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Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients ☆

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ABSTRACT

In this paper, some sufficient conditions are obtained for the oscillation of all solutions of even-order nonlinear neutral differential equations with variable coefficients. Our results improve and generalize known results. In particular, the results are new even when $n = 2$.

1. Introduction

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [1, P292]. Recently, many studies have been made on the oscillatory behavior of solutions of higher-order neutral differential equations; see, for example, [2–6] and the references cited therein.

In this paper, we consider the even-order nonlinear neutral delay differential equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)f[x(\sigma(t))] = 0, \quad n \text{ is even.} \tag{1}$$

Throughout this paper, the following conditions are assumed to hold.

(H1) $0 \le p(t) < 1$, $q(t) \ge 0$ and are continuous on $[t_0, \infty)$;

(H2) $\sigma(t), \tau(t)$ are continuous nonnegative functions with $\lim_{t\to\infty} \sigma(t) = \infty$, $\lim_{t\to\infty} \tau(t) = \infty$, and $\sigma(t) \le t$, $\tau(t) \le t$;

(H3) $f(x)$ is a continuous function on $(-\infty, +\infty)$, $xf(x) > 0$, and $f'(x) \ge 0$ for all $x \ne 0$.

In what follows, we restrict our attention to solutions of (1) which exist on some half-line and are nontrivial for all large t. As is customary, a solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called non-oscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

We notice that, in Eq. (1), if $f(x) = x$, then Eq. (1) can be written as the linear equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)x(\sigma(t)) = 0, \quad n \text{ is even.} \tag{2}$$

In Eq. (2), if $p(t) \equiv 0$, then Eq. (2) can be written as the linear equation

$$x^{(n)}(t) + q(t)x(\sigma(t)) = 0, \quad n \text{ is even.} \tag{3}$$

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In Eq. (3), if $n = 2$, then Eq. (3) can be written as the second-order linear equation

$$x''(t) + q(t)x(\sigma(t)) = 0. \tag{4}$$

In 1986, Koplatadze [7], in 1988, Wei [8], and in 2000, Koplatadze [9], respectively, discussed oscillation criteria for Eq. (4), and obtained some sufficient conditions for Eq. (4) to be oscillatory. In 1998, Zafer [6] generalized the results of Koplatadze for Eq. (2); in 2004, Bai [10] studied oscillation criteria for Eq. (3), and obtained some sufficient conditions for Eq. (3) to be oscillatory. So far, the study of the oscillation behavior of even-order nonlinear neutral differential equations has not been sufficient. The purpose of this article is to improve and generalize the results of Koplatadze, Zafer, Wei and Bai, and show that some results in [6,10,7–9] are special cases of those in this article. In particular, the results are new even when $n = 2$.

The following lemma is well known; see Kiguradge [2, Lemma 2.2.1].

Lemma 1. *Let $u(t)$ be a positive and n-times differentiable function on an interval $[T,\infty)$ with its n-th derivative $u^{(n)}(t)$ nonpositive on $[T,\infty)$ and not identically zero on any interval of the form $[T',\infty)$, $T' \geq T$. Then there exists an integer l, $0 \leq l \leq n-1$, with $n+l$ odd, such that, for some large $T^* \geq T'$,*

$$(-1)^{l+j}u^{(j)} > 0 \quad \text{on } [T^*,\infty)(j = l, l+1, \ldots, n-1)$$
$$u^{(i)} > 0 \quad \text{on } [T^*,\infty)(i = 1, 2, \ldots, l-1) \text{ when } l > 1.$$

The next lemma is given in [2, P169].

Lemma 2. *Let $u(t)$ be as in Lemma 1. If $\lim_{t\to\infty} u(t) \neq 0$, then, for every λ, $0 < \lambda < 1$, there is $T_\lambda \geq t_0$ such that, for all $t \geq T_\lambda$,*

$$u(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}u^{(n-1)}(t).$$

2. Main results

First, we establish a comparison theorem.

