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An Application of the Helly Property to the Partially Ordered Sets

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Quilliot (*Discrete Math.* 1982.) showed that when the bowls of a connected graph satisfy the Helly property it is possible to deduce for this graph some fixed point and homomorphism extension theorems. For a partially ordered set E a special family of subsets is defined which, when it satisfies the Helly property, permits the deductions that every homomorphism from E into E has a fixed point, that every antitone function from E has "almost" a fixed point, and that there exists a simple criterion letting us know when a function f from a subset A of a partially ordered set G can be extended into a homomorphism from G to E.

I. DEFINITIONS

Since $(G, <)$ is a partial order, $x \in G$, we define:

$$\text{Section of } x: \quad x^+ = \text{section}^+ \text{ of } x = [y \in G, \text{ with } y \geqslant x],$$

$$: \quad x^- = \text{section}^- \text{ of } x = [y \in G, \text{ with } y \leqslant x].$$

A code is a finite sequence $(a_1, a_2,\ldots a_n)$ of numbers of the set $[-1, 1]$.

EXAMPLE. $(1, 1, -1, 1, -1)$. If $\sigma = (a_1,\ldots, a_n)$ is a code, we denote by $-\sigma$ the code $(-a_n \times -a_{n-1},\ldots, -a_1)$. Every code $(a_1, a_2,\ldots, a_p)$, $(p < n)$, is called an initial subcode of $\sigma = (a_1,\ldots, a_n)$. If $\sigma = (a_1,\ldots, a_n)$, $vv = (b_1,\ldots, b_m)$ are two codes, we denote by $\sigma \oplus v$ the code $(a_1,\ldots, a_n, b_1,\ldots, b_m)$. We call C_0 the set of all the codes ($\oplus$ is not commutative in C_0).

Let us consider an oriented graph P, without any loop, $P = (X, E)$. Here, P is allowed to have some double-oriented edges. Let $\Gamma = \{x = x_0, x_1,\ldots, x_n = y\}$ a path in P between two vertices x and y $(\forall i \in 0, 1,\ldots, n-1$, x_i and x_{i+1} are adjacent or identical in the corresponding

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undirected graph). We say that the code $(a_1, a_2,..., a_n)$ is associated with Γ if we have

$$\forall i \in 1, 2,..., n;\ a_i = 1 \Rightarrow x_{i-1} = x_i \qquad \text{or} \quad [\overrightarrow{x_{i-1}}, x_i] \in E$$

(which means the edge $[x_{i-1}, x_i]$ is oriented from x_{i-1} to x_i)

$$a_i = -1 \Rightarrow x_{i-1} = x_i \qquad \text{or} \quad [x_i, \overrightarrow{x_{i-1}}] \in E.$$

We say there is no degeneration in this association if all the vertices

$$\left\{ \begin{matrix} n \\ x_i \end{matrix} \right\}_{i=0}$$

are distinct. We denote by $C(x, y)$ the set

$$C(x, y) = \{\sigma \in C_0, \text{such that there exists a path } \Gamma \text{ in } P \text{ between } x \text{ and } y, \text{ with } \sigma \text{ associated to } \Gamma)\}.$$

Note that we have clearly

$$\sigma \in C(x, y) \Leftrightarrow -\sigma \in C(y, x)$$

$$\sigma \in C(x, y), \nu \in C(y, z) \Rightarrow \sigma \oplus \nu \in C(x, z),$$

$$C_0 = C(x, x).$$

For x in X and σ in C_0, we denote by $B(x, \sigma)$ or $B_P(x, \sigma)$ the set of the vertices y of P such that $\sigma \in C(x, y)$. Of course we have $\forall \sigma \in C_0$; $x \in B(x, \sigma)$ and σ' is a subcode of $\sigma \Rightarrow B(x, \sigma') \subset B(x, \sigma)$. If $P = (X, E)$ and $Q = (Y, F)$ are two oriented graphs, we say that a function h from X to Y is a homomorphism from P to Q, if we have:

$$[\overrightarrow{x, y}] \in E \Rightarrow [\overrightarrow{h(x), h(y)}] \in F \qquad \text{or} \quad h(x) = h(y).$$

Note that we also may consider that there is a loop at every vertex in our graph, and therefore a homomorphism may be understood as a function preserving the oriented adjacency.

