On blow-ups and cohomology of almost complex manifolds

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We study the behavior of a special type of almost complex structures, called $C^\infty$ pure and full and introduced by T.-J. Li, W. Zhang (2009) in [10], in relation to the complex blow-up and the symplectic blow-up.

1. Introduction

Let $M$ be a compact oriented manifold of dimension $2n$. A symplectic form $\omega$ compatible with the orientation is a closed 2-form $\omega$ such that the 2n-form $\omega^n$ is a volume form compatible with the orientation. An almost complex structure $J$ on a symplectic manifold $(M, \omega)$ is said to be tamed by $\omega$ if $\omega_x(u, Ju) > 0$, for every $x \in M$ and every non-zero tangent vector $u \in T_x M$. $J$ is called calibrated by $\omega$ or, equivalently, $\omega$ is said to be compatible with $J$ if in addition $\omega_x(Ju, Jv) = \omega_x(u, v)$, for any pair of tangent vectors $u, v \in T_x M$. In this case the pair $(\omega, J)$ is called an almost-Kähler structure.

Let $C(M)$ be the symplectic cone of $M$, i.e. the image of the space of symplectic forms on $M$ compatible with the orientation under the projection to the de Rham cohomology $H^2(M, \mathbb{R})$. In order to study the relation between the $J$-tamed symplectic cone, i.e.

$$K^J_t(M) = \{[\omega] \in H^2(M, \mathbb{R}) | \omega \text{ is tamed by } J \}$$

and the $J$-compatible symplectic cone

$$K^J_c(M) = \{[\omega] \in H^2(M, \mathbb{R}) | \omega \text{ is compatible with } J \}.$$

T.-J. Li and W. Zhang introduced the notion of $C^\infty$ pure and full almost complex structure. More precisely, an almost complex structure $J$ on a compact manifold $M$ is called $C^\infty$ pure and full if and only if the following splitting for the de Rham cohomology holds:

**Keywords:** Pure and full almost complex structure
Cohomology
Symplectic blow-up
\[ H^2(M, \mathbb{R}) = H^{1,1}_J(M)_{\mathbb{R}} \oplus H^{(2,0),(0,2)}_J(M)_{\mathbb{R}}, \]

where
\[ H^{1,1}_J(M)_{\mathbb{R}} = \{ \alpha \mid \alpha \in Z^{1,1}_J \}, \quad H^{(2,0),(0,2)}_J(M)_{\mathbb{R}} = \{ \alpha \mid \alpha \in Z^{(2,0),(0,2)}_J \} \]

and \( Z^{1,1}_J \) and \( Z^{(2,0),(0,2)}_J \) are respectively the closed \( J \)-invariant and the closed \( J \)-anti-invariant forms. If the almost-Kähler structure \((\omega, J)\) is such that \( J \) is integrable, or equivalently if \((\omega, J)\) is a Kähler structure on \( M \), then \( J \) is \( C^\infty \) pure and full.

T. Drăghici, T.-J. Li and W. Zhang proved in [4, Theorem 2.3] that if \( M \) is a compact 4-dimensional manifold, then any almost complex structure on \( M \) is \( C^\infty \) pure and full, namely, for any compact 4-dimensional almost complex manifold, the splitting given by (1) holds. Many other results in this case related to the \( J \)-tamed and \( J \)-calibrated symplectic cones have been obtained in the papers [10,4,5].

If \( \omega \) is a symplectic form on a 2\( n \)-dimensional compact manifold \( M \) and \( J \) is an almost complex structure on \( M \) calibrated by \( \omega \), then \( J \) is \( C^\infty \) pure (see [4,7]). In the particular case that \( J \) is integrable, in [4, Proposition 2.17] it is proved that if \( J \) is an integrable almost complex structure on a compact 4-dimensional manifold \( M \), then \( H^{1,1}_J(M)_{\mathbb{R}} \) and \( H^{(2,0),(0,2)}_J(M)_{\mathbb{R}} \) are isomorphic to the real Dolbeault cohomology groups \( H^{1,1}_\mathbb{R}(M) \cap H^2(M, \mathbb{R}) \) and \( (H^{2,0}_\mathbb{R}(M) \oplus H^{0,2}_\mathbb{R}(M)) \cap H^2(M, \mathbb{R}) \) respectively. In view of this result, the groups \( H^{1,1}_J(M)_{\mathbb{R}} \) and \( H^{(2,0),(0,2)}_J(M)_{\mathbb{R}} \) appear as a natural generalization of the Dolbeault groups to the non-integrable case.

In higher dimension the situation is different, since in [7] we found an example of 6-dimensional nilmanifold which admits a non \( C^\infty \) pure almost complex structure. Another example of non \( C^\infty \) pure integrable almost complex structure will be given in Section 2.

In [1] it has been proved that in general a small deformation of an integrable almost complex \( C^\infty \) pure and full is not \( C^\infty \) pure and full.

Since the Kähler condition on a complex manifold \((M, J)\) is preserved under the blow-up at a point or along a compact submanifold, a natural question is to see if the condition for an integrable almost complex structure to be \( C^\infty \) pure and full is preserved under the blow-up of the complex manifold. In Section 3 we will give a positive answer to the previous question.

In Sections 4 and 5 we will study the behavior under the operation of the symplectic blow-up at a point and along a compact symplectic submanifold of a symplectic manifold \((M, \omega)\) endowed with a \( C^\infty \) pure and full and \( \omega \)-calibrated almost complex structure. The symplectic blow-up along a symplectic submanifold was introduced and used by McDuff in [11] to obtain the first simply-connected example of a non-Kähler symplectic manifold, namely the blow-up of \( \mathbb{C}P^5 \) along a symplectically embedded Kodaira–Thurston surface.

