On spectral bounds for cutsets

J.A. Rodríguez, A. Gutiérrez, J.L.A. Yebra

Departament de Matemática Aplicada IV, Universitat Politècnica de Catalunya, Campus Nord, c/. Gran Capita, s/n Modul C3, 08034 Barcelona, Spain

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Abstract

Let $\Gamma$ be a simple and connected graph. A $k$-vertex separator [$k$-edge separator] is a subset of vertices [edges] whose deletion separates the vertex [edge] set of $\Gamma$ into two parts of equal cardinality, that are at distance greater than $k$ in $\Gamma$. Here we investigate the relation between the cardinality of these cutsets and the laplacian spectrum of $\Gamma$. As a consequence of the study, we obtain the well-known lower bounds for the bandwidth and the bipartition width of a graph.

1. Introduction

Recently, several results bounding distance-related parameters of a graph from the eigenvalues of either its adjacency or its laplacian matrix have appeared in the literature. In this context, parameters such as mean distance, diameter, radius, isoperimetric number, magnifying constant and excess have been studied extensively. See, for instance, the papers of Alon and Milman [1], Biggs [2], Chung [4], Cvetkovic and Doob [6], Delorme and Solé [7], Fiol et al. [11,12,13,14], Fiol and Garriga [8,9,10], Mohar [18,19,20,21], Rodríguez and Yebra [23] and Van Dam and Haemers [24]. Here we investigate the relation between cutsets and the laplacian spectrum of a graph.

We consider simple and connected graphs. Let $\Gamma=(V,E)$ be a graph of order $n$. Let $A$ and $\Delta$ be the adjacency and degree matrices of $\Gamma$. The laplacian matrix of a graph $\Gamma$, denoted by $L=L(\Gamma)$, is defined as $L=\Delta-A$ and its mesh of distinct eigenvalues is denoted by $\mathcal{M} = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_b\}$. We make no use of the multiplicities of
the eigenvalues. We recall that the simple eigenvalue $\lambda_0 = 0$ has $j = (1, 1, \ldots, 1)$ as eigenvector.

We denote by $\hat{d}(u, v)$ the distance between two vertices $u$ and $v$ of $\Gamma$, while the distance between two sets of vertices $U, W \subseteq V(\Gamma)$ is defined as

$$\hat{d}(U, W) = \min_{u \in U, \, v \in V} \{\hat{d}(u, v)\}.$$ 

Let $a$ and $b$ be edges of a graph $\Gamma$. An $ab$-walk of length $k$ between $a$ and $b$ is an alternating sequence

$$u_0, a = a_0, u_1, a_1, \ldots, u_k, a_k = b, u_{k+1}$$

of vertices and edges, such that $a_i = u_iu_{i+1}$ for $i = 0, \ldots, k$. An $ab$-path is an $ab$-walk in which no vertex is repeated. The distance $\hat{c}_e(a, b)$ between two edges $a$ and $b$ is the minimum of the lengths of $ab$-paths, and the distance between two sets of edges $A, B \subseteq E(\Gamma)$ is defined as

$$\hat{c}_e(A, B) = \min_{a \in A, \, b \in B} \{\hat{c}_e(a, b)\}.$$  

A $k$-vertex separator ($k$-edge separator) is a subset $T_k \subset V(\Gamma)$ [$H_k \subset E(\Gamma)$] whose deletion separates $V(\Gamma)$ [$E(\Gamma)$] into two sets of equal cardinality, that are at distance greater than $k$. We denote by $\text{vs}_k(\Gamma)$ [$\text{es}_k(\Gamma)$], the minimum cardinality among all $k$-vertex [$k$-edge] separators, that is,

$$\text{vs}_k(\Gamma) = \min\{|T_k| : T_k \text{ is a } k\text{-vertex separator of } \Gamma\}.$$ 

For simplicity, when $k = 1$ we just say vertex [edge] separator instead of 1-vertex [1-edge] separator and we denote it by $\text{vs}(\Gamma) = \text{vs}_1(\Gamma)$ [$\text{es}(\Gamma) = \text{es}_1(\Gamma)$]. A bisector (see [3]) is a vertex separator $T$ such that $|T| \leq n/3$.