II. First Extension Theorem

Helly Property

We recall that a family S of subsets of a set Y has the Helly property if for every subfamily S' of S such that $\forall A, B \in S', A \cap B \neq \Phi$, we also have

$$\bigcap_{A \in S'} A \neq \Phi.$$

Our purpose here is first to connect this Helly property to the problem of knowing when a function from a subset A of a poset G to a poset G' may be extended into a homomorphism from G to G'. (We already have such a connection established for the simple graphs [7, 8].)

THEOREM I. *Given two oriented graphs $P=(X,E)$, $Q=(Y,F)$, a subset A of X and a function h from A to Y. We suppose that A is finite, X at most countable, and that the family of the subsets of Y*

$$B(y,\sigma) \qquad y\in Y, \quad \sigma\in C_0$$

satisfies the Helly property. Then the following equivalency is true:

h may be extended into a homomorphism h° from P to $Q \Leftrightarrow \forall x, y\in A$, if we have $\sigma\in C(x,y)$ we also have $\sigma\in C(h(x),h(y))$. (1)

In fact there is an equivalency between the Helly property for the subsets $B(y,G)$, $y\in Y$, $\sigma\in C_0$, and the validity of criterion (1) for the existence of h°.

Proof. First let us prove that if the Helly property is not satisfied by the subsets $B(y,\sigma)$, $\sigma\in C_0$, $y\in Y$, then criterion (1) does not work. We have then the existence of $y_1,\ldots, y_n\in Y$, and $\sigma_1,\ldots,\sigma_n\in C_0$ with

$$\forall i,j\in 1,2,\ldots,n\colon B(y_i,\sigma_i)\cap B(y_j,\sigma_j)\neq\Phi \qquad (\alpha)$$

and

$$\bigcap_{i=1}^{n} B(y_i,\sigma_i)=\Phi. \qquad (\beta)$$

Equation (α) also means $\sigma_i\oplus(-\sigma_j)\in C(x_i,x_j)$.

Let us construct a graph $P=(X,E)$, see Fig. 1.

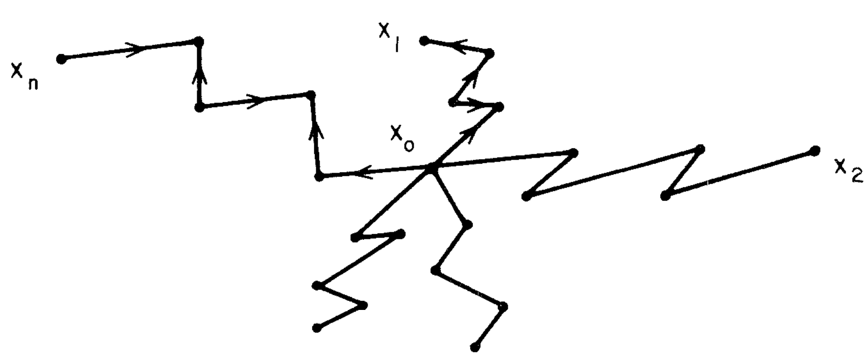

FIGURE 1

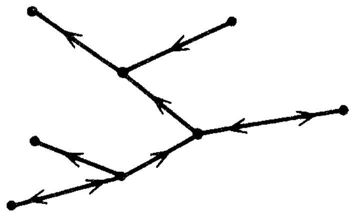

FIGURE 2

Explanation. For $i = 1, 2,..., n$, the path (x_i, x_0) is constructed in such a way that σ_i is associated with this path, with no degeneration. Two paths (x_i, x_0) and (x_j, x_0) $(j \neq i)$ have only x_0 as a common vertex.

It is clear then that if we write $h(x_1) = y_1,..., h(x_n) = y_n$, $A = (x_1,..., x_n)$, Theorem I(1) is satisfied by (A, h) and moreover, it is impossible to construct $h^\circ(x_0)$. Conversely, it is clear that (1) is necessary for the existence of the extension h° and we must in fact construct this extension.

If $x_0 \in X - A$, we must in fact construct $h(x_0)$ in such a way that (1) is satisfied by ($A \cup x_0 = A'$, h so constructed). If we can do this, obviously we may conclude by inductive process on the set X.