2. \( C^\infty \) pure and full almost complex structures

Let \( J \) be an almost complex structure on a compact 2\( n \)-dimensional manifold \( M \). Then \( J \) acts in a natural way on the space \( \Omega^k(M)_{\mathbb{R}} \) of real smooth differential \( k \)-forms, in the following way: given \( \alpha \in \Omega^k(M)_{\mathbb{R}} \), then
\[ J\alpha(X_1, \ldots, X_k) = \alpha(JX_1, \ldots, JX_k), \]
for every \( X_1, \ldots, X_k \) vector fields on \( M \). Therefore, according to this action, \( \Omega^k(M)_{\mathbb{R}} \) decomposes as:
\[ \Omega^k(M)_{\mathbb{R}} = \bigoplus_{p+q=k} \Omega^{p,q}(M)_{\mathbb{R}}, \]

where
\[ \Omega^{p,q}(M)_{\mathbb{R}} = \{ \alpha \in \Omega^{p,q}(M)_{\mathbb{R}} \mid \alpha = \mathbb{R}\}. \]

For a finite set \( S \) of pairs of integers, let
\[ Z^S_j = \bigoplus_{(p,q) \in S} Z^{p,q}_j, \quad B^S_j = \bigoplus_{(p,q) \in S} B^{p,q}_j, \]
where the spaces \( Z^{p,q}_j \) and \( B^{p,q}_j \) are respectively the space of real \( d \)-closed \((p,q)\)-forms and those one of \( d \)-exact \((p,q)\)-forms. There is a natural map
\[ \rho_S : Z^S_j / B^S_j \rightarrow Z^S_j / B \]
where \( B \) is the space of \( d \)-exact forms. As in [10] we will write \( \rho_S(Z^S_j / B^S_j) \) simply as \( Z^S_j / B^S_j \) and we may define the cohomology spaces
\[ H^S_j(M)_{\mathbb{R}} = \{ \alpha \mid \alpha \in Z^S_j \} = \frac{Z^S_j}{B}. \]
Then there is a natural inclusion
\[ H^1_{J}(M)_{\mathbb{R}} + H^2_{J}(M)_{\mathbb{R}} \subseteq H^2(M, \mathbb{R}). \]

As in [10, Definition 4.12] we set the following

**Definition 2.1.** An almost complex structure \( J \) on \( M \) is said to be \( C^\infty \) pure and full if
\[ H^2(M, \mathbb{R}) = H^1_{J}(M)_{\mathbb{R}} \oplus H^2_{J}(M)_{\mathbb{R}}. \]

As an immediate consequence of the definition, an almost complex structure \( J \) is \( C^\infty \) pure if and only if
\[ \frac{\mathcal{Z}^1_{J}}{\mathcal{B}^1_{J}} \cap \frac{\mathcal{Z}^2_{J}}{\mathcal{B}^2_{J}} = 0 \]
and \( J \) is \( C^\infty \) full if and only if
\[ \frac{\mathcal{Z}^2}{\mathcal{B}^2} = \frac{\mathcal{Z}^1_{J}}{\mathcal{B}^1_{J}} + \frac{\mathcal{Z}^2_{J}}{\mathcal{B}^2_{J}}, \]
where \( \mathcal{Z}^2 \) and \( \mathcal{B}^2 \) denote respectively the space of 2-forms which are \( d \)-closed and exact. Let \( \pi_{1,1} : \Omega^2(M, \mathbb{R}) \to \Omega^1_{J}(M)_{\mathbb{R}} \) be the natural projection. If \( J \) is \( C^\infty \) pure and full, then the natural homomorphism
\[ \frac{\mathcal{Z}^1_{J}}{\mathcal{B}^1_{J}} \to \pi_{1,1} \frac{\mathcal{Z}^2}{\mathcal{B}^2} \]
is an isomorphism (see [10, Lemma 4.9]).

We recall the well-known (see e.g. [2]):

**Definition 2.2.** Let \((M, \omega)\) be a symplectic manifold. An almost complex structure \( J \) is said to be tamed by \( \omega \) if \( \omega_x(u, Ju) > 0 \), for every \( x \in M \) and every non-zero tangent vector \( u \in T_x M \). \( J \) is called calibrated by \( \omega \) if in addition \( \omega_x(Ju, Jv) = \omega_x(u, v) \), for every pair of tangent vectors \( u \) and \( v \).

If \( J \) is an \( \omega \)-calibrated almost complex structure on a symplectic manifold \((M, \omega)\), then \( g_J(\cdot, \cdot) := \omega(\cdot, J\cdot) \) is a \( J \)-invariant Riemannian metric on \( M \) and \((\omega, J, g_J)\) is an almost-Kähler structure.

It can be showed that if \( \omega \) is a symplectic form on a \( 2n \)-dimensional compact manifold \( M \) and \( J \) is an almost complex structure on \( M \) calibrated by \( \omega \), then \( J \) is \( C^\infty \) pure (see [4, Proposition 2.8] and [7, Proposition 3.2]).

As a consequence, if \((\omega, J)\) is an almost-Kähler structure, then \( J \) is \( C^\infty \) pure.

By [4, Theorem 2.3] on a compact manifold of real dimension 4 any almost complex structure is \( C^\infty \) pure and full. This is not anymore true in higher dimensions, since an example of compact 6-dimensional manifold with a non \( C^\infty \) pure almost complex structure is given. Compact quotients of 6-dimensional solvable Lie groups with \( C^\infty \) pure and full almost complex structures are obtained in [7].

The next example provides a 6-dimensional nilmanifold endowed with a \( C^\infty \) pure and full almost complex structure \( J \) and a symplectic form calibrating it.