In Section 3 we obtain a lower bound on $\text{vs}_k(\Gamma)$ expressed in terms of the laplacian eigenvalues of $\Gamma$. For $k = 1$ this bound improves the one given by Juvan and Mohar in [16] (see also Helmberg et al. [15]). As a consequence we deduce lower bounds on the bandwidth of the matrix $L^k$. In particular, for $k = 1$ we obtain a new proof of a well-known result of Helmberg et al. [15] concerning the bandwidth of a graph. We also deduce a necessary condition for the existence of bisectors.

Analogously, in Section 4, we extend the results to the study of $k$-edge separators. In particular, for regular graphs and $k = 1$, we obtain a well-known lower bound of Mohar [21] and Merris [17] on the bipartition width of a graph.

Our main tool throughout this paper is the $k$-alternating polynomials, so we begin with a short presentation of them, citing some of their main important properties.

### 2. Alternating polynomials

Some results relating the diameter of a regular graph and its second eigenvalue (in absolute value) have been given by Chung [4], Delorme and Solé [7], and Mohar [20] and other authors. The results in [4,7] admit the following unified presentation: let
\( \lambda_0 > \lambda_1 > \cdots > \lambda_d \) be the distinct (adjacency) eigenvalues of a graph \( G \) of order \( n \) and diameter \( D(G) \), and let \( P \) be a real polynomial. Then,

\[
P(\lambda_0) > \|P\|_\infty (n - 1) \Rightarrow D(G) \leq \text{dgr } P,
\]

where \( \|P\|_\infty = \max_{1 \leq i \leq d} \{|P(\lambda_i)|\} \). The proof basically works as in [4,7]: If \( A \) is the adjacency matrix of \( G \), and \( P(A) \) denotes the matrix obtained when the polynomial \( P \) is evaluated at \( A \), the lefthand side inequality causes the matrix \( P(A) \) to have all its entries different from zero, so that there must exist a path of length \( \leq \text{dgr } P \) between any two vertices of the graph, and this gives the bound on \( D(G) \) in the righthand side. Alternatively, (1) may be thought of as a lower bound on the order of a graph in terms of its eigenvalues and its diameter

\[
D(G) > \text{dgr } P \Rightarrow n \geq 1 + \frac{P(\lambda_0)}{\|P\|_\infty}.
\]

With the formulation (1), Chung [4] considered the case \( P(x) = x^m \), and Delorme and Solé [7] generalized her results by taking \( P(x) = x^m + tx^{m-1} \), \( t \in \mathbb{R}^+ \), which has the advantage of being useful to the case of bipartite biregular graphs (that is, graphs such that vertices in the same vertex class have the same degree.) Other results, using the laplacian matrix, can be found in [5,20]. Besides, in [7] the case of regular digraphs was also considered, and the authors explored the connections of the problem with finite non-Abelian simple groups, primitive association schemes, and coding theory.

However, the formulation in (1) suggests that, to optimize the results, we must face the discrete nature of the problem, and look for the polynomials that maximize the quotient \( P(\lambda_0)/\|P\|_\infty \). Or, alternatively, we should try to maximize \( P(\lambda_0) \) when the considered polynomials are normalized by \( \|P\|_\infty = 1 \). This has been done by Fiol et al. in [11], for not necessarily regular graphs, introducing the alternating polynomials.