Let us consider $\sigma \in C(x, x_0)$; $x \in A$ and $\sigma' \in C(x', x_0)$; $x' \in A$. We have $\sigma \oplus (-\sigma') \in C(x, x')$ and because of (1) $\sigma + (-\sigma') \in C(h(x), h(x'))$. This means in fact, $B(h(x), \sigma) \cap B(h(x'), \sigma') \neq \Phi$. By the Helly property, we may assert

$$\bigcap B(h(x), \sigma) \neq \Phi, \qquad x \in A, \quad \sigma \in C(x, x_0).$$

If we choose $u_0 \in \bigcap B(h(x), \sigma)$, we may write $h(x_0) = y_0$, $x \in A$, $\sigma \in C(x, x_0)$ and clearly for $A \cup x_0$ and h so constructed, hypothesis (1) is still satisfied. We conclude easily.

Some Examples

If we consider a tree whose edges are oriented in an arbitrarily way, we obtain an oriented graph (Y, F) which satisfies the hypothesis of Theorem I (see Fig. 2).

If we consider an elementary cycle without any chord, with length $\geqslant 4$, and whose edges are oriented in an arbitrary way, we obtain (Fig. 3) and oriented graph (Y, F) which never satisfies the hypothesis of Theorem I.

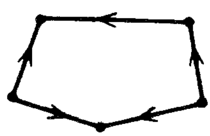

FIGURE 3

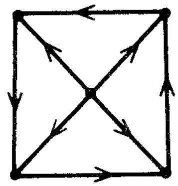

FIGURE 4

Remark. A graph (Y, F) may contain such a cycle as induced subgraph, and satisfy the hypothesis of Theorem I (see Fig. 4).

3. EXTENSION THEOREMS FOR THE PARTIAL ORDERS

Let us consider $(G, >)$ a partial order. It generates an oriented graph $P(G) = (G, E)$ (comparability oriented graph: $[\mathbf{x}, \mathbf{y}] \in E \Leftrightarrow x < y$). A homomorphism between two-order G, G' is in fact a homomorphism between the two graphs $P(G)$ and $P(G')$. In fact, the easy identification between G and $P(G)$ allows us to define for $x, y \in G$, $\sigma \in C_0$ the subset $C(x, y)$ of C_0 and the subset $B(x, \sigma)$ of G. The codes which will be useful for the orders are the code $\sigma = (a_1, \dots, a_n)$, in which no consecutive coefficients are equal. We call C_1 the family of these codes.

EXAMPLE. $((1, -1, 1, -1) \in C_1; (1, 1, -1, 1, -1) \notin C_1)$. We call the number n the length of the code σ. We call σ_n the code

$$\underbrace{(1, -1, 1, \dots)}_{n\text{ times}} \text{ and } \sigma'_n \text{ the code } \underbrace{(-1, 1, -1, 1, \dots)}_{n\text{ times}} \cdot (\sigma_n, \sigma'_n \in C_1).$$

A partial order G is said to be a multilattice if for every $x, y \in G$, there exist $z, z' \in G$, with $z' \leqslant \binom{x}{y} \leqslant z$.

THEOREM II. *Given two partial orders $(G, <)$, $(G', <)$, G at most countable, a finite subset A of G, and a function h from A to G', we suppose that the sections of G' satisfy the Helly property and also that G' is a multilattice. Then the following equivalency is true*:

$$h \text{ may be extended into a homomorphism } h^\circ \text{ from } G \text{ to } G' \Leftrightarrow (\forall x, y \in A, x < y \Rightarrow h(x) \leqslant h(y)) \quad (2)$$

Obviously (2) *is necessary.*

Conversely, we consider $x_0 \in G - A$, and we try to construct $h(x_0)$ in such a way that $A' = A \cup x_0$ and h so completed still satisfy (2). It will be clearly sufficient to do that in order to conclude by an inductive process on the set G. We say

$$u = \{x \in A, x < x_0\}, \qquad v = \{y \in A, y > x_0\}.$$

For example, if u is empty, then it is sufficient to choose $h(x_0) = y_0$ such that $\forall y \in v$, $h(y) \geqslant y_0$. (It will be possible since G' is a multilattice and v is finite.) If both u and v are nonempty, we say that

$$\forall x \in u, \quad y \in v, \quad x < y \Rightarrow h(x)^+ \cap h(y)^- \neq \Phi,$$

$$\forall x, z \in u, \quad x = y \Rightarrow h(x)^+ \cap h(z)^+ \neq \Phi,$$

$$t, y \in v, \quad z < y, \quad x < t \qquad \text{and} \qquad h(y)^- \cap h(t)^- \in \Phi.$$