**Example 2.3.** Consider the real 6-dimensional nilpotent Lie group with structure equations
\[
\begin{align*}
d e^1 &= 0, & j &= 1, 2, 3, \\
d e^4 &= -2 e^{12}, \\
d e^5 &= 2 e^{13}, \\
d e^6 &= 2 e^{23},
\end{align*}
\]
where \( e^i \) denotes the wedge product \( e^i \wedge e^j \). By [3] the previous real Lie group \( G \) can be also described as group of complex matrices of the form
\[
\begin{pmatrix}
1 & z_1 & z_2 & z_3 & 1/z_1 & z_3 \\
0 & 1 & z_1 & 0 & 1/z_1 & z_3 \\
0 & 0 & 1 & 0 & 0 & z_1 \\
0 & 0 & 0 & 1 & z_1 & -z_2 \\
0 & 0 & 0 & 0 & 1 & -z_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
In terms of the coordinates $z_1, z_2, z_3$ one has the product

$$ (z_1, z_2, z_3) \ast (w_1, w_2, w_3) = \left( z_1 + w_1, z_2 + w_2 + \bar{z}_1 w_1, z_3 + w_3 + \frac{1}{2} w_1^2 \bar{z}_1 + w_1 z_2 - \bar{w}_2 z_1 - \frac{1}{2} z_1^2 \bar{w}_1 \right) $$

and that the left-invariant forms on $G$ are given by

$$\begin{align*}
    &e^1 = dz_1, \\
    &e^3 + ie^4 = dz_2 - \bar{z}_1 dz_1, \\
    &e^5 + ie^6 = dz_3 - \bar{z}_2 dz_1 + z_1 dz_2 - \frac{1}{2} z_1^2 dz_1.
\end{align*}$$

The nilpotent Lie group $G$ admits a uniform discrete subgroup $\Gamma$ consisting of those matrices with Gaussian integers as entries. The corresponding nilmanifold $M^6 = \Gamma\backslash G$ admits an invariant almost-Kähler structure $(\omega, J)$ defined by

$$\begin{align*}
    &\omega = 2e^{16} + e^{25} + e^{34}, \\
    &J_1 = \frac{1}{2} e_6, \quad J_2 = e_5, \quad J_3 = e_4.
\end{align*}$$

where $\{e_1, \ldots, e_6\}$ denotes the dual frame to $\{e^1, \ldots, e^6\}$. The almost complex structure $J$ is $C^\infty$ pure and full. Indeed, we have

$$\begin{align*}
    &H^2(M^6, \mathbb{R}) = \text{span}\{[e^{14}], [e^{15}], [e^{16} + e^{25}], [e^{16} + e^{34}], [e^{24}], [e^{26}], [e^{35}], [e^{36}]\}, \\
    &H^2(M^6, \mathbb{R}) = \text{span}\{[e^{15} + 2e^{26}], [e^{16} + e^{25}], [e^{16} + e^{34}], [e^{14} + 2e^{36}], [e^{24} + e^{35}]\}, \\
    &H^2_j(0,0)(M^6, \mathbb{R}) = \text{span}\{[e^{15} - 2e^{26}], [e^{14} - 2e^{36}], [e^{24} - e^{35}]\}.
\end{align*}$$

In general, for an almost complex structure, there is no relation between the integrability of an almost complex structure and the condition of $C^\infty$ pureness.

Indeed, the following gives an example of a non $C^\infty$ pure integrable almost complex structure on a 6-dimensional compact manifold. For another example of integrable almost complex structure which is not $C^\infty$ pure nor $C^\infty$ full, obtained as deformation of an integrable $C^\infty$ pure and full one, see [1].

**Example 2.4.** Let $M^6$ be the nilmanifold associated to the 6-dimensional nilpotent Lie algebra with structure equations

$$\begin{align*}
    &d e^j = 0, \quad j = 1, \ldots, 4, \\
    &d e^5 = -2e^{34} - e^{13} + e^{24}, \\
    &d e^6 = -e^{14} - e^{23}
\end{align*}$$

endowed with the integrable almost complex structure $J$ defined by the $(1,0)$-forms

$$\phi_1 = e^1 + ie^2, \quad \phi_2 = e^3 + ie^4, \quad \phi_3 = e^5 + ie^6.$$  

Then by

$$\begin{align*}
    &d \phi_j = 0, \quad j = 1, 2, \quad d \phi_3 = -i \phi_2 \wedge \bar{\phi}_2 - \phi_1 \wedge \phi_2
\end{align*}\]$$

we get that

$$-e^{13} + e^{24} = d e^5 + 2 e^{34}.$$  

Therefore $J$ is not $C^\infty$ pure, since $[2e^{34}] \in H^2_j(1)(M^6, \mathbb{R})$ and $[-e^{13} + e^{24}] \in H^2_j(0,0)(M^6, \mathbb{R})$.

For complex parallelizable manifolds, i.e. compact quotients of complex Lie groups by discrete Lie subgroups, we can prove the following

**Proposition 2.5.** If $(M = \Gamma\backslash G, J)$ is a complex parallelizable manifold and for the de Rham cohomology we have the isomorphism $H^2(M, \mathbb{R}) \cong H^2(g)$, where $H^2(g)$ denotes the Chevalley Eilenberg cohomology of the Lie algebra $g$ of $G$, then $J$ is $C^\infty$ pure.

**Proof.** By [7, Theorem 5.1] we know that $J$ is $C^\infty$ full and

$$H^2(M, \mathbb{R}) \cong \mathcal{Z}_{inv}^{2,0} \oplus \mathcal{Z}_{inv}^{(2,0)+(0,2)}/\mathcal{B}_{inv}^{(2,0),(0,2)},$$

(2)
where \( Z_{1,1}^{\text{inv}}, Z_{2,0}^{(2,0),(0,2)} \) are respectively the spaces of closed left-invariant forms of type \((1,1)\) and \((0,2) + (2,0)\) and \( B_{2,0}^{(2,0),(0,2)} \) is the space of exact left-invariant forms of type \((2,0) + (0,2)\). Suppose that \( a \in H^1_{1,1} (M) \cap H^1_{2,0} (M) \).

