The elastic $T$-stress for slightly curved or kinked cracks

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This work presents a solution for the elastic $T$-stress at the tip of a slightly curved or kinked crack based on a perturbation approach. Compared to other exact or numerical solutions the present solution is accurate for considerable deviations from straightness. The $T$-stress variation as crack extends along a curved trajectory is subsequently examined. It is predicted that $T$-stress always keeps negative during crack extension when the crack has an initial negative $T$-stress. In the case of a positive $T$-stress and non-zero first and second stress intensity factors initially accompanying the crack, the $T$-stress is not positive with increasing the extension length until a threshold is exceeded. Based on directional stability criterion with respect to the sign of the $T$-stress, this result implies that for a straight crack with a positive $T$-stress, the crack extension path will not turn immediately and instead keep a stable growth until a critical length is reached. This prediction is consistent with experimental observations.

1. Introduction

Curved or kinked cracks are often observed in brittle or ductile materials as a result of mixed mode loading, microstructural inhomogeneities (e.g. defects, inclusions and interfaces), thermal inhomogeneities and so forth. In the past few decades, curved or kinked cracks have been extensively studied using theoretical, experimental and numerical approaches (e.g. Bilby and Cardew, 1975; Lo, 1978; Cotterell and Rice, 1980; Hayashi and Nemati-Nasser, 1981; Karihaloo et al., 1981; He and Hutchinson, 1989; Leblond and Fleck, 2000; Fett et al., 2008). Most of these investigations mainly focused on the evaluation of the stress intensity factors (SIFs), which characterised the singular stress term near the crack tip. However, it has long been recognised that besides SIFs, the non-singular $T$-stress, acting parallel to the crack in the vicinity of the crack tip (e.g. Williams, 1957), is of great importance in practical cases since $T$-stress may play important roles for crack path stability, for crack constraints and even for microstructure evolution at the crack tip. For example, Cotterell and Rice (1980) proposed a stability criterion that the crack path is stable if the $T$-stress is negative and unstable if the $T$-stress is positive. Melin (2002) developed an improved criterion where the crack path stability is related to the magnitude of the $T$-stress rather than the sign. Furthermore, under a small scale yielding condition, the $T$-stress may have significant influences on the shape and size of the plastic zone at the crack tip (e.g. Larsson and Carlsson, 1973; Rice, 1974; Patil et al., 2008). Particularly, it has been found that a negative $T$-stress will provide an enhancement of the fracture toughness (e.g. Du and Hancock, 1991; Tvergaard and Hutchinson, 1994; Tvergaard, 2008), and consequently the $T$-stress at the crack tip has been chosen as a key reference factor in many classic two-parameter fracture criteria (e.g. Betegon and Hancock, 1991; O’Dowd and Shih, 1991, 1992, 1995). It should be noted that the role of $T$-stress examined in these studies is merely with the crack tip plasticity of small scale yielding. In addition, it is also found that $T$-stress of a very short crack can induce an important effect on the dislocation nucleation and emission in BCC materials and consequently is relevant to the study of nano-cracks in these materials (e.g. Belz and Fischer, 2001; Belz and Machaová, 2004).

Many efforts have been made to calculate $T$-stress. For straight cracks, the $T$-stress may be evaluated using the Eshelby technique (e.g. Kfouri, 1986), the weight function method (e.g. Sham, 1989) and other numerical methods (e.g. Broberg, 2005; Ayatollahi et al., 1998; Fett, 1997). For a specially curved crack under remote loading conditions, Chen et al. (2008) proposed a perturbation solution to evaluate the $T$-stress, and for small kinked and forked cracks Fett et al. (2006) proposed a weight function method based on finite element analysis. It is difficult to gain a general solution of $T$-stress for curved or kinked cracks. To date, the $T$-stress can only be assessed by numerical techniques if the crack has an arbitrary shape. The aim of this paper is to provide an explicit and easily calculated $T$-stress solution for non-straight cracks.

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It is well-known that for non-linear problems (e.g., T-stress of curved or kinked cracks) perturbation analysis may typically simplify the equation system and may lead to a remarkably simple formulation. An example is the perturbation analysis performed by Cotterell and Rice (1980) for the SIFs of slightly curved or kinked cracks. Cotterell and Rice (1980) assumed that the curved crack did not deviate far from a straight crack, and by using the complex potential method of Muskhelishvili (1953), they then obtained a first-order solution of the stress intensity factors in an explicit and simple form. Based on the local symmetry criterion, the first-order solution was used to predict the trajectory of crack growth under mixed mode loading. Subsequently, this result led to the establishment of a widely used crack path stability criterion, namely, the crack path is stable for negative T-stress and unstable for positive T-stress. Following Cotterell and Rice (1980), Karbalaloo et al. (1981) proposed a second-order perturbation analysis of the SIFs for curved or kinked cracks, and Gao and Chiu (1992) developed a perturbation scheme of the SIFs for curved or kinked cracks in anisotropic elastic solids.

It is needed to point out that Cotterell and Rice (1980) incompletely solved complex-potentials of Muskhelishvili (1953) and these incomplete complex-potentials are sufficient to represent the stress intensity factors indicating the normal and shear stress acting across the crack prolongation. This approach is not capable of directly solving the T-stress as the T-stress acts parallel across the crack prolongation. However, it is inspired from Cotterell and Rice (1980) that the solution for T-stress may rely on completely solving the complex-potentials of Muskhelishvili (1953). This paper addresses such an attempt and is laid out as follows: Section 2 proposes the perturbation method for the T-stress solution of a slightly curved crack. Section 3 shows a comparison between the present solutions and the exact solution for circular crack as well as some applications including kinked crack problems and a parent crack with a small curved extension. In Section 4, concluding remarks are made.

