\left((\mathbf{x}-\mathbf{y})^2,\ l,l'\right) = \Phi\left(\frac{(\mathbf{x}-\mathbf{y})^2}{ll'}, \frac{l'}{l}\right). \tag{5}$$

The conformal group of symmetry consists of the scale subgroup plus invariance to the special conformal transformation. In two dimensions, arbitrary analytic function of complex variable induces some conformal transformation, that *locally* looks as the scale one. In three dimensions, there exists the only independent transformation of space that locally looks as the scale one. This transformation is the inversion relative the unit sphere $\mathbf{x} \to \mathbf{x}/x^2$.

The inversion tansforms the radius l and the center $\mathbf{a}$ of the sphere $(\mathbf{x}-\mathbf{a})^2 \le l^2$, over which the density was smoothed, as

$$\mathbf{a}_1 = \frac{\mathbf{a}}{a^2 - l^2}, \qquad l_1 = \frac{l}{|a^2 - l^2|}. \tag{6}$$

Let us suppose that function (5) remains unchanged when all spatial points and scales are transformed according to (6). The function Φ may be dependent only on the conformal invariant $[l^2 + l'^2 - (\mathbf{x}-\mathbf{y})^2]/(ll')$,

$$\Phi\left(\mathbf{x},\mathbf{y},l,l'\right) = \Phi\left[\frac{(\mathbf{x}-\mathbf{y})^2 - l^2 - l'^2}{ll'}\right]. \tag{7}$$

4. SIMPLE SELF-SIMILAR GAUSSIAN PROCESS

In this section, the random field φ is assumed to be Gaussianly distributed. Let us average solution (4). Taking into account that $\langle\rho(\mathbf{x},l)\rangle = \rho_0$ at any $l,\mathbf{x}$, we obtain

$$1 = \left\langle \exp\left[-\int_l^L \varphi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right]\right\rangle. \tag{8}$$

For the Gaussian field $f(l)$ and for arbitrary regular function θ, we have the following equality [7]:

$$\begin{aligned}&\left\langle \exp\left[-i\int_l^L \theta(l_1)\, f(l_1)\, dl_1\right]\right\rangle \\ &= \exp\left[-i\int_l^L \theta(l_1)\langle f(l_1)\rangle\, dl_1 - \frac{1}{2}\int_l^L dl_1 \int_l^L dl_1 \theta(l_1)\,\theta(l_2)\,\langle f(l_1)\, f(l_2)\rangle_c\right].\end{aligned} \tag{9}$$

Choosing $\theta(l) = -i/l,\ f(l_1) = \varphi(\mathbf{x},l)$, we have from (8),(9),

$$\int_l^L \frac{dl_1}{l_1}\int_l^L \frac{dl_2}{l_2}\langle\varphi(\mathbf{x},l_1)\varphi(\mathbf{x},l_2)\rangle_c = 2\int_l^L \langle\varphi(\mathbf{x},l_1)\rangle\frac{dl_1}{l_1}. \tag{10}$$

From (3),(10), we obtain the formula $\langle \ln^2\rho(\mathbf{x},l)\rangle = \ln^2\rho_0 + 2\int_l^L \langle\varphi(\mathbf{x},l_1)\rangle\, \frac{dl_1}{l_1}$. Let us divide the interval of integration into (l, L_0) and (L_0, L), where L_0 is the upper bound of the similarity range. We obtain

$$\left\langle \ln^2\rho(\mathbf{x},l)\right\rangle = A + 2\overline{\varphi}\ln\left(\frac{L_0}{l}\right), \tag{11}$$

where A is some nonuniversal contribution that comes from large scales $L_0 < l_1 < L$.

In order to obtain from (4) another correlation function of density ρ, one needs the second correlation of the Gaussian field φ. The scale symmetry determines them up to the universal function of two dimensional arguments (5), and the conformal symmetry retains the only dimensionless complex (see (7)). This enables us to find the general form of the correlation functions of ρ.