By the previous isomorphism \((2)\) we have that \( a = [\alpha] = [\beta] \), with \( \alpha \in Z^{1,1}_{\text{inv}} \) and \( \beta \in Z^{(2,0),(0,2)}_{\text{inv}} \). Thus

\[
\alpha = \beta + d\gamma,
\]

where the exact 2-form \( d\gamma \) is left-invariant. By using the property that \( M \) admits a bi-invariant volume form and by applying the symmetrization process of \( [6, \text{Theorem 2.1}] \) we get that \( a \) is holomorphic.

**Proof.** By assumption there exist holomorphic coordinates \((z_1, \ldots, z_n)\) on a neighborhood of \( p \) with \( \pi(p) = \emptyset \). Let \( \tilde{M} \) be the blow-up of \( M \) at \( p \). If \( M \) is \( C^\infty \) pure and full, then \( \tilde{M} \) is \( C^\infty \) pure and full.

As an application, if \((M, J)\) is a complex parallelizable nilmanifold, then \( J \) is \( C^\infty \) pure.

### 3. Complex blow-up

In this section we study the behavior of the property for an integrable almost complex structure to be \( C^\infty \) pure and full under the blow-up process. We start with a preliminary result

**Proposition 3.1.** Let \((M, J)\) be a 2n-dimensional compact almost complex manifold such that \( J \) is integrable in a neighborhood of a point \( p \in M \). Let \((\tilde{M}, \tilde{J})\) be the complex blow-up of \((M, J)\) at \( p \). If \( J \) is \( C^\infty \) pure and full, then \( \tilde{J} \) is \( C^\infty \) pure and full.

**Proof.** By assumption there exist holomorphic coordinates \((z_1, \ldots, z_n)\) on a neighborhood \( V \) of \( \pi(p) \). Therefore, we can blow-up \((M, J)\) at \( p \). Since \( J \) is \( C^\infty \) pure and full, we have that

\[
H^2(M, \mathbb{R}) = H^2_J (M) \oplus H^2_J (M).
\]

We have then to prove that

\[
H^2(\tilde{M}, \mathbb{R}) = H^2_{\tilde{J}} (\tilde{M}) \oplus H^2_{\tilde{J}} (\tilde{M}).
\]

The de Rham cohomology groups of the two manifolds \( M \) and \( \tilde{M} \) are related by

\[
H^2(\tilde{M}, \mathbb{R}) = H^2(M, \mathbb{R}) \oplus H^2 (\mathbb{CP}^{n-1}, \mathbb{R}).
\]

The forms of type \((p, q)\) with respect to \( J \) are also of type \((p, q)\) with respect to \( \tilde{J} \) and \( H^2(\mathbb{CP}^{n-1}) = \langle \omega_{FS} \rangle \), where \( \omega_{FS} \) is the Kähler form on \( \mathbb{CP}^{n-1} \) associated to the Fubini–Study metric, we get that \( \tilde{J} \) is \( C^\infty \) pure and full.

The following proposition shows that the blow-up of a compact complex manifold along a compact complex submanifold preserves the \( C^\infty \) pure and full property.

**Proposition 3.2.** Let \((M, J)\) be a compact complex manifold of complex dimension \( n \) and \( Y \subset M \) be a compact complex submanifold of \( M \) of complex dimension \( k \). Let \((\tilde{M}, \tilde{J})\) be the complex blow-up of \((M, J)\) along \( Y \). If \( J \) is \( C^\infty \) pure and full, then \( \tilde{J} \) is \( C^\infty \) pure and full.

**Proof.** Let \( \pi : (\tilde{M}, \tilde{J}) \to (M, J) \) be the holomorphic projection. By construction \( \pi : \tilde{M} \setminus \pi^{-1}(Y) \to M \setminus Y \) is a biholomorphism and \( \pi^{-1}(Y) \cong \mathbb{P}(N_{Y|M}) \), where \( \mathbb{P}(N_{Y|M}) \) is the projectified of the normal bundle of \( Y \).

It is known that

\[
H^*(\tilde{M}, \mathbb{C}) = \pi^* H^*(M, \mathbb{C}) \oplus H^* (\mathbb{P}(N_{Y|M}), \mathbb{R}) / \pi^* H^*(Y, \mathbb{R})
\]

(see for instance \([9, p. 606]\)) and that the cohomology ring \( H^*(\mathbb{P}(N_{Y|M}), \mathbb{R}) \), is generated, as an \( H^*(Y, \mathbb{R}) \) algebra, by the Chern class \( \xi = c_1 (T) \) of the tautological bundle on \( \mathbb{P}(N_{Y|M}) \), with the relation

\[
\xi^n - c_1(N_{Y|M}) \xi^{n-k-1} + c_2(N_{Y|M}) \xi^{n-k-2} + \cdots + (-1)^{n-k-1} c_{n-k-1}(N_{Y|M}) \xi + (-1)^{n-k} c_{n-k}(N_{Y|M}) = 0.
\]

Then in particular we get that

\[
H^2(\tilde{M}, \mathbb{R}) = \pi^* H^2(M, \mathbb{R}) \oplus \text{span}(\xi),
\]

and therefore if \( J \) is \( C^\infty \) pure and full, we have that \( \tilde{J} \) is \( C^\infty \) pure and full, since \( \xi \) is of type \((1,1)\) with respect to \( \tilde{J} \) and \( \pi \) is holomorphic.