2. Perturbation method

2.1. The complete first-order perturbation solution for a slightly curved crack

Following the perturbation method of Cotterell and Rice (1980), we consider an infinite plane with zero stress at infinity and containing a slightly curved crack. As shown in Fig. 1, crack tips are located at $x = \pm a$, crack surfaces are subjected to normal and shear tractions $T_n$, $T_s$, which are equal on top and bottom surfaces, and the deviation of the actual crack from straightness is described by the function $\lambda(x)$. Based on the complex-potential representation of Muskhelishvili (1953), the stress field, for plane strain or generalized plane stress, can be expressed in terms of two analytic functions $\Phi(z)$, $\Psi(z)$ as

\begin{align}
\sigma_{xx} + \sigma_{yy} &= 2\left[\Phi(z) + \Phi'(z)\right], \\
\sigma_{yy} - 2i\sigma_{xy} &= 2\left[z\Phi(z) + \Psi(z)\right],
\end{align}

where $z = x + iy$, and correspondingly the boundary condition is expressed as

$$\Phi(z) + e^{-2i\theta}|z\Phi(z) + \Psi(z)| = -(T_n - iT_s).$$

Introducing the analytic function

$$\Omega(z) = \Phi(z) + z\Phi(z) + \Psi(z),$$

Eq. (2) can be rewritten as

$$\Phi(z) + e^{-2i\theta}|z\Phi(z) - \Psi(z) + \Omega(z)| = -(T_n - iT_s).$$

It is assumed that there are two analytical functions $F(z)$, $W(z)$ which are analytic outside of the straight cut between two crack tips, and their boundary values are $F^+(t)$, $W^+(t)$ on the upper and lower surfaces. $F(z)$ and $W(z)$, in the manner of small parameter $\|\lambda(x)\|$, approximate the functions $\Phi(z)$ and $\Omega(z)$ which have their cut along the actual crack. Using a perturbation expansion in the function $\lambda$, the functions $F(z)$, $W(z)$ can be expressed as

\begin{align}
F(z) &= F_0(z) + F_1(z) + O(\|\lambda\|^2), \\
W(z) &= W_0(z) + W_1(z) + O(\|\lambda\|^2),
\end{align}

where $F_0$ and $W_0$ are of zero-order with respect to $\|\lambda(x)\|$ and $F_1(z)$ and $W_1(z)$ are of the first-order. Following the similar procedures as that of Cotterell and Rice (1980), then, the zero- and first-order terms of the boundary values can be extracted as follows:

\begin{align}
F_0^+(t) + W_0^+(t) &= -(T_n - iT_s), \\
F_1^+(t) + W_1^+(t) &= -2i\lambda(F_0^+(t) - W_0^+(t)).
\end{align}

As shown by Cotterell and Rice (1980), the zero-order complex-potential solutions can be given by

$$F_0(z) = W_0(z) = \frac{1}{2\pi i X(z)} \int_{-a}^{a} \frac{-(T_n - iT_s)X^+(t)}{t - z} dt,$$

where $X(z) = \sqrt{\lambda^2 - \lambda'^2}$ is limited to the branch for which $X(z)/z \to 1$ as $z \to \infty$. Note that this solution has satisfied the single-valuedness condition of displacements given by Muskhelishvili (1953). Similarly, the boundary values of functions $F_1(z) + W_1(z)$, $F_1(z) - W_1(z)$ are solved from Eq. (6a) as

\begin{align}
[F_1^+(t) + W_1^+(t)]^+ - [F_1^+(t) - W_1^+(t)]^- &= 2i\lambda(T_n - iT_s)^{1/2} - 2i(t/T_s)^{1/2}, \\
[F_1^+(t) - W_1^+(t)]^+ - [F_1^+(t) - W_1^+(t)]^- &= -4i\lambda R(F_0^+(t) - F_0^+(t))^1.
\end{align}
where (9) denotes real part. Noting that the boundary values of both \( F_1(z) + W_1(z) \) and \( F_1(z) - W_1(z) \) are given here, instead of the treatment employed by Cotterell and Rice (1980) where only Eq. (8a) is applied and derived for evaluation of stress intensity factors. In present perturbation scheme, the complete solutions of \( F_1(z) \) and \( W_1(z) \) will be derived by applying the boundary values in Eq. (8). It is believed that this procedure plays important role to get the T-stress solution which cannot be derived from the incomplete complex-potentials given by Cotterell and Rice (1980). Using the method of Muskhelishvili (1953), the functions \( F_1(z) \) and \( W_1(z) \) can be expressed with respect to \( X(z) \) as

\[
F_1(z) = \frac{1}{2\pi i X(z)} \int_a^a \left[ \frac{[i(\lambda(T_n - iT_T)]y - 2(i(\lambda T_T)]X'(t)}{t - z} \right] dt + \frac{1}{2\pi i X(z)} \int_a^a \frac{2i(i\lambda F'_0(t) - F_0(t))'}{t - z} dt - \frac{1}{2\pi i X(z)} \int_a^a \frac{C}{X(z)} dt, \tag{9}
\]

\[
W_1(z) = \frac{1}{2\pi i X(z)} \int_a^a \left[ \frac{[\lambda(T_n - iT_T)]y - 2(i(\lambda T_T)]X'(t)}{t - z} \right] dt + \frac{1}{2\pi i X(z)} \int_a^a \frac{2i(i\lambda F'_0(t) - F_0(t))'}{t - z} dt - \frac{1}{2\pi i X(z)} \int_a^a \frac{C}{X(z)} dt, \tag{10}
\]

where \( C \) is a complex constant to be determined from the single-valuedness condition of displacements. Since the function \( F_0(z) \) has been obtained, the difference \( F_0 - F_0' \) at the straight cut can be determined by the Plemelj formulae (e.g. Muskhelishvili, 1977) which yields

\[
-2i\lambda\Re[F'_0(t) - F_0(t)] = \frac{2i\lambda}{X(t)} \int_a^a X(\xi) T_\lambda(\xi) d\xi. \tag{11}
\]

As \( \lambda(\pm a) \) and \( \lambda'(\pm a) \) exist and the integral on the right-hand side of Eq. (11) is the principal value integral, the values of the above function at the crack tips \( t = \pm a \) will vanish. Hence, the second term in Eqs. (9) and (10) can be simplified, after integration by parts, as

\[
-\frac{2i\lambda\Re[F'_0(t) - F_0(t)']}{t - z} dt = \frac{d}{dz} \int_a^a \frac{2i\lambda\Re[F'_0(t) - F_0(t)]}{t - z} dt. \tag{12}
\]

Therefore, it is determined that \( C = 0 \) from the single-valuedness condition of displacements given by Muskhelishvili (1953). Substituting (12) with (11) into (9) and (10) and then combining with Eq. (7), the complete first-order perturbation solution equation (5b) is explicitly given in terms of analytic functions \( F(z) \) and \( W(z) \) by

\[
F(z) = \frac{1}{2\pi i X(z)} \int_a^a \frac{[i(\lambda(T_n - iT_T)]y - 2(i(\lambda T_T)]X'(t)}{t - z} dt - \frac{1}{2\pi i X(z)} \int_a^a \frac{1}{X(t)} \left[ \int_a^a X(\xi) T_\lambda(\xi) d\xi \right] dt, \tag{13}
\]

\[
W(z) = \frac{1}{2\pi i X(z)} \int_a^a \frac{[\lambda(T_n - iT_T)]y - 2(i(\lambda T_T)]X'(t)}{t - z} dt - \frac{1}{2\pi i X(z)} \int_a^a \frac{1}{X(t)} \left[ \int_a^a X(\xi) T_\lambda(\xi) d\xi \right] dt. \tag{14}
\]