By the Helly property, we can conclude

$\bigcap h(x)^+ \cap h(y)^- \neq \Phi$, $x \in u$, $y \in v$, and we choose $y_0 = h(x_0)$ with $y_0 \in \bigcap h(x)^+ \cap h(y)^-$, $x \in u$, $y \in v$.

Remark. All the lattices satisfy the property of being a multilattice, with sections satisfying the Helly property. (See [2] or [3].)

If $\{x_i^+\}_{i \in I}$, $\{y_j^-\}_{j \in J}$ (I and J are finite) is a family of pairwise intersecting sections of a lattice, it is clear that both $\bigvee_{i \in I} x_i$ and $\bigwedge_{j \in J} y_j$ are in $\bigcap_{i \in I} x_i^+$, $\bigcap_{j \in J} y_j^-$. There are other examples, however. The set of the compact intervals of the real line, with the dominating order

$\Leftrightarrow x < y$

also satisfies this property, without being a lattice. (If $x \cap y \neq \Phi$, there is no smallest interval among the intervals which dominate both x and y.)

THEOREM III. *We consider two partial orders* $(G, >)$, $(G', >)$, G *at most countable, a finite subset* A *of* G, *and we suppose that the family of the subsets of* G', $B(y, \sigma)$, $y \in G'$, $\sigma \in C_1$, *satisfies the Helly property. Then a function* h *from* A *to* G' *being given, the following equivalency is true*:

h may be extended into a homomorphism from G to $G' \Leftrightarrow [\forall x, y \in A, \sigma \in C(x, y), \sigma \in C_1 \Rightarrow \sigma \in C(h(x), h(y))]$. (3)

In fact there is an equivalency between the Helly property for the subsets $B(y, G)$, $y \in G'$, $\sigma \in C_1$, *and the validity of criterion* (3).

The connection between Theorems I and III is obvious. If $(a_1, ..., a_n)$ is equal to a code in C_0, we call σ^0 the code obtaned by confusing every maximal sequence of identical coefficients in σ in one unique coefficient with the same value.

$$\sigma = (1, 1, 1, -1, 1, -1, -1, 1) \Rightarrow \sigma^\circ = (1, -1, 1, -1, 1),$$

σ° is in C_1. We only have to say then that in a partial order G, the following equivalency is true:

$$\sigma \in C(x, y) \Leftrightarrow \sigma^\circ \in C(x, y) \qquad (x, y \in G).$$

The conclusion is then obvious by the application of Theorem I.

Of course every partial order G' satisfying the hypothesis of Theorem II (multilattice, with sections having the Helly property), satisfies Theorem III also. We may find some other orders.

THEOREM IV. *For $n \in N$, we give the partially ordered sets T_n, T'_n, G_n in Fig. 5. The representation graphs for T_n, T'_n, G_n in Fig. 5 are defined as follows: There is an arrow from x to $y \Leftrightarrow x < y$ and there is no z with $x < z < y$. Then T_n, T'_n, G_n satisfy the hypothesis on G' in Theorem* III.

Proof. It is obvious in the case of T_n and T'_n (the sets $B(x, \sigma)$, $\sigma \in C_1$, $x \in T_n$, are in fact intervals of the set totally ordered $(0, 1, 2, \ldots, n)$. In the case of G_n, we must proceed by induction on n. If $\sigma = (a_1, \ldots, a_n$ is a code of