For holomorphic mappings between complex manifolds satisfying more suitable assumptions, one can show the following:
Proposition 3.3. Let \( \pi : (\tilde{M}, \tilde{J}) \rightarrow (M, J) \) be a proper, surjective, holomorphic mapping between complex manifolds of the same dimension. If \( \tilde{J} \) is \( C^\infty \) pure, then \( J \) is also \( C^\infty \) pure.

Proof. According to [14, Theorem 3.1] we have that

\[
\pi^* : H^*(M, \mathbb{R}) \rightarrow H^*(\tilde{M}, \mathbb{R})
\]

is injective. Assume that \( J \) is not \( C^\infty \) pure, then there exists \( 0 \neq a \in H^1_\beta(M) \setminus H^1_\beta(\mathbb{R}) \). Therefore, \( a = [\alpha] = [\beta] \), where \( \alpha \in H^1_\beta(M) \) and \( \beta \in H^1(\mathbb{R}) \). Hence, \( \pi^* \alpha = \pi^* \beta + d(\pi^* \gamma) \) and, consequently, \( 0 \neq [\pi^* \alpha] = [\pi^* \beta] \in H^1_\beta(M) \setminus H^1(\mathbb{R}) \). This is a contradiction with the assumption that \( \tilde{J} \) is \( C^\infty \) pure. \( \square \)

4. Almost complex structures and symplectic blow-up at a point

Denote by \((\mathbb{B}^{2n}(R), \omega_0, J_0)\) the 2n-dimensional ball with center in 0 and of radius \( R \) endowed with the standard symplectic structure \( \omega_0 \) and the standard \( \omega_0 \)-calibrated complex structure \( J_0 \).

We recall that to blow up a general complex manifold \((M, J)\) at a point \( p \in M \) one proceeds in the following way. Choose a complex embedding \( \iota : B^{2n}(R) \rightarrow M \) such that \( \iota(0) = p \) and consider the \( r \)-ball subbundle \( U_r \) of the total space of the tautological line bundle over \( \mathbb{C}P^{n-1} \):

\[
U_r = \{(z, [w_1 : \ldots : w_n]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z \in \mathbb{C}w, |z| < r \}.
\]

Then the complex blow-up \((\tilde{M}_p, \tilde{J})\) is defined as the union

\[
\tilde{M}_p = (M \setminus \{p\}) \cup U_r/\sim,
\]

where the equivalence relation \( \sim \) identifies the point \((z, [z_1 : \ldots : z_n]) \) \( \in U_r \setminus \mathbb{C}P^{n-1} \) with \( \iota(z) \) \( \in M \). Then there exists a natural embedding

\[
\iota : \mathbb{C}P^{n-1} \rightarrow \tilde{M}_p,
\]

given by the composition of the embedding of the zero section

\[
\mathbb{C}P^{n-1} \rightarrow U_r, [w_1 : \ldots : w_n] \mapsto (0, [w_1 : \ldots : w_n])
\]

with the inclusion \( U_r \rightarrow \tilde{M}_p \). The image of the embedding \( \tilde{\iota} \) is the exceptional divisor \( E = \tilde{\iota}(\mathbb{C}P^{n-1}) \subset \tilde{M}_p \). Moreover, there is a holomorphic projection \( \pi : \tilde{M}_p \rightarrow M \) such that \( E = \pi^{-1}(p) \) and \( \pi \) restricts to a diffeomorphism from \( \tilde{M}_p \setminus E \) to \( M \setminus \{p\} \).

Given a symplectic manifold \((M, \omega)\), a symplectic embedding \( \iota : \mathbb{B}^{2n}(R) \rightarrow M \) and \( J \) an almost complex structure on \( M \) such that \( \iota^*J = J_0 \), then one can do the previous construction since \( J \) is integrable near \( p \). The almost complex structure \( J \) on \( M \) induces a unique almost complex structure \( \tilde{J} \) on \( \tilde{M}_p \), with respect to which the projection \( \pi : \tilde{M}_p \rightarrow M \) is pseudoholomorphic.

By [13, Proposition 9.3.3] (see also [12]) if \((M, \omega)\) is a symplectic manifold, \( \iota : \mathbb{B}^{2n}(R) \rightarrow M \) a symplectic embedding and \( J \) an \( \omega \)-calibrated almost complex structure such that \( \iota^*J = J_0 \), then for any \( r < R \), there exists a symplectic form \( \tilde{\omega}_r \) on the complex blow-up \((\tilde{M}_p, J)\) of \((M, J)\) at \( p = \iota(0) \) such that

1. \( \pi^* \omega = \tilde{\omega}_r \) on \( \pi^{-1}(M \setminus \iota(\mathbb{B}^{2n}(R))) \subset \tilde{M}_p \);
2. \( \pi^* \tilde{\omega}_r = r^2 \omega_{FS} \), where \( \omega_{FS} \) denotes the Fubini–Study form;
3. \( \tilde{J} \) is \( \tilde{\omega}_r \)-calibrated;
4. for every smooth map \( \tilde{u} : \Sigma \rightarrow \tilde{M}_p \) from a Riemann surface \( \Sigma \) to \( \tilde{M}_p \),

\[
\int_{\Sigma} (\pi \circ \tilde{u})^* \omega = \int_{\Sigma} \tilde{u}^* \tilde{\omega}_r + \pi r^2 (\tilde{u} \cdot E),
\]

where \( E \) is the exceptional divisor of the complex blow-up \( \pi : \tilde{M}_p \rightarrow M \).

We can prove that if \( J \) is \( C^\infty \) pure and full, then the almost complex structure \( \tilde{J} \) on the complex blow-up \((\tilde{M}_p, \tilde{J})\) of \((M, J)\) at \( p = \iota(0) \) is \( C^\infty \) pure and full.