2.2. Stress intensity factor and T-stress

The asymptotic behavior of functions \( F(z) \) and \( W(z) \) near the crack tip is relevant to the crack tip field. Then using the formulae given in Appendix A, the asymptotic expansion of \( F(z) \) and \( W(z) \) near the crack tip \( z = a \) can be written by

\[
F(z) = \left( \frac{G_0}{2\pi i\sqrt{2a}} - \frac{H_1}{2\pi} \right) (z - a)^{\frac{1}{2}} + \frac{G_1}{2\pi i\sqrt{2a}} + \frac{H_1}{2\pi} + C[(z - a)^{\frac{3}{2}}]. \tag{15a}
\]

\[
W(z) = \left( \frac{G_0}{2\pi i\sqrt{2a}} + \frac{H_1}{2\pi} \right) (z - a)^{\frac{1}{2}} + \frac{G_1}{2\pi i\sqrt{2a}} + \frac{H_1}{2\pi} + C[(z - a)^{\frac{3}{2}}]. \tag{15b}
\]

where \( G_0, G_1, H_0, H_1, H_2, H_3 \) and \( C \) are constants with the explicit expressions as given in Appendix B.

For the problem of traction free on crack surfaces as shown in Fig. 2, the first stress invariant at the vicinity of the crack tip can be written as (e.g. Williams, 1957)

\[
\sigma_{11} + \sigma_{12} = \frac{4}{a^2} \lim_{z \to z_{tip}} \frac{a + \phi}{a - \phi} \sqrt{z - z_{tip}} \tag{16}
\]

where \( K_I \) and \( K_II \) are the stress intensity factors, \( T \) is the non-singular elastic T-stress and \( (r, \phi) \) are the polar coordinates. Substituting Eq. (14a), the first stress invariant is given in Cartesian coordinates, into Eq. (16), where the first stress invariant is given in polar coordinates, and noting the relation \( r_{tip} = (z - z_{tip}) e^{it} \), where \( \delta \) is the angle between the x-axis and the tangent to the crack at its tip, the SIFs and the elastic T-stress at crack tip can be expressed as, only in association with the complex-potential \( \Phi(z) \),

\[
K_{I} - iK_{II} = 4\sqrt{\frac{\pi}{2}} \lim_{z \to z_{tip}} \frac{\Phi(z) \sqrt{z - z_{tip}}}{d^2}, \tag{17a}
\]

\[
T = 4\sqrt{\frac{\pi}{2}} \lim_{z \to z_{tip}} \frac{d}{dz} \frac{\Phi(z) \sqrt{z - z_{tip})}}{d^2}. \tag{17b}
\]

When the crack surfaces are subjected to loads, see e.g. Fig. 1, Eq. (17a) for SIFs still holds, whereas the expression of T-stress is affected and we discuss this later. Substituting Eq. (15a) with Eq. (8b) into Eq. (17a) where \( \Phi(z) = F(z) \), \( z_{tip} = a \) and \( e^{-it/2} = 1 - iL_\lambda(a)/2 \), and only retaining the first-order terms, the first-order solution of SIFs at the tip of a slightly curved crack can be given by

\[
K_{I} - iK_{II} = \frac{1}{\sqrt{\pi a}} \int_a^a \frac{q_i - iq_0}{a + \phi} \sqrt{a + \phi} d\phi, \tag{18}
\]

where

\[
q_i = T_n - \frac{3}{2} \lambda a T_s + \lambda a T_T, \tag{19a}
\]

\[
q_0 = T_s + \lambda a T'_T + \frac{1}{2} \lambda a T_n. \tag{19b}
\]

The stress intensity factor results are identical to that of Cotterell and Rice (1980).
Because in the problem treated here the crack surfaces are not traction free as shown in Fig. 1. The near-tip stress asymptotic expression, Eq. (16), is no longer applicable. However, it can be updated as follows. The crack problem with tractions \( T_x \) and \( T_y \) exerted on the crack surfaces will remain unchanged if it is superposed by the following two problems. The first problem is a perfect infinite plane bearing remote stresses such that it has shear and normal stresses \( T_x(a) \) and \( T_y(a) \), respectively, at position \( x = a \) of a virtual crack surface. The second problem has contrary remote stresses to the first one. Therefore, the superposition of the non-free-traction crack problem and the first problem has a zero traction at the crack tip \( x = a \). Consequently according to the result of Williams (1957) this combination problem has a near crack tip stress expression equation (16). The additional contributions to the crack tip stress field from the second problem only lie in two non-singular terms, namely \( V \)-stress \( V \) and \( S \)-stress \( S \), which are given by
\[
V = -T_y(a), \quad S = -T_x(a).
\]
(20)

The \( V \)-stress is the normal stress acting vertical to the crack at its tip and the \( S \)-stress is the shear stress acting parallel to the crack at its tip. Therefore, for the problem shown in Fig. 1, the overall first-order solution can be given as a virtual form that may give an improved accuracy. A curved crack tip may be considered, where the crack tip is aligned with the \( x \)-axis with the coordinate origin fixed at the tip, and the surface tractions are given in terms of the components \( T_x \) and \( T_y \).

2.3. An alternative T-stress solution

Furthermore, the first-order solution can be given in an alternative form that may give an improved accuracy. A curved crack shown in Fig. 3 is considered, where the crack tip is aligned with the \( x \)-axis with the coordinate origin fixed at the tip, and the surface tractions are given in terms of the components \( T_x \) and \( T_y \).

The coordinates used in the previous solution are given by, in terms of the new coordinates as shown in Fig. 3,
\[
t = a - r \cos \omega - \eta \sin \omega,
\]
\[
\lambda = \eta \cos \omega - r \sin \omega.
\]
(25)

In order to keep first-order accuracy, the term of \( \eta \sin \omega \) should be omitted and this yields
\[
t = a - r \cos \omega,
\]
\[
\lambda = \eta \cos \omega - r \sin \omega.
\]
(26)
circular arc only with surface tractions, the T-stress solution to first-order at crack tip \( x = a \) reads
\[
T = \sigma_{22}^0 - \sigma_{33}^0 - 4\sigma_{23}^0 x,
\]
which is identical to the first-order solution obtained by letting \( x \) go to zero in the exact solution Eq. (30). The results including the first-order solution and the exact solution are compared in Figs. 5 and 6 for shear and biaxial stress loading at infinity. In the case of biaxial loading at infinity the approximate solution gives a value which is accurate to within 10\% for \( a \leq 10 \). In the case of shear stress loading at infinity, the T-stress solution is accurate to 10\% for \( a \leq 22 \).