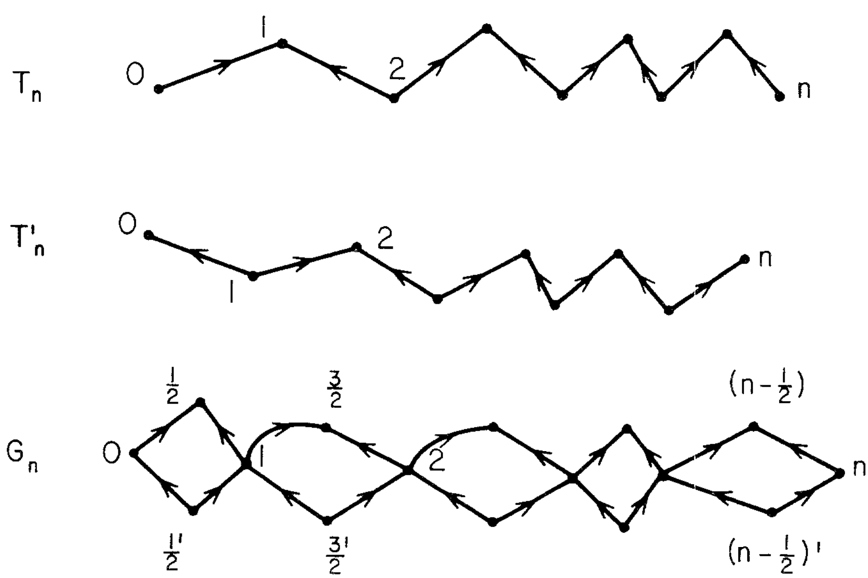

FIGURE 5

C_1, we call σ_1 the code $(a_2,..., a_n)$ also in C_1. We identify the suborder of G, $G-(n, n-\frac{1}{2}, (n-\frac{1}{2})')$ with G_{n-1}. We say that

$$\begin{aligned}\text{Length of } \sigma \geqslant 1 \Rightarrow B(n-\tfrac{1}{2}, \sigma) \cap G_{n-1} &= B(n-1, \sigma) \cap G_{n-1}, \\ \Rightarrow B((n-\tfrac{1}{2})', \sigma) \cap G_{n-1} &= B(n-1, \sigma) \cap G_{n-1}, \\ \text{Length of } \sigma \geqslant 2 \Rightarrow B(n, \sigma) \cap G_{n-1} &= B(n-1, \sigma_1) \cap G_{n-1}.\end{aligned}$$

These remarks clearly permit the conclusion.

4. Application to the Notion of Dimension

As in the case of the simple graphs, we may deduce from this several concepts of dimension. Given a partial order $(G, >)$ (at most countable), $x, y \in G$, with $x < y$, we see that the function h from $\{x, y\}$ to Z (set of the relative integers) defined by $h(x) = 0$, $h(y) = 1$, may be extended into a homomorphism from G to Z, (Theorem II). If $x \sim y$ (x incomparable with y), the two functions h, h' from $\{x, y\}$ to Z defined by: $h(x) = 0$, $h(y) = 1$, $h'(x) = 1$, $h'(y) = 0$ may be extended into 2 homomorphisms h°, h'° from G to Z.

We deduce a coefficient $\alpha(G)$ defined by

$\alpha(G)$ = smallest number of homomorpisms $h_1, h_2,..., h_p$ from G to Z such that, for every x, y in G, we have $x < y \Rightarrow$ there exists $i \in (1, 2,..., n)$ with $h_i(x) < h_i(y)$, $x \sim y \Rightarrow$ there exists $i, j \in (1, 2,..., n)$ with $h_i(x) < h_i(y)$, $h_j(y) < h_j(x)$. (A)

If G is infinite, $\alpha(G)$ may be infinite.

Recall: Dimension of a Poset

A realizer of a poset $(G, <)$ is a family of linear order relations on G, (linear extensions), $R_1, R_2,..., R_k$, such that $x < y (x, y \in G) \Leftrightarrow \forall i \in 1, 2,..., k$, $xR_i y$. The dimension of $(G, <)$ is then the smallest cardinality k of a realizer of $(G, <)$. (Ref. (Dushnik and Miller [4], Trotter [11]). Therefore we may assert

PROPOSITION I. *Here, $\alpha(G)$ as defined above is in fact the dimension of the poset $(G, <)$.*