Let us consider the second correlations of ρ. Formula (4) gives

$$\langle\rho(\mathbf{x},l)\rho(\mathbf{x}+\mathbf{r},l)\rangle = \rho_0^2\left\langle \exp\left[-\int_l^L \left[\varphi(\mathbf{x},l_1)+\varphi(\mathbf{x}+\mathbf{r},l_1)\right]\frac{dl_1}{l_1}\right]\right\rangle.$$

Using equation (9) with $f(l_1)=\varphi(\mathbf{x},l_1)+\varphi(\mathbf{x}+\mathbf{r},l_1)$, $\theta(l)=-i/l$ and taking into account (10), we have

$$\langle\rho(\mathbf{x},l)\rho(\mathbf{x}+\mathbf{r},l)\rangle = \rho_0^2\exp\left\{\int_l^{L_0}\frac{dl_1}{l_1}\int_l^{L_0}\frac{dl_2}{l_2}\Phi\left(\frac{l_1^2+l_2^2-r^2}{l_1l_2}\right)\right\}. \tag{12}$$

The integral is analyzed in the polar coordinates λ,χ in l_1,l_2 plane: $\lambda=\sqrt{l_1^2+l_2^2}$, $\sin\chi = l_2/\sqrt{l_1^2+l_2^2}$. The region of integration is divided in the three subregions,

(1) $l\le\lambda\le r$,
(2) $r\le\lambda\le L_0$,
(3) $L_0\le\lambda\le L$.

The integral is evaluated asymptotically at $r,l\ll L_0<L$.

The last region gives some nonuniversal contribution α_3. In that region, the distance $r\ll\lambda$ and may be omitted. Thus, the dimensionless contribution α_3 does not depend on r. The scale and conformal invariance give in the polar coordinates

$$\langle\rho(\mathbf{x},l)\rho(\mathbf{x}+\mathbf{r},l)\rangle = \rho_0^2\exp\left\{\alpha_3+2\int_l^{L_0}\frac{d\lambda}{\lambda}\int_0^{\pi/2}\frac{d\chi}{\sin 2\chi}\Phi\left[\frac{4}{\sin^2 2\chi}\left(\frac{\lambda^2-r^2}{\lambda^2}\right)^2\right]\right\}. \tag{13}$$

In Subregion 2, the main logarithmical divergent term is extracted. In the remainder convergent contribution, the upper limit is replaced by ∞. That approximation gives an error of the order of $O(r^2/L_0^2)$.

The result of the integration is

$$\begin{gathered}\langle\rho(\mathbf{x},l)\rho(\mathbf{x}+\mathbf{r},l)\rangle \approx C\rho_0^2\left(\frac{L_0}{r}\right)^{2\overline{\varphi}},\\ C=\exp\left[\sum_{i=1}^{3}\alpha_i\right],\end{gathered} \tag{14}$$

where α_i, $i=1,2,3$ are determined by integrals over Subregions 1, 2, and 3.

$$\begin{aligned}\alpha_1(A) &= 2\int_{l/r}^1\frac{ds}{s}\int_0^{\pi/2}\frac{d\chi}{\sin 2\chi}\Phi\left[4\frac{s^2-1}{s^2\sin^2 2\chi}\right]\approx 2\int_0^1\frac{ds}{s}\int\frac{d\chi}{\sin 2\chi}\Phi\left[4\frac{s^2-1}{s^2\sin^2 2\chi}\right]=\text{const},\\ \alpha_2(A) &= 2\int_1^\infty\frac{ds}{s}\int_0^{\pi/2}\frac{d\chi}{\sin 2\chi}\left[\Phi\left(4\frac{s^2-1}{s^2\sin^2 2\chi}\right)-\Phi\left(\frac{-4}{\sin^2 2\chi}\right)\right]=\text{const},\\ \alpha_3(A) &= 4\int_{L_0}^L\frac{d\lambda}{\lambda}\int_0^{\pi/4}\frac{d\chi}{\sin 2\chi}\langle\varphi(\mathbf{x},\lambda\cos\chi)\varphi(\mathbf{x}+\mathbf{r},\lambda\sin\chi)\rangle.\end{aligned} \tag{15}$$

The value C is not universal because it contains the contribution α_3. On the contrary, the power index in (14) is universal. According to the experimental data [1], $2\overline{\varphi}\sim 0.3$.