Theorem 4.1. If \((M, \omega)\) is a symplectic manifold, \( \iota : (\mathbb{B}^{2n}(R), \omega_0) \rightarrow (M, \omega) \) is a symplectic embedding and \( J \) is an \( \omega \)-calibrated \( C^\infty \) pure and full almost complex structure such that \( \iota^*J = J_0 \), then for any \( r < R \), there exists a symplectic form \( \tilde{\omega}_r \) on the complex blow-up \((\tilde{M}_p, \tilde{J})\) of \((M, J)\) at \( p = \iota(0) \) such that
(1) \( \pi^* \omega = \tilde{\omega}_r \) on \( \pi^{-1}(M \setminus \iota(B^{2n}(R))) \subset \tilde{M}_p \);
(2) \( i^* \tilde{\omega}_r = r^2 \omega_{FS} \), where \( \omega_{FS} \) denotes the Fubini–Study form and \( i \) is the embedding given by (3);
(3) \( J \) is \( C^\infty \) pure and full and it is \( \tilde{\omega}_r \)-calibrated;
(4) for every smooth map \( \tilde{u} : \Sigma \to \tilde{M}_p \) from a Riemann surface \( \Sigma \) to \( \tilde{M}_p \),
\[
\int_{\Sigma} (\pi \circ \tilde{u})^* \omega = \int_{\Sigma} \tilde{u}^* \tilde{\omega}_r + \pi r^2 (\tilde{u} \cdot E),
\]
where \( E \) denotes the exceptional divisor of the complex blow-up \( \pi : \tilde{M}_p \to M \).

**Proof.** The existence of \( \tilde{\omega}_r \) and \( J \) satisfying previous conditions follows by [13, Proposition 9.3.3], that we reviewed before. Therefore, we have only to prove that if \( J \) is \( C^\infty \) pure and full, then also \( J \) is \( C^\infty \) pure and full.

Since \( J \) is \( C^\infty \) pure and full, we have that
\[
H^2(M, \mathbb{R}) = H^1_{\tilde{J}}(M) \oplus H^2_{\tilde{J}}(M),
\]

Since \( J \) is calibrated by \( \tilde{\omega}_r \), by [4,7] \( J \) is \( C^\infty \) pure. We have then to prove that
\[
H^2(\tilde{M}_p, \mathbb{R}) = H^1_{\tilde{J}}(\tilde{M}_p) \oplus H^2_{\tilde{J}}(\tilde{M}_p),
\]
where \( (\tilde{M}_p, J) \) is the complex blow-up of \( (M, J) \) at the point \( p = \iota(0) \). By using Proposition 3.1 we get that \( J \) is \( C^\infty \) full. \( \square \)

The assumption in the last theorem that \( i^* J = J_0 \) is not so restrictive: indeed, starting with a calibrated almost complex structure, one can modify it in order that the new one satisfies the previous condition, as showed in the following:

**Theorem 4.2.** Let \( (M, \omega) \) be a compact 2n-dimensional symplectic manifold, \( 2n \geq 4 \). \( J \) be an \( \omega \)-calibrated almost complex structure, \( p \in M \) and \( \iota : (\mathbb{B}^{2n}(R), \omega_0) \to (V, \omega) \) be a symplectomorphism such that \( \iota(0) = p \), where \( V \) is an open neighborhood of \( p \) in \( M \). Let \( W = \iota(\mathbb{B}^{2n}(R')), \) for \( 0 < R' < R \). Then there exists an almost complex structure \( \tilde{J} \) on \( M \) such that

(1) \( \tilde{J} = J \) on \( M \setminus W \);
(2) \( \tilde{J} = \iota_* \circ J_0 \circ \iota_*^{-1} \) on an open set \( U \subset W \);
(3) \( \tilde{J} \) is \( \omega \)-calibrated.

**Proof.** By assumption we have that \( \iota^* \omega = \omega_0 \). Let \( \tilde{J} \) be the almost complex structure on \( W \) defined by \( \tilde{J} = \iota_* \circ J_0 \circ \iota_*^{-1} \). By construction \( \tilde{J} \) is \( \omega \)-calibrated on \( W \). Then, by [2] there exists an endomorphism \( \tilde{\mathbb{L}} \) of the tangent bundle \( TW \) such that

(a) \( \iota \tilde{J} + J \tilde{\mathbb{L}} = 0 \);
(b) \( \iota \tilde{\mathbb{L}} = \tilde{\mathbb{L}} \);
(c) \( \| \tilde{\mathbb{L}} \| < 1 \)

and
\[
\tilde{\mathbb{L}} = (I + \tilde{\mathbb{L}})(I + \tilde{\mathbb{L}})^{-1}
\]
on \( W \), where \( I \) denotes the identity. Let \( \rho : M \to \mathbb{R} \) be a \( C^\infty \)-function with \( \text{supp} (\rho) \subset W \) and such that \( \rho \geq 0 \) on \( M \) and \( \rho = 1 \) on a neighborhood \( U \) of \( p \) with \( U \subset W \). Consider the endomorphism of the tangent bundle \( TM \) defined by
\[
\tilde{J} = (I + \rho \tilde{\mathbb{L}})(I + \rho \tilde{\mathbb{L}})^{-1}.
\]

Then \( \tilde{J}^2 = -I \),
\[
\tilde{J} = J \quad \text{on} \ M \setminus W,
\]
\[
\tilde{J} = \tilde{J} = \iota_* \circ J_0 \circ \iota_*^{-1} \quad \text{on} \ U
\]
and \( \tilde{J} \) is \( \omega \)-calibrated. \( \square \)
5. Almost complex structures and symplectic blow-up along a submanifold

We briefly recall the definition of the symplectic blow-up of a symplectic manifold along a symplectic submanifold; for more details, see [11] and [8]. Let \((M, \omega)\) be a 2n-dimensional symplectic manifold and consider a symplectic embedding \(i : (Y, \sigma) \hookrightarrow (M, \omega)\) of a compact symplectic submanifold \((Y, \sigma)\) of codimension 2k and such that the normal bundle of \(Y\) in \(M\) reduces to \(U(k)\). We will identify \(Y\) with the image \(i(Y) \subset M\). The normal bundle \(\mathcal{N}_{\mathcal{Y}|M}\) of \(i(Y)\) in \(M\) may be identified with the symplectic orthogonal bundle of the tangent bundle \(TY\) in \(TM\). Therefore, \(\mathcal{N}_{\mathcal{Y}|M}\) carries a canonical symplectic bundle structure, given by the restriction of \(\omega\) to each fibre, and hence a homotopically unique calibrated almost complex structure.