For the case of the laplacian eigenvalues, these polynomials can be defined as follows: let \( \mathcal{M} = \{\mu_1 < \cdots < \mu_b\} \) be a mesh of \( b \) real numbers. For any \( k = 0, 1, \ldots, b - 1 \) let \( Q_k \) be the \( k \)-alternating polynomial associated to \( \mathcal{M} \). That is, the polynomial of \( \mathbb{R}_k[x] \) with norm \( \|Q_k\|_\infty = \max_{1 \leq i \leq b} \{|Q_k(\mu_i)|\} \leq 1 \), such that

\[
Q_k(\mu) = \sup\{q(\mu): q \in \mathbb{R}_k[x], \|q\|_\infty \leq 1\},
\]

where \( \mu \) is any real number smaller than \( \mu_1 \). In [11] it was shown that, for any \( k = 0, 1, \ldots, b - 1 \),

- There is a unique \( Q_k \) which, moreover, is independent of the value of \( \mu( < \mu_1) \);
- \( Q_k \) has degree \( k \);
- \( Q_k \) takes \( k + 1 \) alternating values \( \pm 1 \) at the mesh points;
- \( Q_0(\mu) = 1 < Q_1(\mu) < \cdots < Q_{b-1}(\mu) \).

It is easy to compute \( Q_k \) for \( k = 0, 1 \) and \( b - 1 \), while the problem of finding \( Q_k \) for other values of \( k \), can be translated into a linear programming problem, see [11].
Example 1. The graph on 6 vertices obtained by joining one vertex of a triangle to some vertex of another triangle has laplacian eigenvalues:

\[
\mu_0 = 0, \quad \mu_1 = \frac{5 - \sqrt{17}}{2}, \quad \mu_2 = 3, \quad \mu_3 = \frac{5 + \sqrt{17}}{2}
\]

from which we obtain

\[
Q_1(\mu_1) = 1, \quad Q_1(\mu_3) = -1 \Rightarrow Q_1(\mu) = \frac{5\mu_3 + \mu_1 - 2\mu}{\mu_3 + \mu_1} = \frac{5 - 2\mu}{\sqrt{17}},
\]

\[
Q_2(\mu_1) = 1, \quad Q_2(\mu_2) = -1, \quad Q_2(\mu_3) = 1 \Rightarrow Q_2(\mu) = 5\mu^2 - 5\mu/2 + 2.
\]

3. Vertex separators

Our main tool in this section is the following result that appears in [22].

Lemma 1. Let \( \Gamma \) be a simple and connected graph of order \( n \) and let \( Q_k \) be the \( k \)-alternating polynomial associated to the mesh of the Laplacian eigenvalues of \( \Gamma \). Let \( U, W \subset V(\Gamma) \) be such that \( |U| = |W| = t \) and \( \partial(U, W) > k \), then,

\[
t \leq \frac{n}{Q_k(0) + 1}.
\]

Proof. Let \( v_1, v_2, \ldots, v_n \) be a labeling of the vertex set \( V(\Gamma) \) of \( \Gamma \), and let

\[
u = \sum_{v_i \in U} e_i \quad \text{and} \quad w = \sum_{v_i \in W} e_i
\]

be vectors associated to the sets \( U \) and \( W \), where \( e_i \) is the \( i \)th unit vector of \( \mathbb{R}^n \). From the following decompositions of the vectors \( u \) and \( w \)

\[
u = \frac{t}{n} j + z_u, \quad w = \frac{t}{n} j + z_w, \quad (\text{where } z_u, z_w \in j^\perp)
\]

we have

\[
t = \|u\|^2 = \frac{t^2}{n} + \|z_u\|^2 \Rightarrow \|z_u\| = \sqrt{t - \frac{t^2}{n}}
\]

and similarly

\[
\|z_w\| = \sqrt{t - \frac{t^2}{n}}.
\]