4.1. Spatial Correlations of Higher Order

Formula (4) gives for the correlation function of n^{th} order,

$$\left\langle \prod_{i=1}^{n} \rho(\mathbf{x}_i, l) \right\rangle = \rho_0^n \left\langle \exp\left[-\int_l^L \sum_{i=1}^{n} \varphi(\mathbf{x}_i, l_1) \frac{dl_1}{l_1} \right] \right\rangle . \tag{16}$$

With the help of equation (9), the mean of the exponent is written as the exponent of mean value of an expression. Using (8), we have

$$\left\langle \exp\left[-\int_l^L \sum_{i=1}^{n} \varphi(\mathbf{x}_i, l_1) \frac{dl_1}{l_1} \right] \right\rangle$$
$$= \exp\left\{ -n\overline{\varphi} \ln \frac{L}{l} + \frac{1}{2} \int_l^L \frac{dl_1}{l_1} \int_l^L \frac{dl_2}{l_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \varphi(\mathbf{x}_i, l_1) \varphi(\mathbf{x}_j, l_2) \rangle_c \right\}$$
$$= \frac{1}{2} \int_l^L \frac{dl_1}{l_1} \int_l^L \frac{dl_2}{l_2} \sum_{i \neq j}^{n} \Phi \left[\frac{l_1^2 + l_2^2 - r_{ij}^2}{l_1 l_2} \right],$$

where $r_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2$.

The integral is of the same kind as considered above. Similar straightforward algebra leads to

$$\left\langle \prod_{i=1}^{n} \rho(\mathbf{x}_i, l) \right\rangle = C^n \rho_0^n \prod_{i \neq j} \left(\frac{L_0}{r_{ij}} \right)^{2\overline{\varphi}},$$

where $r_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2$.

4.2. Non-Gaussian Conformal Fields of Higher Order

Let us consider the correlation function of the conformal symmetric field φ of the third order,

$$\Phi_3 (\mathbf{x}_1, l_1, \mathbf{x}_2, l_2, \mathbf{x}_3, l_3,) = \langle \varphi (\mathbf{x}_1, l_1) \varphi (\mathbf{x}_2, l_2) \varphi (\mathbf{x}_3, l_3) \rangle_c .$$

There are three conformal invariants of the same kind as in equation (7). The conformal symmetric function has to be equal to

$$\Phi_3 (\mathbf{x}_1, l_1, \mathbf{x}_2, l_2, \mathbf{x}_3, l_3,) = \Phi'' \left(\frac{l_1^2 + l_2^2 - x_{12}^2}{l_1 l_2}, \frac{l_1^2 + l_3^2 - x_{13}^2}{l_1 l_3}, \frac{l_3^2 + l_2^2 - x_{23}^2}{l_3 l_2} \right).$$

The generalization to the correlation functions of more high order is obvious. The correlation functions have to depend on all independent conformal invariants.

5. CONCLUSIONS

We started from the modified Kolmogorov theory in terms of the ratios of smoothed fields. The scale symmetry determines the correlation functions of those fields as universal functions of dimensionless arguments. The differential equation (3) expresses the usual fields in terms on the scale invariant ones. Conformal symmetry diminished the number of dimensional arguments. For the Gaussian φ, the formulae of the log-normal model follows with definite expressions for its parameters in terms of the integrals of correlations of the conformal field φ.

REFERENCES

1. M. Sahimi, Flow phenomena in rocks: From continuum models, to fractals, percolation, cellular automata, and simulated annealing, *Reviews of Modern Physics* **65**, 1393–1534 (1993).
2. G. Molchan, Turbulent cascades: Multifractal characteristics, *Comput. Seismology* **29**, 155–167 (in Russian) (1997).
3. A. Kolmogorov, The local structure of turbulence in incompressible viscous fluids at very high Reynolds numbers, *Dokl. Akad. Nauk. SSSR* **30**, 301–305 (1941).
4. A.N. Kolmogorov, On log-normal distribution of sizes of fragmented particles, *Dokl. A.N. SSSR* **31**, 99–101, (in Russian) (1941).
5. A. Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number, *J. Fluid Mech.* **13**, 82–85 (1962).
6. G. Mack and A. Salam, Finite-component field representations of the conformal group, *Ann. Phys.* **53**, 173–202. (1969).
7. A. Monin and A. Yaglom, *Statistical Fluid Mechanics, Vol. 1*, MIT Press, Cambridge, (1975).