The alternative solution equation (28) can also lead to same result. For a circular arc, the function \( g(r) \) as shown in Fig. 3 can be given by
\[
\eta(r) = \frac{L}{\sin^2 \alpha} - \frac{L}{\sin 2\alpha}.
\]
Hence, substituting Eq. (34) into Eq. (29) and retaining first-order with respect to \( \sin 2\alpha \) yield
\[
\eta(r) = -r \frac{\sin 2\alpha}{2L} \int_0^r \frac{1}{(\zeta - r)\sqrt{(L - \zeta)}} d\zeta.
\]

Based on singular integration results of Muskhelishvili (1953) \( \eta(r) = 0 \). Following Eq. (28) the alternative T-stress solution is given by
\[
T = \sigma_{22}^0 - \sigma_{33}^0 = \left( \sigma_{22}^0 - \sigma_{33}^0 \right) \cos 2\alpha - 2\sigma_{23}^0 \sin 2\alpha,
\]
which is identical to Eq. (33) to first-order with respect to \( \alpha \).

### 3.2. T-stress for kinked cracks

When a straight crack is loaded asymmetrically, the new crack initiates at an angle to the old one. Most of the past literatures focus on solving the stress intensity factors of a kinking crack and there are very little attempts on the T-stress in this case. Our first-order solution may provide a more explicit formula to evaluate the T-stress of a kinked crack. Indeed, problem might arise with the application of the present alternative solution for a kinked crack, since the crack profile of kinked crack is not differentiable at the corner point such that the singularity will be introduced into the exact complex-potential solutions of a kinked crack (see, Sumi, 1991; Amestoy and Leblond, 1992; Wu, 1979). The present first-order complex-potential solutions do not allow the singularity there, because a prerequisite is that the curved crack profile represented by \( \tilde{\eta}(x) \) has a continuous first-order derivative. Regardless of the deficiency of the method developed by Cotterell and Rice (1980) for kinked crack, however, it is found that the approach for stress intensity factors is correct to first-order (see, Sumi, 1991). Thus, the correctness of the approach of Cotterell and Rice (1980) for a kinked crack is quite probable but not fully established yet. As an extension of this approach, the present alternative solution for T-stress may be still correct to first-order for a kinked crack.

Fig. 7 shows that the problem of crack with an infinitesimal kink can be decomposed into the sum of a straight crack subproblem and a subproblem of kinked crack subjected to distribution forces on the kink surfaces. Therefore, the T-stress at the tip of an infinitesimally kinked crack can be divided into two parts.
\[ T = T_s + T_k, \]  
\[
T_1 = \frac{k_1}{\sqrt{2\pi r}} \left( \frac{5}{4} \cos \frac{\alpha}{2} + \frac{1}{4} \cos^3 \frac{3\alpha}{2} \right) 
+ \frac{k_0}{\sqrt{2\pi r}} \left( \frac{5}{4} \sin \frac{\alpha}{2} + \frac{3}{4} \sin^3 \frac{3\alpha}{2} \right) + T_0 \cos^2 \alpha. \]  
\[ (37) \]
\[ (38) \]

To calculate the distribution forces of the kinked crack subproblem, it should be noted that these forces are in balance with the stresses in the line of the putative kink of straight crack subproblem. Therefore, these distribution forces can also be written, with respect to the stress intensity factors \( k_1 \) and \( k_0 \) and T-stress \( (T_0) \) of the main crack (see Williams, 1957), by

\[
T_y = C_{11} \frac{k_1}{\sqrt{2\pi s}} + C_{12} \frac{k_0}{\sqrt{2\pi s}} + T_0 \sin^2 \alpha, \]
\[
T_x = C_{21} \frac{k_1}{\sqrt{2\pi s}} + C_{22} \frac{k_0}{\sqrt{2\pi s}} - T_0 \sin \alpha \cos \alpha, \]  
\[ (39) \]

where \( s \) denotes the distance from the tip of kinked crack surface to the tip of main crack and

\[
C_{11} = \frac{1}{4} \left( 3 \cos \frac{\alpha}{2} + \cos^3 \frac{3\alpha}{2} \right), \]
\[
C_{12} = -\frac{3}{4} \left( \sin \frac{\alpha}{2} + \sin^3 \frac{3\alpha}{2} \right), \]
\[
C_{21} = \frac{1}{4} \left( \sin \frac{\alpha}{2} + \sin^3 \frac{3\alpha}{2} \right), \]
\[
C_{22} = \frac{1}{4} \left( \cos \frac{\alpha}{2} + 3 \cos^3 \frac{3\alpha}{2} \right). \]  
\[ (40) \]

Based on the first-order solution, Eq. (28), \( T_k \) can be written by

\[
T_k = -T_y |_{\ell = \bar{r}} + \frac{4}{\sqrt{\pi}} \int_0^\ell T_x' (r) \frac{1}{\sqrt{r}} \, dr \]  
\[ (41) \]

where \( T_y \) and \( T_x \) given in Eq. (39) are seen as functions with respect to the variable \( s \). Between variables \( s \) and \( r \) the relation \( s = \ell - r \) holds and

\[
\gamma' (r) = \lim_{L \to \infty} \gamma (r) \sqrt{L}, \]  
\[ (42) \]

where \( L \) is the total length of the kinked crack as shown in Fig. 3. It should be noted that here the limit for an infinitesimal kink is treated as the limit for a finite kinked crack with a straight main crack infinitely long.