With respect to this structure, we consider the projectivization \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\). Choose a tubular neighborhood \(W\) of \(Y\) in \(M\). There exists a closed 2-form \(\rho\) on \(\mathcal{N}_{\mathcal{Y}|M}\) which restricts to \(\sigma\) along the zero section and to the canonical symplectic form on each fibre, and with respect to which \(W\) may be symplectically identified with a neighborhood \(V\) of the zero section of \(\mathcal{N}_{\mathcal{Y}|M}\) (see [11, Lemma 3.1]). Let

\[
\mathcal{L} = \{ (l, \epsilon) \in \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M}) \times \mathcal{N}_{\mathcal{Y}|M}, l \in H^1 \}
\]

be the tautological line bundle over \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\). This is called the blow-up of \(\mathcal{N}_{\mathcal{Y}|M}\) along \(Y\). It fibres over \(Y\) and each fiber is a line bundle over \(\mathbb{C}^{p+1}\).

Let \(q : \mathcal{L} \to \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\) be the bundle projection and \(\varphi : \mathcal{L} \to \mathcal{N}_{\mathcal{Y}|M}\) be the natural projection, then we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\varphi} & \mathcal{N}_{\mathcal{Y}|M} \\
q \downarrow & & \downarrow q \\
\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M}) & \xrightarrow{p} & Y.
\end{array}
\]

Since \(\varphi\) is an isomorphism outside the zero section of \(\mathcal{L}\), one can give the following definition (see [11]).

**Definition 5.1.** Let \(\tilde{V} := \varphi^{-1}(V)\) be a disc sub-bundle (with fibres real 2-discs) of the complex line bundle \(\mathcal{L}\). The symplectic blow-up \(\tilde{M}\) of \(M\) along \(Y\) is the manifold

\[
\tilde{M} := M \setminus W \cup_{\tilde{\varphi}} \tilde{V}.
\]

Then \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\) can be considered as the zero section of the disc bundle \(\tilde{V} \subset \mathcal{L}\) and \(Y\) can be viewed as the zero section of \(V \subset \mathcal{N}_{\mathcal{Y}|M}\). Therefore, by using \(\varphi\), one has an identification of \(\tilde{V} \setminus \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\) with \(V \setminus Y\). Therefore, one may also obtain \(\tilde{M}\) by identifying \(M\) \(Y\) and \(\tilde{V}\) along \(W \setminus Y \equiv V \setminus Y \equiv \tilde{V} \setminus \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\). Moreover, there is a natural inclusion \(i : \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M}) \to \tilde{M}\) and \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\) is the exceptional divisor of the blow-up.

By [11, Proposition 2.4] there is a short exact sequence

\[
0 \to H^*(M, \mathbb{R}) \to H^*(\tilde{M}, \mathbb{R}) \to A^* \to 0,
\]

where \(A^*\) is a free module over \(H^*(V, \mathbb{R})\) with generators \(a, \ldots, a^{k-1}\), where \(a \in H^2(\mathcal{L})\) is the first Chern class of the dual of the tautological line bundle over \(\mathcal{L}\). Then

\[
H^2(\tilde{M}, \mathbb{R}) = H^2(M, \mathbb{R}) \oplus \text{span}(a).
\]

McDuff showed in [11, Proposition 3.7] that, if \(Y\) is compact, then the symplectic blow-up \(\tilde{M}\) of \(M\) along \(Y\) has a symplectic form \(\tilde{\omega}\), which equals \(\varphi^*\omega\) outside a neighborhood of \(\varphi^{-1}(Y)\). Such a form is constructed in the following way. On \(M \setminus W\), she sets \(\tilde{\omega} = \omega\) and \(\tilde{\omega}\) on \(V\) must be a symplectic form on \(V\) which coincides with \(\varphi^*\rho\) near \(\partial V\). To construct this form, she considers a closed 2-form \(\alpha\) on \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\) that restricts to the canonical symplectic form on each fibre of \(p\), and that pulls back under \(q\) to a form on \(\mathcal{L}\) that is exact outside the zero section \(\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M}) \subset \mathcal{L}\).

Since \(q^*\alpha\) is exact away from the zero section of \(\mathcal{L}\), one finds a 1-form \(\beta\) such that \(q^*\alpha = d\beta\) on \(\mathcal{L} \setminus \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})\). There exists an \(\epsilon > 0\), depending on \(\rho\) and \(\alpha\), such that for \(\epsilon \in (0, \epsilon]\) and with \(\lambda\) a radial bump function on \(V\) which equals 0 near the boundary, the 2-form

\[
\tilde{\rho} = \begin{cases} 
\varphi^*\rho + \epsilon q^*\alpha & \text{on } T\tilde{M}|_{\mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})}, \\
\varphi^*\rho + \epsilon d(\lambda, \beta) & \text{on } V \setminus \mathbb{P}(\mathcal{N}_{\mathcal{Y}|M})
\end{cases}
\]

is non-degenerate on \(\tilde{V}\), and the form

\[
\tilde{\omega} = \begin{cases} 
\omega & \text{on } M \setminus W, \\
\tilde{\rho} & \text{on } \tilde{V}
\end{cases}
\]

is a symplectic form on \(\tilde{M}\).
By [8] one has a map \( f : \tilde{M} \to M \) by
\[
f = \begin{cases} 
\text{id} & \text{on } M \setminus W, \\
\varphi & \text{on } \tilde{V}.
\end{cases}
\]
(6)