Proof. Let us call $\dim(G)$ the coefficient of dimension of G. Obviously, we have $\dim(G) \geqslant \alpha(G)$, since every linear order on G, compatible with our relation $>$, may be interpreted in terms of homomorphisms from G to Z. Conversely, if we consider a homomorphism h from G to Z, it is easy to consider it as a homomorphism from G to R (the real numbers) and to modify it slightly in a homomorphism h° from G to R with

$$\forall x \neq y \in G,\ h^\circ(x) \neq h^\circ(y), \qquad h(x) < h(y) \Rightarrow h^\circ(x) < h^\circ(y),$$

h° induces at its turn a homomorphism from G to Z, which we also call h°. If $h_1, \ldots, h_n$ is a family of homomorphisms from G to Z defining $\alpha(G)$ as in (A), we see that $h_1^\circ, \ldots, h_n^\circ$ has the same property, and may also be interpreted as linear orders on G defining the Dushnik–Miller coefficient. So we conclude.

We may proceed the same way, replacing Z by any lattice, and also replacing Z by the set of the compact intervals of R, with the dominating order. In that case we return to the coefficient of interval dimension of the poset G, denoted by Int dim(G). (See Trotter and Moore [10].) We may also define a coefficient as follows: Given a finite order $(G, <)$, $x, y \in G$, a homomorphism h from G to T_n, T'_n, or G_n is said to be separating for x and y if we have

Case 1. $h(x) = 0$; $h(y) = n$; $\sigma_n \in C(x, y)$.

Case 2. $h(x) = 0$; $h(y) = n$; $\sigma'_n \in C(x, y)$.

Case 3. $h(x) = 0$; $h(y) = n$; σ'_{n+1} and $\sigma_{n+1} \in C(x, y)$.

A family $h_1, \ldots, h_p$ of homomorphisms, each one from G to one of the orders of the family $\{T_n\}_{n \in N} \cup \{T'_n\}_{n \in N} \cup \{G_n\}_{n \in N} = F$ will be called a separating family for G, if we have $\forall x, y \in G$, there exists $i \in 1, 2, \ldots, p$, with h_i is separating for x, y. Then we define $\gamma(G)$ as equal to a strong dimension of G equals to the smallest cardinality of a separating family for G.

5. Contractibility: Fixed Point Problems

We shall say that a finite poset $(G, <)$ is a Helly poset if its comparability graph is connected and if the family of subsets $B(x, \sigma)$, $x \in G$, $\sigma \in C_1$, satisfies the Helly property. We shall also say that $x_0 \in G$ stops down $y_0 \in G$ if we have

$$x_0 < y_0; \qquad y \in G, \quad y < y_0 \Rightarrow y \leqslant x_0,$$

that $x_0 \in G$ stops up $y_0 \in G$ if we have

$$x_0 > y_0; \qquad y \in G, \quad y > y_0 \Rightarrow y \geqslant x_0,$$

that x_0 stops y_0 if x_0 stops down or stops up y_0. If we refer to the terminology of Rival [9], we see that we get $y_0 \in G$ is irreducible in

$G \Leftrightarrow$ there exists x_0 such that x_0 stops y_0. We shall say that $(G, <)$ is contractible or based on Rival [9], that $(G, <)$ is dismantlable by being ireducible if we may order the elements of G, $G = (x_1,..., x_n)$ in such way that $\forall i \leqslant 1, 2,..., n-1$, x_i is irreducible in $G - (x_1,..., x_{i-1})$.

PROPOSITION II. *Every poset T_n, T'_n, G_n $(n \geqslant 0)$, is contractible.*

It is a simple verification. We now connect our notion of a Helly poset to this notion of contractability.

THEOREM V. *Helly Poset is contractible.*

We know [9], that if G does not contain any crown (alternating cycle), G must be contractible. It does not seem obvious that a Helly poset cannot contain any crown in spite of the fact that G itself cannot be equal to a crown. In fact, we shall consider a Helly poset $(G, <)$, and prove our assertion by induction on $|G|$. Let us consider $x_0 \in G$ and a code $u = (a_1,..., a_p) \in C_p) \in C_1$ such that

$$a_1 = I; \qquad B(x_0, u) = G; \qquad \text{if } u' = (a_1,..., a_{p-1}), \quad B(x_0, u') \neq G.$$

It is obvious that u exists.