In order to evaluate \( \gamma' (r) \), it should be noted that for a finitely kinked crack the function \( \eta (r) \) in Eq. (29) can be specified as

\[
\eta (r) = \begin{cases} 
0, & \text{if } 0 \leq r \leq \ell, \\
(r - \ell) \tan \alpha, & \text{if } \ell \leq r \leq L. 
\end{cases} \]  
\[ (43) \]

Then, substituting Eq. (29) into Eq. (42) yields

\[
\gamma' (r) = \frac{\ell}{\ell - \ell} \lim_{L \to \infty} \sqrt{L} \int_0^\ell \left( \frac{\xi - \ell}{(\xi - r)^2} \right) \sqrt{L - \xi} \, d\xi. \]  
\[ (44) \]

Further calculations of the Cauchy-type integration, the function \( \gamma' (r) \) can be expressed in an explicit form as

\[
\gamma' (r) = \tan \alpha \left[ (1 - \frac{r - \ell}{r}) \ln \frac{\sqrt{\ell - r}}{\sqrt{\ell + r}} + \frac{1}{\sqrt{\ell}} \left( \frac{4}{3} - 2 \frac{r}{r} \right) \right]. \]  
\[ (45) \]

It should be noted that \( \gamma' (r) \) has removable singular points: \( r = 0 \) and \( r = \ell \). Thus the integrand in Eq. (41) has weak singularities at \( r = 0 \) and \( r = \ell \) and consequently it is integrable. By substituting the expressions (39) of \( T_y \) and \( T_x \) and Eq. (45) into Eq. (41), the T-stress for a kinked crack with tractions loaded on the kink can be readily obtained. Thus, the final T-stress at the tip of an infinitesimal free branch can be obtained, in terms of the stress intensity factors and T-stress at the tip of the parent crack, as

\[
T = T_k + T_s = b_1 \frac{k_1}{\sqrt{2\pi \ell}} + b_2 \frac{k_0}{\sqrt{2\pi \ell}} + b_3 T_0, \]  
\[ (46) \]

where the \( \alpha \)-dependences of \( b_1 \), \( b_2 \) and \( b_3 \) are given by

\[
b_1 = \sin \alpha \sin \frac{\alpha}{2} + (\frac{4}{3\pi} - \frac{1}{3}) \tan \alpha \left( \frac{\sin^2 \frac{3\alpha}{2}}{2} - \frac{\sin \frac{\alpha}{2}}{2} \right), \]
\[
b_2 = (1 + 3 \cos \alpha) \sin \frac{\alpha}{2} + (\frac{4}{3\pi} \frac{1}{2}) \tan \alpha \left( \frac{\cos^2 \frac{3\alpha}{2}}{2} + \cos \frac{3\alpha}{2} \right), \]
\[
b_3 = \cos 2 \alpha + (\frac{1}{2} - \frac{2}{3\pi^2} \frac{8}{3\pi^2}) \sin^2 \alpha, \]  
\[ (47) \]

which are accurate to the first-order in \( \alpha \). Fig. 8 shows the comparison of present results with those of Fett et al. (2006). It can be seen that a good agreement is achieved for a kink angle of up to 45°. The numerical results of Fett et al. (2006) are based on a finite element analysis assisted weight function determination and an approximate semi-infinite parent crack was employed for the case of infinitesimal kink. The numerical results of Fett et al. (2006) are believed to be accurate for a kink angle of up to 30°. Hence Eq. (47) may pro-
provide a reliable and explicit prediction to the $\lambda$-dependences of $b$’s for small kink angle $x$ and the present perturbation solution is further verified through its application on the problem of infinitesimally kinked cracks.

In addition, the $T$-stress of a parent crack with finite kink in a constant remote stress field can be analytically evaluated. First, the target problem should be decomposed into a constant stress problem and a crack problem with tractions on surfaces. The kinked crack subjected to tractions is then directly solved according to Eqs. (28) and (29) with a kink shape specification as given by Eq. (43). Fig. 9 shows the $T$-stress against kink angle for a finite-kink crack in a biaxial stress field. It can be seen that as the biaxial stress field tends to be bitensile the $T$-stress varies after a small extension of the parent crack.

$K_{II}$ and Goree (1998) could not cover the cases of crack with a small kinked length ratio. It should be pointed out that the key issue in the application of the modified first-order solution is to calculate the function, Eq. (29). This Cauchy type singular integral is only related to the geometrical shape of crack. For several special configurations of curved cracks, for examples, the polynomial configurations, Muskhelishvili (1953) has given a simple method to analytically calculate the integral.

3.3. The $T$-stress variation after a small extension of the parent crack

In the perturbation analysis of Cotterell and Rice (1980), by imposing the criterion of $K_{II} = 0$ the subsequent crack growth can be approximately described, after correction of a misprint, as

\[ \lambda(x) = \theta_0 x (1 + \alpha x^2) + \epsilon(x^3), \tag{48} \]

where

\[ \theta_0 = -2 \frac{k_0}{k_0}, \quad c = \frac{8}{3} \sqrt{2} \frac{T_0}{\pi k_0}. \tag{49} \]

This growth profile tends to deflect the crack back toward the initial path if $T_0 < 0$, and deflect the crack further away from the initial path if $T_0 > 0$.

Now, we are more concerned about how the $T$-stress varies when the crack advances along this profile. Based on the analysis of Cotterell and Rice (1980), we consider a semi-infinite crack which at the origin is tangential to the $x$-axis and has its tip at the point $(\ell, \lambda(\ell))$, see Fig. 11. When the tractions are applied to the small extension part, the approximately transformed version of Eq. (28) for the semi-infinite crack is obtained by letting $L \to \infty$, $\alpha \lambda/L \to 0$, and by writing

\[ \eta = \lambda(x) - \lambda(\ell) - \lambda'(\ell)(x - \ell). \tag{50} \]

Thus, the $T$-stress at the extending crack tip, due to tractions loaded on the small extension part, can be expressed as

\[ T_1 = -T_0|_{\lambda=\ell} + \lambda'(\ell)T_0|_{\lambda=\ell} + \frac{4}{\pi^2} \int_0^\ell \frac{\eta(\xi)\sqrt{\ell}}{\xi - x} \, d\xi, \tag{51} \]

where

\[ \tilde{\eta}'(\xi) = (\ell - \xi) \lim_{L \to \infty} \frac{\eta(\xi)\sqrt{L}}{(\ell - \xi)^2 \sqrt{(\ell - \xi)(\ell - \xi + L)}} \, d\xi. \tag{52} \]

Substitution of Eq. (48) into Eq. (50), noting that $\lambda(x) = 0$ for $x \leq 0$, yields

\[ \eta(x) = \left\{ \begin{array}{ll} (\ell - x)\lambda'(\ell) - \lambda(\ell), & \text{if } -\infty < x < 0, \\ \lambda'(\ell)(\ell - x)^2 + \epsilon[(\ell - x)^3], & \text{if } 0 \leq x < \ell. \end{array} \right. \tag{53} \]