Now, if for two vertices \( v_i, v_j \in V(\Gamma) \) we have \( \partial(v_i, v_j) > k \), then \( (L^k)_{ij} = 0 \). Hence, with \( Q_k(L) \) denoting the matrix obtained when the polynomial \( Q_k \) is evaluated at \( L \),
we obtain
\[
\hat{c}(U, W) > k \Rightarrow 0 = \langle Q_k(L)u, w \rangle = \left( \frac{1}{n} Q_k(0) + Q_k(L)z_u, \frac{t}{n} j + z_w \right) = \frac{t^2}{n} Q_k(0) + \langle Q_k(L)z_u, z_w \rangle.
\]
Then, by the Cauchy–Schwarz inequality
\[
\frac{t^2}{n} Q_k(0) \leq \|Q_k(L)z_u\| \|z_w\|
\]
\[
\leq \|z_u\| \|Q_k\|_{\infty} \|z_w\|
\]
\[
= \sqrt{t - \frac{t^2}{n}} \sqrt{t - \frac{t^2}{n}} = t - \frac{t^2}{n}
\]
and the result follows. \(\square\)

**Theorem 2.** Let \(\Gamma = (V, E)\) be a simple connected graph of order \(n\). Then,
\[
\text{vs}_k(\Gamma) \geq n \frac{Q_k(0) - 1}{Q_k(0) + 1},
\]
where \(Q_k\) is the \(k\)-alternating polynomial associated to the mesh of the laplacian eigenvalues of \(\Gamma\).

**Proof.** Let \(T_k \subset V(\Gamma)\) be a \(k\)-vertex separator and let \(U, W \subset V(\Gamma)\) be the corresponding subsets of equal cardinality \(t\) such that \(\hat{c}(U, W) > k\). Therefore, \(|T_k| = n - 2t\). By Lemma 1, \(t \leq n/(Q_k(0) + 1)\) and the result immediately follows. \(\square\)

The above bound is tight for different values of \(k\), as we can see in the following examples.

1. Firstly, we shall need the following binary graph operations. Let \(I_1\) and \(I_2\) be vertex disjoint graphs. Denote by \(I_1 \cup I_2\) their union, and let \(I_1 \ast I_2\) be their joint (obtained from \(I_1 \cup I_2\) by joining every vertex of \(I_1\) with every vertex of \(I_2\)). The laplacian characteristic polynomial \(\phi(\Gamma, \mu)\) of the resulting graphs can be expressed in terms of the laplacian characteristic polynomials of \(I_1\) and \(I_2\), as follows:
\[
\phi(I_1 \cup I_2, \mu) = \phi(I_1, \mu)\phi(I_2, \mu),
\]
\[
\phi(I_1 \ast I_2, \mu) = \frac{\mu(\mu - n_1 - n_2)}{(\mu - n_1)(\mu - n_2)} \phi(I_1, \mu - n_1)\phi(I_2, \mu - n_2),
\]
where \(n_i = |V(I_i)|, \; \text{for} \; i = 1, 2\). Consider the class of graphs of the form \(\Gamma = (I_1 \cup I_1) \ast I_2\), such that \(n_2 \leq 2n_1\). By the above formulas we obtain \(\mu_k(\Gamma) = 2n_1 + n_2\) and \(\mu_1(\Gamma) = n_2\). For \(k = 1\), since \(Q_1(0) = (\mu_1 + 1)/(\mu_1 - \mu_1)\), we obtain the bound \(\text{vs}(\Gamma) \geq n_2\) which is tight.
2. A graph \( \Gamma \) is call \( r \)-antipodal when the relation “\( u \sim v \Leftrightarrow \hat{d}(u,v)=D(\Gamma) \)” is an equivalence relation and each equivalence class has exactly \( r \) vertices. For \( r \)-antipodal distance regular graphs we have \( Q_{b-1}(0) = 2n/r - 1 \) (see [14]). Thus, \( vs_{b-1}(\Gamma) \geq n - r \) and the bound is trivially attained when \( r \) is even, \( r = 2t \).

3. For the class of laplacian \( k \)-boundary graphs of diameter \( k + 1 \), see [22], each diametral vertex has exactly one vertex at distance \( k + 1 \). Besides, \( Q_k(0) = n - 1 \), so that the corresponding bound \( vs_k(\Gamma) \geq n - 2 \) is tight.