This map is a diffeomorphism outside \( \mathbb{P}(N_{Y|M}) \), and \( \mathbb{P}(N_{Y|M}) = f^{-1}(Y) \). In particular, one has the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(N_{Y|M}) & \xrightarrow{i} & \tilde{M} \\
p \downarrow & & \downarrow f \\
Y & \xrightarrow{i} & M.
\end{array}
\]

The normal bundle of \( i(\mathbb{P}(N_{Y|M})) \) in \( \tilde{M} \) is isomorphic to \( L \), i.e. one has the short exact sequence of vector bundles:
\[
0 \to T\mathbb{P}(N_{Y|M}) \to i^*TM \to L \to 0
\]
and by construction \( i : \mathbb{P}(N_{Y|M}) \to \tilde{M} \) is symplectic.

**Remark 5.2.** Note that the construction of the symplectic form \( \tilde{\omega} \) on the symplectic blow-up of the symplectic manifold \((M, \omega)\) depends on the choice of the 2-forms \( \rho \) and \( \alpha \), a bump function \( \lambda \), and an \( \varepsilon \) in an open interval depending on \( \rho \) and \( \alpha \). However, the definition of \( M \) depends on the choice of the complex bundle structure on \( N_{Y|M} \).

In the case that \( \tilde{J} \) and \( J \) are almost complex structures on \( \tilde{M} \) and \( M \), respectively, such that the map \( f : (\tilde{M}, \tilde{J}) \to (M, J) \) defined above is holomorphic, then we have the following

**Theorem 5.3.** Let \((M, \omega)\) be a compact symplectic manifold and \( i : (Y, \sigma) \to (M, \omega) \) be a symplectic embedding of a compact submanifold \( Y \). Consider the symplectic blow-up \( \tilde{M} \) of \((M, \omega)\) along \( Y \). Let \( J \) and \( \tilde{J} \) respectively almost complex structures on \( M \) and \( \tilde{M} \) such that the map \( f : (\tilde{M}, \tilde{J}) \to (M, J) \), given by (6), is pseudoholomorphic.

If \( J \) is \( C^\infty \) pure and full and it is \( \omega \)-calibrated, then \( \tilde{J} \) is \( C^\infty \) pure and full.

**Proof.** Since \( J \) is \( C^\infty \) pure and full we have that
\[
H^2(M, \mathbb{R}) = H^{1,1}_j(M) \oplus H^{(2,0), (0,2)}_j(M) \oplus \text{span}(\alpha).
\]
Moreover, by [11] (see formula (5))
\[
H^2(\tilde{M}, \mathbb{R}) = H^2(M, \mathbb{R}) \oplus \text{span}(\alpha).
\]
Since \( f \) is pseudoholomorphic we have that \( \tilde{J} = J \) on \( M \setminus W \) and \( J \circ \varphi_* = \varphi_* \circ \tilde{J} \) on \( \tilde{V} \). This implies that a form of type \((p, q)\) on \( M \) with respect to \( J \) is also of type \((p, q)\) with respect to \( \tilde{J} \). Since \( J \) is calibrated by \( \omega \), then \( \alpha \) is of type \((1, 1)\) with respect to \( \tilde{J} \). Therefore, the theorem is proved. \( \square \)

Starting with an almost complex structure on \( M \) preserving the submanifold \( Y \), the assumption in **Theorem 5.3** that there exist almost complex structures on \( \tilde{M} \) and \( M \) such that \( f \) is pseudoholomorphic is satisfied. Indeed,

**Proposition 5.4.** Let \((M, \omega)\) be a symplectic manifold and \( i : (Y, \sigma) \to (M, \omega) \) be a symplectic embedding of a compact submanifold \( Y \). Consider the symplectic blow-up \( \tilde{M} \) of \((M, \omega)\) along \( Y \). Let \( J \) be an \( \omega \)-calibrated almost complex structure on \( M \) such that \( J|_Y : TY \to TY \). Then there exists an almost complex structure \( \tilde{J} \) on \( \tilde{M} \) such that \( f : (\tilde{M}, \tilde{J}) \to (M, J) \), given by (6), is pseudoholomorphic.

**Proof.** By the assumption that the submanifold \( Y \) is \( J \)-invariant and \( J \) is \( \omega \)-calibrated, \( J \) naturally induces an almost complex structure \( \tilde{J} \) on the normal bundle \( N_{Y|M} \), and, consequently, on its projectivization, also denoted by \( \tilde{J} \). As already recalled, \( \varphi : \tilde{V} \setminus \mathbb{P}(N_{Y|M}) \to W \setminus Y \) is a diffeomorphism. Set
\[
\tilde{J} = \begin{cases} 
\tilde{J} & \text{on } M \setminus W, \\
\varphi_*^{-1} \circ J \circ \varphi_* & \text{on } \tilde{V} \setminus \mathbb{P}(N_{Y|M}), \\
\tilde{J} & \text{on } \mathbb{P}(N_{Y|M}).
\end{cases}
\]

Then \( \tilde{J} \) gives rise to an almost complex structure on \( \tilde{M} \) such that \( f : (\tilde{M}, \tilde{J}) \to (M, J) \) is pseudoholomorphic. \( \square \)
Acknowledgement

We would like to thank Daniele Angella for useful comments on the paper.

References