Then substituting Eq. (53) into Eq. (52) yields

\[ \tilde{\eta}'(\xi) = \frac{1}{\sqrt{\ell}} \left[ \lambda'(\ell)\tilde{\eta}_1'(\xi) + \frac{\lambda'(\ell)}{L} \tilde{\eta}_2'(\xi) + L\lambda''(\ell)\tilde{\eta}_3'(\xi) \right]. \tag{54} \]
where

\[
\tilde{g}_1(x) = \frac{\sqrt{\ell}}{\sqrt{x - \ell}} \ln \left( \frac{\sqrt{x - \ell} - x}{\sqrt{x - \ell} + x} \right) + 2, \\
\tilde{g}_2(x) = -\frac{1}{(x - \ell)^2} \ln \left( \frac{\sqrt{x - \ell} - x}{\sqrt{x - \ell} + x} \right) - \frac{2}{3} \frac{\sqrt{x - \ell}}{x - \ell}, \\
\tilde{g}_3(x) = -\frac{\sqrt{\ell - x}}{\sqrt{\ell}} \ln \left( \frac{\sqrt{\ell - x} - x}{\sqrt{\ell - x} + x} \right).
\]

(55)

In the analysis of Cotterell and Rice (1980), the tractions on a crack extension part from the origin to a point \((\ell, \lambda \ell)\) can be derived from the stress field on the x-axis near the parent crack tip and to first-order in \(\lambda\), they are

\[
T_y = \frac{1}{2\pi x} \left[ k + \frac{\tilde{\zeta}(x)}{2x} k_0 + \tilde{\zeta}(x) k_0 \right], \\
T_x = \frac{1}{2\pi x} \left[ k + \frac{\tilde{\zeta}(x)}{2x} k_0 - \tilde{\zeta}(x) k_0 \right] - \tilde{\zeta}(x) T_0.
\]

(56)

Thus, substitution of these tractions and the function \(\tilde{\gamma}(x)\) (54) into Eq. (51) yields the following first-order expression for \(T\)-stress at the tip of the extending crack loaded by the tractions \(T_y, T_x\):

\[
T_1 = -k_1 \frac{k_0}{2\pi \ell} \left( c_1 + 2 \right) \frac{\tilde{\gamma}(x)}{\ell} + \left( c_2 + \frac{1}{2} \right) \frac{\tilde{\gamma}(x)}{\ell} + c_3 \frac{\tilde{\gamma}(x)}{\ell},
\]

(57)

where

\[
c_1 = \frac{4}{\pi^2} \int_0^\ell \frac{\tilde{g}_1(x)}{\sqrt{x} - \lambda \ell} \, dx = \frac{8}{\pi} - 4, \\
c_2 = \frac{4}{\pi^2} \int_0^{2\ell} \frac{\tilde{g}_2(x)}{\sqrt{x} - \lambda \ell} \, dx = \frac{8}{3\pi} + 2, \\
c_3 = \frac{4}{\pi^2} \int_0^\ell \frac{\tilde{g}_3(x)}{\sqrt{x} - \lambda \ell} \, dx = \frac{8}{\pi}.
\]

After considering the contribution to the \(T\)-stress from the transverse stress at the putative tip \((\ell, \lambda \ell)\) due to the stress field near the parent crack tip (see, e.g. Williams, 1957), the final \(T\)-stress at the extending crack tip can be given, to first-order, by

\[
T = T_0 + k_0 \frac{\tilde{\gamma}(x)}{\ell} \left[ \frac{8}{\pi} \frac{\tilde{\gamma}(x)}{\ell} - \frac{8}{3\pi} \frac{\tilde{\gamma}(x)}{\ell} + \frac{8}{\pi} \frac{\tilde{\gamma}(x)}{\ell} \right].
\]

(59)

Substituting the crack growth profile (48) into (59) yields

\[
T = T_0 + \frac{k_0/d_0}{2\pi \ell} \left[ \frac{16}{\pi^2} \frac{10}{\pi^2} \frac{c_1 \tilde{\gamma}(x)}{\ell} \right],
\]

(60)

where constants \(d_0\) and \(c\) are given by Eq. (49). It needs to be noted that \(k_0\) and \(d_0\) have opposite signs such that \(k_0/d_0\) is non-positive always. The prediction equation (60) reveals that the \(T\)-stress has the square-root singularity with respect to the crack extending length \(\ell\). For any negative values of \(k_0/d_0\), apparently, this singularity will lead to a significant change of \(T\)-stress subsequent to a sufficiently small distance of crack propagation. In case of absence of initial kink \((k_0 = 0)\), it is assumed that a very small initial kink often occurs due to accidental reasons such that \(k_0/d_0\) is always non-zero. Thus, when \(\ell\) tends to zero, the \(T\)-stress tend to negative infinity by all means. On the other hand, when \(\ell\) increases, the effect of the initial \(T\)-stress \(T_0\) on the sign of the final \(T\)-stress must be taken into account.

When the initial non-singular \(T\)-stress, \(T_0\) is negative, the sign of constant \(c\) will be negative based on Eq. (49). In spite of the extending length \(\ell\), therefore, the non-singular \(T\)-stress is always negative. Thus, the crack propagation path is stable based on the stability criterion of Cotterell and Rice (1980) who pointed out that the crack path is not stable for negative \(T\)-stress and unstable for positive \(T\)-stress. Noting that to re-examine the crack path stability after an appropriate length of propagation here, it is assumed that the criterion of Cotterell and Rice (1980) is still applicable, although this criterion is originally developed for predicting the stability prior to crack propagation. In case of a positive initial non-singular stress \((T_0 > 0)\), there exists a critical length \(\ell_c\) given by

\[
\ell_c = \left[ \frac{64\ell_0^2}{(9\pi^2 + 120\ell_0^2)} \right] \frac{1}{c^2},
\]

(61)

such that \(T < 0\) holds when \(\ell < \ell_c\) and \(T > 0\) when \(\ell > \ell_c\). Therefore, the stability criterion of Cotterell and Rice (1980), applied at the kink point, predicts instability because the sign of the “curvature parameter” \(\theta_0\), in Eq. (48) is identical to that of the kink angle \(\theta_0\).

4. Concluding remarks

In this study, the \(T\)-stress of a slightly curved or kinked crack is examined by employing a perturbation method. The main conclusions of this work are summarised below.

1. The perturbation solution of the complex-potentials for a slightly curved crack is completely solved by extending the perturbation scheme of Cotterell and Rice (1980) and applying the single-valuedness condition of displacement given by Mushkelishvili (1953).

