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Algorithm of Finding Hypo-Critical Path in Network Planning

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Abstract

Network planning technology could be used to represent project plan management, such Critical Path Method (CPM for short) and Performance Evaluation Review Technique (PERT for short) etc. Aiming at problem that how to find hypo-critical path in network planning, firstly, properties of total float. free float and safety float are analyzed, and total float theorem is deduced on the basis of above analysis; and secondly, simple algorithm of finding the hypo-critical path is designed by using these properties of float and total theorem, and correctness of the algorithm is analyzed. Proof shows that the algorithm could realize effect of whole optimization could be realized by part optimization. Finally, one illustration is given to expatiate the algorithm.

*Keywords:*project management; operational research; network planning; total float theorem; optimization algorithm; float

1. Introduction

The critical path method (abbreviate CPM) network planning [1,2] was founded in 1956. It is the most common technology in graphics mode to take order with project plan. This technology prompt decision-maker to notice that they should focus their attention on critical path, as activities in this path are usually considered to be the most critical part for one project [3]. Definitely speaking, use the technology could scientifically work out the float of each activity in project, and then identify the critical path, point out the critical tache of project, and measure importance of each activity, thereby improving decision-making ability and management level of decision-maker. For measure importance of activity, importance of arc need to be measured in Activity-on-Arc representation network, and importance of node need to be measured in Activity-on-Node representation network [4]. The float (means mobile time) is the core of CPM network, and its start gone with the appearance of CPM network planning. The conceptions of total float, safety float, free float, and interference float were proposed by Battersby and Thomas [5] in 1967 and 1969 respectively, and the conclusion was deduced that the path whose total float is zero is the critical path. The conception of node float [5-9] was proposed by Elmaghraby in 1977, and these floats were analyzed by him[7]. In addition, for calculating the time parameters, the non-uniqueness of Activity-on-Arc representation of CPM network planning may lead to non-unique time parameters. To solve this problem, one algorithm was proposed by Elmaghraby and Kamburowski [10-12] to amend time parameters

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of in-dummy node whose immediate predecessor activities are all dummy activities and out-dummy node whose immediate successor activities are all dummy activities.

However, generally in fact, not only need the critical path to be found out [3,13], but also some important non-critical path [14] . For example, the duration of one project is 1000 days, viz. the length of critical path is 1000 days, if the length of the second longest path which could be named hypo-critical path is 990 days, then it may turn to critical path easily. Therefore, much attention also need be paid on activities in the hypo-critical path. Another interrelated case is time-cost trade off problem [15-19]. It is very important to ensure the maximum effective value of shortening activities' duration for every step of shortening duration process. On one hand, for reduce the steps of shortening, it is necessary to shorten duration as much as possible in every step. On the other hand, useless shortening should be avoided in shortening process, which not only couldn't shorten the period of a project but also increase cost of shortening. The maximum effective shortening is determined by the margin between the different lengths of paths contain critical path and non-critical path.

In this paper, firstly, some important conclusions are deduced base on the conception of total float, free float and safety floa; and secondly, on the basis of these conclusions, one simple algorithm of finding the hypo-critical path is designed. Use the algorithm could obtain the effect that optimizing in whole could be realized by optimizing in local, and the workload of resolving problem could be reduced greatly.

2. Algorithms of calculating time parameter in CPM network planning

2.1 Algorithm of Calculating Time Parameter of Node

In any activity-on-arc representation network, suppose that $P(j)$ and $S(j)$ denote the set of immediate predecessor and immediate successor activities of node (j) respectively, ET_j and LT_j denote the earliest and latest realization time of node (j) respectively, and T_{ij} denotes the duration of activity (i,j). ET_j is defined as the maximum of the earliest completion times of the activities which terminate at node (j), while LT_j is defined as the minimum of the latest allowable start times of the activities which start at node (j). From these, one algorithm of calculating and correcting time parameter of node in activity-on-arc representation network could be designed as follows:

Step 1 Algorithm of calculating time parameter of node.