Let us suppose $a_p = -1$. (The reasoning will be the same if $a_{p-1} = 1$.) We consider $y_0 \in B(x_0, u) - B(x_0, u')$, maximal with this property. If $y > y_0$, we obviously have $B(y_0, 1) \cap B(y, -1) \neq \Phi$; $B(y_0, 1) \cap B(x_0, u') \neq \Phi$; (Since y_0 is maximal in $B(x_0, u) - B(x_0, u')$.) $B(y, -1) \cap B(x_0, u') \neq \Phi$; Therefore $\bigcap_{y > y_0} B(y, -1) \cap B(y_0, 1) \cap B(x_0, u') \neq \Phi$.

If z_0 is in this intersection, we clearly have z_0 stops down up y_0. In order to achieve the proof, we only have to verify that $G - y_0$, if $(b_1,..., b_k) \in C_1$ and is associated with a path T in the comparability graph of G, then $(b_1,..., b_k)$ is also associated with the path in the comparability graph of $G - y_0$, which is obtained from T by replacing y_0 with z_0 every time it appears in T.

COROLLARY I. *If $(G, <)$ is a Helly poset, every homomorphism from G into itself has a fixed point.*

It is sufficient to apply Theorem V and a theorem of Rival [9], which asserts that if a finite poset is contractible, then every homomorphism from this poset into itself possesses a fixed point. We are going to show now that it is possible to obtain a similar result concerning the antitone maps from the poset G into itself. We are going to show now that it is possible to obtain a similar result concerning the antitone maps from the poset G into itself.

Remark. A function h from G to G is an antitone map (G is a partial order), if we have $x, y \in G$, $x \leqslant y \Rightarrow h(x) \geqslant h(y)$.

THEOREM VI. *If G is a finite contractible partial order, and if h is an antitone map from G to G, we have one of two possibilities which is true*:

(1) *There exists $\alpha \in G$, with $h(\alpha) = \alpha$.*

(2) *There exists $\alpha, \beta \in G$, with $\alpha < \beta$, $h(\alpha) = \beta$, $h(\beta) = \alpha$.*

Remark. Conditions (1), (2) may be summarized. There exist $\alpha, \beta \in G$, with $\alpha \leqslant \beta$ and $h(\alpha) = \beta$, $h(\beta) = \alpha$.

We proceed by induction on G. We consider $x_0, y_0 \in G$, with x_0 steps (for instance stops down) y_0. We define the following retraction R from G to $G - y_0$:

$$x \neq y_0 \Rightarrow R(x) = x; \qquad R(y_0) = x_0.$$

By induction there must occur in $G - y_0$ one of the following:

(1) There exists $\alpha' \in G - y_0$ with $R \circ h(\alpha') = \alpha'$.

(2) There exist α', $\beta' \in G - y_0$ with $\alpha' < \beta'$; $R \circ h(\alpha') = \beta'$; $R \circ h(\beta') = \alpha'$.

Case 1. The only problem rises when we have $h(x_0) = y_0$; $h(y_0) \neq x_0, y_0$. Clearly, however, we deduce that since h is antitone, $x_0 < y_0 \Rightarrow h(y_0) < h(x_0) = y_0$, and since x_0 stops down y_0, $h(y_0) < x_0$. Repeating this, we get $h^2(y_0) \geqslant h(x_0) = y_0$, $h^3(y_0) \leqslant h(y_0)$, $h^4(y_0) \geqslant h^2(y_0)$, and so on. We get two sequences

$$y_0 \leqslant h^2(y_0) \leqslant h^4(y_0) \leqslant \cdots \leqslant h^{2n}(y_0) \leqslant \cdots,$$
$$x_0 \geqslant h(y_0) \geqslant h^3(y_0) \geqslant \cdots \geqslant h^{2n+1}(y_0) \geqslant \cdots.$$

Since G is finite, we must get, when n is great enough

$$\underset{\alpha}{h^{2n}(y_0)} = h^{2n+2}(y_0) \geqslant \underset{\beta}{h^{2n+1}(y_0)} = h^{2n+3}(y_0).$$

Clearly, we get our result.