2. Under this perturbation scheme, the \(T\)-stress of a slightly curved crack is explicitly determined using integral representations (see, Eq. (22), or alternatively Eq. (28)). Compared to other exact or numerical solutions the first-order solutions are accurate for considerable deviations from straightness.

3. Using the first-order solution of \(T\)-stress, an explicit formula is given to describe the \(T\)-stress dependence for a crack with an infinitesimal kink on the stress intensity factors and the \(T\)-stress of the crack before kinking (see, Eq. (46)).

4. Based on the directional stability criterion of Cotterell and Rice (1980), it is predicted that the crack path will not turn immediately to be unstable when \(T > 0\) and instead it will first keep a stable growth till reaching a critical length given by Eq. (61), then an unstable growth occurs.

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Appendix A. The asymptotic expansion of Cauchy integral

Let path \(L\) be a straight cut occupying the interval \([-a, a]\) and let \(f(t)\) denote a function defined on \(L\) as

\[
f(t) = g(t)\sqrt{t - a},
\]

(A-1)
where \( g(t) \) has a continuous first-order derivative \( g'(t) \) in the interval \([-a, a]\) and \( \sqrt{t-a} \) is limited to any definite branch that varies continuously on \( L \). Next we examine

\[
F'(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t)\sqrt{t-a}}{t-z} \, dt. \tag{A-2}
\]

where \( z \) is near \( a \) but not on \( L \). Because \( g(t) \) has a continuous first-order derivative, \( F'(a) \) exists and is given by

\[
F'(a) = \frac{1}{2\pi i} \int_{L} \frac{g(t)\sqrt{t-a}}{t-a} \, dt. \tag{A-3}
\]

Therefore,

\[
F'(z) - F'(a) = F''(z)(z-a), \tag{A-4}
\]

where

\[
F''(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{\sqrt{t-a}(t-z)} \, dt. \tag{A-5}
\]

In order to obtain the asymptotic behavior of \( F'(z) \) near \( a \), the integral \( A-5 \) should be examined. Now rewrite \( F''(z) \) in the following manner

\[
F''(z) = \frac{g(a)}{2\pi i} \int_{L} \frac{1}{\sqrt{t-a}(t-z)} \, dt + \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{\sqrt{t-a}(t-z)} \, dt, \tag{A-6}
\]

and examine the behavior of the two terms near \( a \), respectively.

1. The first term

\[
F''_1(z) = \frac{1}{2\pi i} \int_{L} \frac{g(a)}{\sqrt{t-a}(t-z)} \, dt. \tag{A-7}
\]

For \( z \) near \( a \), namely \( |z-a| < 2a \), the Cauchy integral formula gives

\[
2F''_1(z) + \frac{1}{2\pi i} \int_{L} \frac{g(a)}{\sqrt{t-a}(t-z)} \, dt = \frac{g(a)}{\sqrt{z-a}}. \tag{A-8}
\]

where \( \gamma \) is a circle whose center, starting point and end point are \( a \), \( -a \) and \( a+0i \), respectively. Therefore,

\[
F''_1(z) = \frac{g(a)}{2\sqrt{z-a}} - \frac{1}{4\pi i} \int_{\gamma} \frac{g(a)}{\sqrt{t-a}(t-z)} \, dt. \tag{A-9}
\]

Apparently, the second term is analytic at \( z=a \), then

\[
\frac{1}{4\pi i} \int_{\gamma} \frac{g(a)}{\sqrt{t-a}(t-z)} \, dt = \frac{g(a)}{\sqrt{2a}} + \mathcal{O}(z-a). \tag{A-10}
\]

Thus, for \( z \) near \( a \), the asymptotic expansion of function \( F''_1(z) \) is given by

\[
F''_1(z) = \frac{g(a)}{2\sqrt{z-a}} - \frac{g(a)}{\sqrt{2a}} + \mathcal{O}(z-a). \tag{A-11}
\]

2. The second term

\[
F''_2(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{\sqrt{t-a}(t-z)} \, dt. \tag{A-12}
\]

Introduce the following function

\[
\tilde{g}(t) = \frac{g(t) - g(a)}{t-a}. \tag{A-13}
\]

Because \( g(t) \) has a continuous first-order derivative, \( \tilde{g}(t) \) is continuous on the interval \([-a, a]\) and \( \tilde{g}(a) = g'(a) \). Then \( F''_2(z) \) can be rewritten as

\[
F''_2(z) = -\frac{1}{2\pi i} \int_{L} \frac{\tilde{g}(t)\sqrt{t-a}}{t-z} \, dt. \tag{A-14}
\]

Also, \( F''_2(a) \) exists and is of the form

\[
F''_2(a) = \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{(t-a)^{3/2}} \, dt. \tag{A-15}
\]

Therefore,

\[
F''_2(z) - F''_2(a) = F''_2^*(z)(z-a), \tag{A-16}
\]

where

\[
F''_2^*(z) = \frac{1}{2\pi i} \int_{L} \frac{\tilde{g}(t)}{\sqrt{t-a}(t-z)} \, dt. \tag{A-17}
\]

Following the results given by Muskheilishvili (1977) and noting that the point \( z \) is near \( a \) but not on \( L \), \( F''_2^*(z) \) can be given by

\[
F''_2^*(z) = \frac{\tilde{g}(a)}{2\sqrt{z-a}} + \mathcal{O}(1). \tag{A-18}
\]

where

\[
|F_0(z)| < \frac{C}{|z-a|^\alpha}, \quad 0 < \alpha < \frac{1}{2}. \tag{A-19}
\]

Thus, substituting Eq. (A-18) into Eq. (A-16) yields

\[
F''_2^*(z) = F''_2(a) + \mathcal{O}(\sqrt{z-a}). \tag{A-20}
\]

Substituting of Eqs. (A-11) and (A-20) into (A-6) yields

\[
F''(z) = \frac{g(a)}{2\sqrt{z-a}} \frac{g(a)}{\pi\sqrt{2a}} + \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{(t-a)^{3/2}} \, dt + \mathcal{O}(\sqrt{z-a}). \tag{A-21}
\]

Then, by substituting Eq. (A-21) into Eq. (A-4), the asymptotic expansion of the function \( F'(z) \) is obtained

\[
F'(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{\sqrt{t-a}} \, dt + \frac{g(a)}{2\sqrt{z-a}} \frac{g(a)}{\pi\sqrt{2a}} + \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{(t-a)^{3/2}} \, dt + \mathcal{O}(\sqrt{z-a}). \tag{A-22}
\]