(1) For $j = 2,3,\cdots,n$, do

$$\begin{cases} ET_1 = 0 \\ ET_j = \max\limits_{(i)\in P(j)} \left\{ ET_i + T_{ij} \right\} \end{cases} \tag{1}$$

(2) Then for $j = n-1, n-2, \cdots, 1$, do

$$\begin{cases} LT_n = ET_n \\ LT_j = \max\limits_{(k)\in S(j)} \left\{ LT_k - T_{jk} \right\} \end{cases} \tag{2}$$

Step 2 Algorithm of correcting time parameter of node.

(1) For $j = 2,3,\cdots,n$, if node (j) is in-dummy node whose immediate predecessor activities are all dummy, do

$$LT_j = \max_{(i)\in P(j)} \{LT_i\}, j = 2,3,\cdots,n \tag{3}$$

(2) Then for $j = n-1, n-2,\cdots,1$, if node (j) is out-dummy node whose immediate successor activities are all dummy, do

$$ET_j = \min_{(k)\in S(j)} \{ET_k\}, j = n-1, n-2,\cdots,1 \tag{4}$$

2.2 Algorithm of Calculating Time Parameter of Activity

In activity-on-arc representation network, for any one activity (i, j), it suppose that ES_{ij} , EF_{ij} , LS_{ij} and LF_{ij} denote the earliest start time, the earliest completion time, the latest start time and the latest completion time of the activity respectively. Then these time parameters could be computed as follows:

$$\begin{aligned} ES_{ij} &= ET_i \\ EF_{ij} &= ES_{ij} + T_{ij} = ET_i + T_{ij} \\ LF_{ij} &= LT_j \\ LS_{ij} &= LF_{ij} - T_{ij} = LT_j - T_{ij} \end{aligned} \tag{5}$$

3. Basic Conception

For the main content of the paper could be described effectively, several relevant conceptions are proposed as follows.

Path The path which marked as μ denotes one pass from start node to terminal node along with directional arc in activity-on-arc representation network. And the length of a path which marked as $L(\mu)$ represents the sum of all activities duration on the path. The longest path which has the maximal length of all is named critical path which marked as μ^{∇} , and activities and nodes on critical path are named critical activities and critical nodes respectively. The path which is only shorter than critical path is named hypo-critical path, and marked as $\mu^{\nabla 1}$

Total Float The total float of activity (i, j) which marked as TF_{ij} is defined as:

$$TF_{ij} = LT_j - ET_i - T_{ij} \tag{6}$$

The total float denotes the time an activity can be delayed without causing a delay in the project.

Safety Float The safety float of activity (i, j) which marked as SF_{ij} is computed as:

$$SF_{ij} = LT_j - LT_i - T_{ij} \tag{7}$$

The safety float of an activity represents the number of periods by which the duration of the activity may be prolong furthest when all its predecessor activities complete at the latest completion time without increasing the completion time of the project.

Free Float The free float of activity (i, j) which marked as FF_{ij},

$$FF_{ij} = ET_j - ET_i - T_{ij} \tag{8}$$

The free float defines the allowable delay in the activity finish time without affecting the earliest start time of its immediate successor activities.

4. Main Theorem

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In the paper, we will find the hypo-critical path by using float, especially total float. Therefore, we study relation between float and path length firstly, and deduce total float theorem as follows:

Lemma 1 Free floats of activities on the longest path $\mu_{1\to i}^{\nabla}$ from start node (1) to any node (i) are all zero, and length $L\left(\mu_{1\to i}^{\nabla}\right)$ of the path is equal to the earliest time ET_i of the node (i), and also equal to the earliest start time ES_{ij} of immediate successor activity (i, j) of the node (i), viz.