4. For the graph of Example 1, \( Q_2(0) = 2 \). Thus \( vs_2(\Gamma) \geq 2 \). Obviously, the set with the two adjacent vertices of different triangles is a 2-cutset, which shows that, in this case, the bound is tight.

When \( k = 1 \), since \( Q_1(0) = (\mu_b + \mu_1)/(\mu_b - \mu_1) \), as a particular case of Theorem 2, we obtain the following corollary.

**Corollary 3.** Let \( \Gamma \) be a simple and connected graph, and let \( \mu_1 \) and \( \mu_b \) be its second smallest and largest eigenvalues. With \( a = \lceil n/\mu_1 \rceil \), we have

\[
vs(\Gamma) \geq \begin{cases} 
  a & \text{if } n-a \text{ is even}, \\
  a+1 & \text{if } n-a \text{ is odd}.
\end{cases}
\]

In [15] the following lower bound was established for the cardinality of those cutsets \( S_3 \) that partition the vertex set of \( \Gamma \) in subsets \( S_1 \) and \( S_2 \),

\[
|S_3| \geq \frac{4\mu_b\mu_2|S_1||S_2|}{n(\mu_b - \mu_1)^2}.
\] (3)

It can easily be checked that, when \( |S_1| = |S_2| \), the above corollary improves (3).

Let \( T \subset V(\Gamma) \) be a bisector. Since \( n/3 \geq |T| \geq vs(\Gamma) \), we have the following necessary condition for the existence of bisectors.

**Corollary 4.** Let \( \Gamma = (V,E) \) be a simple connected graph of order \( n \). Let \( \mu_1 \) and \( \mu_b \) be the second smallest and the largest laplacian eigenvalues of \( \Gamma \). A necessary condition for \( \Gamma \) to have a bisector is that \( 3\mu_1 \leq \mu_b \).

Therefore, a graph such that \( 3\mu_1 > \mu_b \) can have no bisector. This is the case of the Petersen graph \( \mathcal{P} \), because \( \mu_1(\mathcal{P}) = 2 \) and \( \mu_b(\mathcal{P}) = 5 \). Note, however, that the converse is not true, as the line graph \( L(H_3) \) of the 3-cube shows.

### 3.1. Bounding the bandwidth

A symmetric matrix \( M \) is said to have a bandwidth \( w(M) \) if \( (M)_{ij} = 0 \) for all \( i,j \) satisfying \( |i - j| > w(M) \). The bandwidth \( w(\Gamma) \) of a graph \( \Gamma \) is the smallest possible bandwidth of its adjacency or laplacian matrix. It is not difficult to check that \( vs(\Gamma) \leq w(\Gamma) \), and more generally that \( vs_k(\Gamma) \leq w(L^k) \). Thus, Theorem 2 gives a lower bound for \( w(L^k) \). In particular, for \( k = 1 \) we obtain the well-known inequality of Helmberg et al. [15], concerning the bandwidth of \( \Gamma \): \( w(\Gamma) \geq n\mu_1/\mu_b \).
We emphasize that there are graphs for which vs(\(\Gamma\)) = w(\(\Gamma\)), such as the 3-cube where vs(\(\Gamma\)) = w(\(\Gamma\)) = 4, while in other cases w(\(\Gamma\)) can be much larger than vs(\(\Gamma\)) as the graph \(\Gamma = (K_n \cup K_n) \ast K_1\), for which w(\(\Gamma\)) = n while vs(\(\Gamma\)) = 1 shows.

4. Edge separators

In this section, we extend the previous results to the study of \(k\)-edge cutsets.