Case 2. The only problem arises from the situation

$$\exists x \in G, x \neq y_0 \text{ with } \begin{cases} h(x) = y_0, \\ h(x_0) = x, x \text{ and } x_0 \text{ comparable.} \end{cases}$$

Let us suppose that $x_0 < x$. Then we have $h(x_0) \geqslant h(x)$, which means $x \geqslant y_0 > x_0$ and we get a contradiction. Thus we have $x_0 > x$ and

$y_0 > x_0 > x$. We also have $h(y_0) \leqslant h(x_0) = x$ (h antitone), $h^2(y_0) \geqslant h(x) = y_0$, $h^3(y_0) \leqslant h(y_0)$, and so on. We proceed as in Case 1 and conclude.

COROLLARY II. *If $(G, <)$ is a Helly poset and if h is an antitone map from G into itself, then there $x_0 \in G$ such that*

$$h(x_0) \leqslant x_0; \qquad h^2(x_0) = x_0.$$

It is an obvious consequence of Theorems V *and* VI.

Note that this result may be compared to the fixed-edge theorem of Nowakowski and Rival [7]: Given a finite graph $G = (X, E)$, with loops and no multiple edges (reflexive graph), for every homomorphism h from G into itself, we may find $[a, b] \in E$ such that $\{h(a), h(b)\} = \{a, b\}$, if and only if G is connected and without any cycle (G is a tree).

We may also propose an other proof of Corollary III which does not use Theorem V. Let us consider a separating family of homomorphisms $f_1, ..., f_p$, from G to posets of the family

$$(T_n)_{n \in N} \cup (T'_n) \cup (G_n) = F.$$

It gives rise to a homomorphism f from G to an ordered set H, which is a product of posets of the class F. Here, H is contractible. (The product of two contractible posets is contractible. See Baclawski and Bjorner [1] or Duffus and Rival [5].) In fact, we may consider G as a suborder of H. By the definition of a separating family, we see that

$$u \in C(f(x), f(y)), \qquad x, y \in G \Leftrightarrow u \in C(x, y), \qquad (u \in C_1).$$

This means that considering G as a suborder of H, through the embedding f, we may apply Theorem III to the identical function I from G to G and extend it into a retraction r from H to G. Theorem VI gives us $\exists x_0 \in H$, with

$$h(r(x_0)) \leqslant x_0; \qquad \text{homomor}(x_0) = x_0.$$

Clearly, this means that $x_0 \in G$, and therefore that x_0 is the solution to our problem. Note that we could have applied the same reasoning to prove Corollary I. This proof technique using the notion of retraction and embedding may be connected to the work about the retractions found in [5].

Let us now give a last fixed-point result which involved the automorphism group of a Helly poset. If $(G, <)$ is a finite poset, an automorphism of G is a one-to-one homomorphism from G into itself. These automorphism form a group for the composition of the homomorphisms, denoted by $A(G)$. We get

THEOREM VII. *Given $(G, <)$ a Helly poset and $A(G)$ its automorphism group. Then there exists in G a chain which is invariant under the action of $A(G)$. (In particular there exists a point in G which is invariant under the action of $A(G)$.)*

Proof. We proceed by induction on $|G|$, and suppose that G is not a total order (a trivial case). Let us consider a code $u = (a_1, ..., a_p) \in C_1$ such that

$$\forall x, y \in G, B(x, u) \cap B(y, u) \neq \Phi. \tag{I}$$

Then (I) does not hold if we replace u with $u' = (a_1, ..., a_{p-1})$. (Obviously u exists and $p \geqslant 1$.) We can pose

$$G' = \bigcap_{x \in G} B(x, u) \neq \Phi$$

(Helly property). Obviously $|G'| < |G|$. Since G is not a total order. Also G' is globally invariant under the action of $A(G)$. It will be easy to conclude if we can prove that G' is a Helly poset. For this we only have to prove that if $x_0, y_0 \in G'$ and if $v \in C_1$ is in $C_G(x_0, y_0)$, it is also in $C_{G'}(x_0, y_0)$. Let us write $v = (b_1, ..., b_k)$, $v' = (b_1, ..., b_{k-1})$, and proceed by induction on k, where $B(x_0, v') \cap B(y_0, -b_k) \neq \Phi$ and $x_0, y_0 \in \bigcap_{x \in G} B(x, u) \Rightarrow$ (Helly property), there exists $z_0 \in G'$ with $v' \in C_G(x_0, z_0)$, $b_k \in C_G(z_0, y_0)$.

Clearly, we see that this process gives us the result.

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