$$L\left(\mu_{1\to i}^{\nabla}\right) = ET_i = ES_{ij} \tag{9}$$

Proof In activity-on-arc representation network, according to conception and algorithm of time parameter of activity, the earliest start time of any activity is equal to the maximal earliest finish time of immediate predecessor activity of the activity, viz.

$$ES_{ij} = \max\left\{EF_{k_1 i},\ EF_{k_2 i},\ \cdots,\ EF_{k_n i}\right\} \tag{10}$$

Suppose that

$$ES_{ij} = EF_{ki} \tag{11}$$

it means that for all immediate predecessor activities of any activity (i, j), there are at least one activity (k, i) whose the earliest finish time being equal to the earliest start time of activity (i, j).

Suppose that any path between start node (1) and node (i) of activity (i, j) is marked as $\mu_{1\to i} = (1) \to (a) \to (b) \to (c) \to \cdots \to (e) \to (f) \to (g) \to (i)$, for duration of any activity (u, v) could be computed as $T_{uv} = EF_{uv} - ES_{uv}$, length $L(\mu_{1\to i})$ of the path $\mu_{1\to i}$ could be computed as follows:

$$\begin{aligned} L(\mu_{1\to i}) &= T_{1a} + T_{ab} + T_{bc} + \cdots + T_{ef} + T_{fg} + T_{gi} \\ &= (EF_{1a} - ES_{1a}) + (EF_{ab} - ES_{ab}) + \cdots \\ &\quad + (EF_{fg} - ES_{fg}) + (EF_{gi} - ES_{gi}) \\ &= (EF_{1a} - ES_{ab}) + (EF_{ab} - ES_{bc}) + \cdots \\ &\quad + (EF_{gi} - ES_{ij}) + (ES_{ij} - ES_{1a}) \end{aligned} \tag{12}$$

According to formula (10),

$$EF_{1a} - ES_{ab} \le 0,\ EF_{ab} - ES_{bc} \le 0,\ \cdots ,$$
$$EF_{fg} - ES_{gi} \le 0,\ EF_{gi} - ES_{ij} \le 0$$

Then according to formula (12),

$$L(\mu_{1\to i}) \le ES_{ij} - ES_{1a}$$

In activity-on-arc representation network, $ES_{1a} = 0$, therefore

$$L(\mu_{1i}) \le ES_{ij} \tag{13}$$

According to formulae (12) and (13), one path whose length being equal to ES_{ij} could be found out, viz.

$$EF_{1a} - ES_{ab} = 0,\ EF_{ab} - ES_{bc} = 0,\ \cdots ,$$
$$EF_{fg} - ES_{gi} = 0,\ EF_{gi} - ES_{ij} = 0$$

According to conception of free float, free floats of activities on the path are all zero. And then deduced by formula (11), this path is the longest path between start node (1) and node (i), therefore its length which marked as $L(\mu_{1\to i}^{\nabla})$ is equal to ES_{ij} .

Lemma 2 Safety floats of activitis on the longest path $\mu_{i\to n}^{\nabla}$ from start node (1) to any node (i) are all zero, and length $L(\mu_{i\to n}^{\nabla})$ of the path is equal to value of length $L(\mu^{\nabla})$ of critical path minus the latest time of node (j) , and also equal to margin of $L(\mu^{\nabla})$ minus the latest finish time of immediate predecessor activity (i,j) of node (j) , viz.

$$\begin{aligned} L(\mu_j^{\oplus}) &= L(\mu^{\nabla}) - LT_j \\ &= L(\mu^{\nabla}) - LF_{ij} \end{aligned} \tag{14}$$

Proof The proof is similar with proof of Lemma 1, and the details will not be deduced.