**Lemma 5.** Let \(\Gamma = (V,E)\) be a simple connected graph. Let \(m = |E|\) be the cardinality of \(E = E(\Gamma)\) and let \(P_k\) be the \(k\)-alternating polynomial associated to the mesh of the laplacian eigenvalues of the line graph \(L(\Gamma)\) of \(\Gamma\). Let \(A,B \subset E(\Gamma)\) be such that \(|A| = |B| = b\) and \(\ell_k(A,B) > k\), then,

\[
b \leq \frac{m}{P_k(0) + 1}.
\]

**Proof.** Apply Lemma 1 to the line graph of \(\Gamma\) and the result immediately follows.

For \(\delta\)-regular graphs the above upper bound can be obtained from the \(k\)-alternating polynomial \(P_k\) associated to the mesh \(M = M^* \cup \{2\delta\}\), where \(M^*\) is the mesh of non null laplacian eigenvalues of \(\Gamma\).

As in Theorem 2, by applying the above lemma we have

**Theorem 6.** Let \(\Gamma = (V,E)\) be a simple connected graph on \(m\) edges and let \(P_k\) be the \(k\)-alternating polynomial associated to the mesh of the laplacian eigenvalues of the line graph \(L(\Gamma)\) of \(\Gamma\). Then,

\[
es_k(\Gamma) \geq \left\lfloor \frac{m}{P_k(0) + 1} \right\rfloor.
\]

In particular, for \(k = 1\) the above inequality becomes

\[
es(\Gamma) \geq \left\lfloor \frac{m}{\mu_1(L)} \right\rfloor,
\]

where \(\mu_1(L)\) and \(\mu_b(L)\) are the second and the largest laplacian eigenvalues of the line graph \(L(\Gamma)\) of \(\Gamma\).

For the line graph \(L(\Gamma)\) of the graph \(\Gamma\) of Example 1 we have \(\mu_1(L(\Gamma)) = 2 - \sqrt{2}\) and \(\mu_b(L(\Gamma)) = 4 + \sqrt{2}\). Thus (5) leads to \(es(\Gamma) \geq 1\) and the bound is attained.

In the case of \(\delta\)-regular graphs on \(n\) vertices, since \(\mu_1(L) = \mu_1\) and \(\mu_b(L) = 2\delta\), we get

\[
es(\Gamma) \geq \left\lfloor \frac{n}{4} \right\rfloor.
\]

The hypercubes \(H_k\), \((k \geq 2)\) show that the above upper bound is tight. In this case \(n = 2^k\) and \(\mu_1 = 2\), so that (6) gives \(es(H_k) \geq 2^{k-1}\). Clearly \(H_k \cong H_{k-1} \times K_2\), and an
optimal edge separator $H$ can be chosen as the set of edges with ends in different copies of $H_{k-1}$. More generally, let $I_1$ be a regular graph on $n$ vertices, such that $\mu_1(I_1) \geq 2$. Then the class of graphs $\Gamma = I_1 \times K_2$ shows that bound (6) is tight. This can be checked from the fact that $\mu_1(K_2) = \mu_1(\Gamma) = 2$.

4.1. Relation with the bipartition width

The bipartition width $bw(\Gamma)$ of a graph $\Gamma$ on $n$ vertices is defined as

$$bw(\Gamma) = \min \left\{ e(S, \bar{S}) : \ S \subset V(\Gamma), \ |S| = \left\lfloor \frac{n}{2} \right\rfloor \right\},$$

where $e(S, \bar{S})$ denotes the cardinality of the set of those edges that have one endvertex in $S$ and the other one in $\bar{S}$. It equals the minimum number of edges whose deletion disconnects the vertex set of $\Gamma$ into two parts of the same cardinality (up to one vertex).

It is not difficult to check that, for regular graphs of even order $n$, $es(\Gamma) = bw(\Gamma)$. Thus, (6) gives

$$bw(\Gamma) \geq \left\lfloor n \frac{\mu_1}{4} \right\rfloor. \quad (7)$$

This inequality has been established by Mohar [21] and Merris [17] for not necessarily regular graphs. Note that for nonregular graphs, we may have $es(\Gamma) \neq bw(\Gamma)$.

References