For the point \( z = t_0 \) lying on \( L \), using the Plemelj formulæ, we have

\[
\frac{1}{2\pi i} \int_{L} \frac{g(t)\sqrt{t-a}}{t-t_0} \, dt = (F')^{-1}(t_0) - \frac{1}{2}g(t_0)\sqrt{t_0-a}, \tag{A-23}
\]

where \( (F')^{-1}(t_0) \) can be directly calculated from Eq. (A-22), for \( t_0 \) near \( a \), as

\[
(F')^{-1}(t_0) = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{\sqrt{t-a}} \, dt + \frac{g(a)}{2\sqrt{t_0-a}} \frac{g(a)}{\pi\sqrt{2a}} + \frac{1}{2\pi i} \int_{L} \frac{g(t) - g(a)}{(t-a)^{3/2}} \, dt (t_0-a) + \mathcal{O}((t_0-a)^{3/2}). \tag{A-24}
\]

Note that, for \( t_0 \) near \( a \), \( g(t_0) = g(a) + g'(a)(t_0-a) + \mathcal{O}((t_0-a)^2) \), then substituting it together with Eq. (A-24) into Eq. (A-23) yields

\[
\frac{1}{2\pi i} \int_{L} \frac{g(t)\sqrt{t-a}}{t-t_0} \, dt = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{\sqrt{t-a}} \, dt + \mathcal{O}(t_0-a). \tag{A-25}
\]

In addition, it is noted that

\[
\int_{L} \frac{g(t)}{\sqrt{t-a}(t-t_0)} \, dt = \frac{1}{t_0-a} \left[ \int_{L} \frac{g(t)\sqrt{t-a}}{t-t_0} \, dt - \int_{L} \frac{g(t)}{\sqrt{t-a}} \, dt \right]. \tag{A-26}
\]

Thus, for \( t_0 \) near the end \( a \), the magnitude of the above expression can be evaluated, by substituting (A-25) into (A-26), as

\[
\int_{L} \frac{g(t)}{\sqrt{t-a}(t-t_0)} \, dt = \mathcal{O}(1). \tag{A-27}
\]
Appendix B. The coefficients of \( G_0, G_1, H_0, H_2 \) and \( H_1 \)

According to Eq. (A-22), the asymptotic expansion of the first integral term in Eq. (13) is given by

\[
\int_a^b \frac{\tilde{\lambda}(z - T_n + iT_s)'}{(z - T_n + iT_s)'} e^{i\frac{z}{C_0}} \, dz = H_1 g(z) + c(z - a),
\]

(B-1)

with

\[
H_1 = \int_a^b \frac{\tilde{\lambda}(z - T_n + iT_s)'}{(z - T_n + iT_s)'} e^{i\frac{z}{C_0}} \, dz.
\]

(B-2a)

\[
G_0 = \int_a^b \frac{\hat{\lambda}(z - T_n + iT_s)'}{(z - T_n + iT_s)'} e^{i\frac{z}{C_0}} \, dz,
\]

(B-2b)

In Eq. (13), the second integral term subsequent to the differential sign in Eq. (13) can be written by

\[
\int_a^b \frac{1}{z - t} \frac{\tilde{\lambda}(t)}{(z - t)} \int_a^b \frac{X^*(z)T_i(z)}{(z - t)} \, d\zeta \, dt = \int_a^b \frac{g(t)}{z - t} \, dt - H_0 + H_1(z - a) \}

(B-3)

with

\[
H_0 = \int_a^b \frac{\tilde{\lambda}(t)}{(z - t)} \int_a^b \frac{X^*(z)T_i(z)}{(z - t)} \, d\zeta \, dt,
\]

(B-6a)

\[
H_1 = \int_a^b \frac{\hat{\lambda}(a) - \tilde{\lambda}'(a)}{a - t} \int_a^b \frac{X^*(z)T_i(z)}{a - t} \, d\zeta \, dt.
\]

(B-6b)

Then, according to Eq. (B-4a), the function \( g(z)g(z) - g_2(a)/z \) can be written as

\[
\frac{g(z)g(z) - g_2(a)}{t - a} = 2\pi i \frac{\tilde{\lambda}(t)}{(z - t)} \int_a^b \frac{\tilde{\lambda}(t)}{(z - t)} \, d\zeta.
\]

(B-7)

where, from Eq. (B-4c).

\[
\int_a^b \frac{\tilde{\lambda}(t)}{(z - t)} \, dt = \int_a^b \frac{\tilde{\lambda}(z)T_i(z)}{(z - a) - \tilde{\lambda}(z)} \, d\zeta.
\]

(B-8)

Applying Eq. (A-27) to the right-hand side of Eq. (B-8), it can be seen that the above function exists at \( t = a \) is equal to \( T_i(a) \). Then, substituting Eq. (B-7) into Eq. (B-6c) yields

\[
H_1 = \int_a^b \frac{\tilde{\lambda}(t)\gamma_1(t)}{\sqrt{a - t} \, dt} + \gamma_2 \int_a^b \frac{\hat{\lambda}(t)}{a - t} \, dt,
\]

(B-9)

where

\[
\gamma_1(t) = -\int_a^b \frac{\tilde{\lambda}(z)T_i(z)}{(z - a)} \, d\zeta,
\]

(B-10a)

\[
\gamma_2 = -\int_a^b \frac{\hat{\lambda}(t) + \tilde{\lambda}(z) - 2\tilde{\lambda}(z)}{(z - a)\sqrt{a - z}} \, d\zeta - 2\tilde{\lambda}(a).
\]

(B-10b)

By applying the formula of Poincaré-Bertrand (e.g. Muskhelishvili, 1977) on Eq. (B-9), an alternative expression of the coefficient \( H_1 \) is given by

\[
H_1 = -\int_a^b \frac{\tilde{\lambda}(z)T_i(a)}{(z - a)\sqrt{a - z}} \, d\zeta.
\]

(B-11)

\[
\gamma_1(t) = -\int_a^b \frac{\tilde{\lambda}(z)T_i(z)}{(z - a)} \, d\zeta,
\]

(B-10a)

\[
\gamma_2 = -\int_a^b \frac{\hat{\lambda}(t) + \tilde{\lambda}(z) - 2\tilde{\lambda}(z)}{(z - a)\sqrt{a - z}} \, d\zeta - 2\tilde{\lambda}(a).
\]

(B-10b)

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