On the basis of above Lemmas, theorem which represent relations between total float and path's length are deduced as follows:

Total Theorem The total float of any activity (i,j) is equal to margin of length $L(\mu^{\nabla})$ of the critical path minus length $L(\mu_{ij}^{\nabla})$ of the longest path marked as μ_{ij}^{∇} which passes the activity (i,j), viz.

$$TF_{ij} = L(\mu^{\nabla}) - L(\mu_{ij}^{\nabla}) \tag{15}$$

Proof According to Lemma1 and 2, length $L(\mu_{ij}^{\nabla})$ of the longest path which pass the activity (i,j) is equal to

$$\begin{aligned} L(\mu_{ij}^{\nabla}) &= L(\mu_{1\to i}^{\nabla}) + T_{ij} + L(\mu_{j\to n}^{\nabla}) \\ &= ES_{ij} + T_{ij} + \left(L(\mu^{\nabla}) - LF_{ij}\right) \\ &= EF_{ij} + L(\mu^{\nabla}) - LF_{ij} \\ &= L(\mu^{\nabla}) - \left(LF_{ij} - EF_{ij}\right) \end{aligned}$$

According to conception of total float, then

$$L(\mu_{ij}^{\nabla}) = L(\mu^{\nabla}) - TF_{ij}$$

viz.

$$TF_{ij} = L(\mu^{\nabla}) - L(\mu_{ij}^{\nabla})$$

The theorem is correct.

5. Algorithm of Finding Hypo-Critical Path

5.1 Description of the Algorithm

According to above theories, the algorithm of finding hypo-critical path could be designed as follows:

Step 1 Compute time parameters of each node and activity by using formulae (1)~(5).

Step 2 Compute total float of each activity by using formula (6), and find out non-critical activity with positive minimal total float which could be marked as (i,j).

Step 3 Find out the longest path $\mu_{1\to i}^{\nabla}$ between start node (1) and node (i).

(1) Compute free floats of immediate predecessor activities of node (i), and find out activity with zero free float which could be marked as (h,i);

(2) Compute free floats of immediate predecessor activities of node (h), and find out activity with zero free float which could be marked as (g,h);

……

This process won't stop until to start node (1), and find out the longest path $\mu_{1\to i}^{\nabla}$ between start node (1) and node (i) which composed by these activities.

Step 4 Find out the longest path $\mu_{i\to n}^{\nabla}$ between node (j) and terminal node (n).

(1) Compute safety floats of immediate successor activities of node (j), and find out activity with zero safety float which could be marked as (j,k);

(2) Compute safety floats of immediate successor activities of node (k), and find out activity with zero safety float which could be marked as (k,l);

……

This process won't stop until to terminal node (n), and find out the longest path $\mu_{i\to n}^{\nabla}$ between node (j) and terminal node (n) which composed by these activities.

The path $\mu_{ij}^{\nabla} = \mu_{1\to i}^{\nabla} \to (i,j) \to \mu_{j\to n}^{\nabla}$ is the hypo-critical path $\mu^{\nabla 1}$, and its length is $L(\mu^{\nabla 1}) = L(\mu^{\nabla}) - TF_{ij}$.

5.2 Analysis on Correctness of Algorithm

According to total float theorem, if total float TF_{ij} is positive minimal, it means that $L(\mu^{\nabla}) - L(\mu_{ij}^{\nabla})$ is positive minimal, therefore path μ_{ij}^{∇} is the longest non-critical path which is only shorter than critical μ^{∇} in network, viz. the hypo-critical path $\mu^{\nabla 1}$. Therefore the activity with positive minimal total float need be found out firstly, and it proves step 1 and 2 are correct.

Then according to lemma 1 and 2, the path which found out by step 3 and 4 is the longest path μ_{ij}^{∇} which passes activity (i,j). And for total float TF_{ij} of activity (i,j) is positive minimal, the path μ_{ij}^{∇} is the hypo-critical path $\mu^{\nabla 1}$.

Therefore, by running step 1~4, the hypo-critical path $\mu^{\nabla 1}$ could be found out. The algorithm is correct.

6. Algorithm of Finding Hypo-Critical Path

The CPM network planning of one project is described in Figure 1. Try to find out the hypo-critical path.

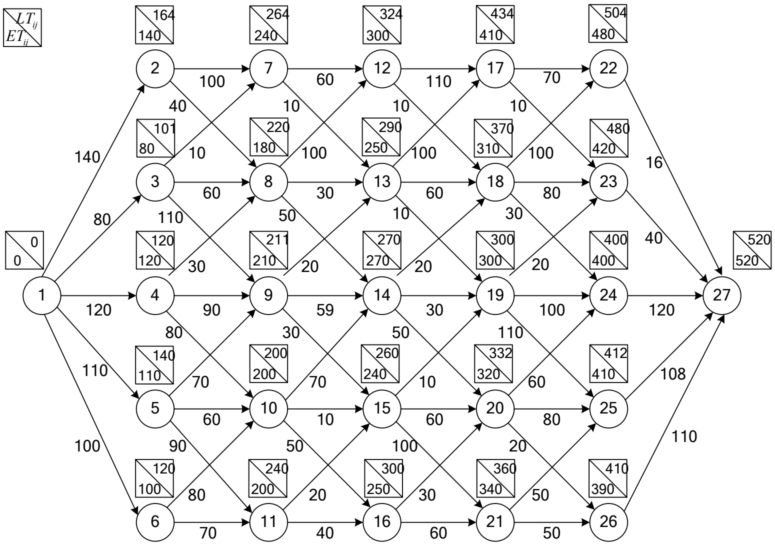

Figure 1. Network planning

Step 1 Compute time parameters of each node and activity by using formula (1)~(5), which showed in Figure 1.

Step 2 Compute total float of each activity by using formula (6), and total floats of non-critical activities $(4,9)$ and $(9,14)$ are positive minimal ones.

Step 3 Find out the longest path $\mu_{1\to 4}^{\nabla}$ and $\mu_{1\to 9}^{\nabla}$ that

$$\mu_{1\to 4}^{\nabla}=(1)\to(4)$$
$$\mu_{1\to 9}^{\nabla}=(1)\to(4)\to(9)$$

Step 4 Find out the longest path $\mu_{9\to 27}^{\nabla}$ and $\mu_{14\to 27}^{\nabla}$ that

$$\mu_{9\to 27}^{\nabla}=(9)\to(14)\to(19)\to(24)\to(27)$$
$$\mu_{14\to 27}^{\nabla}=(14)\to(19)\to(24)\to(27)$$

Therefore, the hypo-critical path $\mu^{\nabla 1}$ is

$$\begin{aligned}\mu^{\nabla 1}&=\mu_{4,9}^{\nabla}=\mu_{9,14}^{\nabla}\\&=(1)\to(4)\to(9)\to(14)\to(19)\to(24)\to(27)\end{aligned}$$

and its length is

$$\begin{aligned}L\left(\mu^{\nabla 1}\right)&=L\left(\mu^{\nabla}\right)-TF_{4,9}\\&=520-1=519\end{aligned}$$

7. Conclusion

If an algorithm is designed to resolve the problem of finding the hypo-critical path in network planning, it helps to resolve many problems in management, and particularly provide an important idea to resolve other relevant problems. Base on these, it could accelerate development of optimization theory of network planning and advance level of project management effectively.

The maximal advantage of the algorithm used to find the hypo-critical path is that the longest paths which found out from few paths are also the longest paths in the whole network. It means that the effect that optimizing in whole could be realized by optimizing in local could be achieved by using the algorithm, and the workload of calculation could be reduced greatly. As the orientation of study in future, we should improve the algorithm to make it be more feasible.

The academic base of the algorithm is total float theorem. Not only is the theorem very important to resolve the problem of finding the hypo-critical path, but also is the base of studying other relevant problems, and provide new theory and idea to study network planning. Seeing from point of development, the theorem and algorithm in this paper have important significance.

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