Theorem 1. *Let $|f(x)| \geq |x|$, for all $|x| \geq x_0 > 0$. Assume that there exists a constant $0 < \lambda_0 < 1$, such that the first-order delay differential equation*

$$z'(t) + \frac{\lambda_0}{(n-1)!}q(t)\sigma^{n-1}(t)[1 - p(\sigma(t))]z(\sigma(t)) = 0 \tag{5}$$

is oscillatory. Then (1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of (1), say $x(t) > 0$ and $x(\sigma(t)) > 0$, when $t \geq t_0$. Let

$$y(t) = x(t) + p(t)x(\tau(t)). \tag{6}$$

Then, from (H1), (H2) and (H3), there exists $t_1 \geq t_0$ such that

$$y(t) > 0 \quad \text{and} \quad y^{(n)}(t) \leq 0 \text{ for all } t \geq t_1.$$

By Lemma 1, there exists $t_2 \geq t_1$ and an odd integer $l \leq n-1$ such that, for some large $t_3 \geq t_2$,

$$\begin{aligned} &(-1)^{l+j}y^{(j)}(t) > 0, \quad j = l, l+1, \ldots, n-1, t \geq t_3, \\ &y^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l-1, t \geq t_3. \end{aligned} \tag{7}$$

Thus, from (7), $y'(t) > 0$ and $y^{(n-1)}(t) > 0$ for $t \geq t_3$. Hence, $\lim_{t\to\infty} y(t) \neq 0$. By Lemma 2, for every λ, $0 < \lambda < 1$, there exists T_λ such that, for all $t \geq T_\lambda$,

$$y(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}y^{(n-1)}(t). \tag{8}$$

From (6), $x(\sigma(t)) = y(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))$, and consequently, we have

$$y^{(n)}(t) + q(t)f[y(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))] = 0, \quad \text{for all large } t.$$

Noting that $|f(x)| \geq |x|$, for all $|x| \geq x_0 > 0$, we obtain

$$y^{(n)}(t) + q(t)[y(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))] \leq 0, \quad \text{for all large } t.$$

Noting that $y(t) > x(t)$ and $y'(t) > 0$, we obtain

$$y^{(n)}(t) + q(t)[1 - p(\sigma(t))]y(\sigma(t)) \leq 0, \quad \text{for all sufficiently large } t.$$

Now, using (8), we have that, for every $0 < \lambda < 1$,

$$y^{(n)}(t) + \frac{\lambda}{(n-1)!} q(t)\sigma^{n-1}(t)[1 - p(\sigma(t))]y^{(n-1)}(\sigma(t)) \le 0, \quad \text{for all large } t.$$

Let $u(t) = y^{(n-1)}(t)$. Thus, for t large enough, $u(t)$ satisfies

$$u'(t) + \frac{\lambda}{(n-1)!} q(t)\sigma^{n-1}(t)[1 - p(\sigma(t))]u(\sigma(t)) \le 0, \quad \text{for every } 0 < \lambda < 1. \tag{$*$}$$

Since the differential inequality $(*)$ has the nonoscillatory solution $u(t)$ by a well-known result (see Corollary 3.2.2 in [4], P67), the differential equation

$$z'(t) + \frac{\lambda}{(n-1)!} q(t)\sigma^{n-1}(t)[1 - p(\sigma(t))]z(\sigma(t)) = 0,$$

also has an eventually positive solution for every $0 < \lambda < 1$. This contradicts the fact that (5) is oscillatory. In the case that $x(t)$ is an eventually negative solution, $-x(t)$ will be an eventually positive solution. The proof of Theorem 1 is complete. □

It is well known (see [3,4]) that if either

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\mathrm{d}s > \frac{1}{e} \tag{9}$$

or $\sigma(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\mathrm{d}s > 1, \tag{10}$$

then

$$x'(t) + q(t)x(\sigma(t)) = 0 \tag{11}$$

is oscillatory.

Thus, from Theorem 1, we can obtain the following result.

Corollary 1. *Eq. (1) is oscillatory if either*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > \frac{(n-1)!}{e} \tag{12}$$

or $\sigma(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > (n-1)!. \tag{13}$$

Proof. From (12) and (13), one can choose a positive constant $0 < \lambda_0 < 1$ such that

$$\liminf_{t\to\infty} \lambda_0 \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > \frac{(n-1)!}{e}$$

or

$$\limsup_{t\to\infty} \lambda_0 \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > (n-1)!.$$

By Theorem 1, and in view of (9) and (10), the conclusion of Corollary 1 is obtained. The proof of Corollary 1 is complete. □

Remark 1. In [6], the author obtains sufficient conditions for (2) to be oscillatory, namely, if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > \frac{(n-1)2^{(n-1)(n-2)}}{e} \tag{14}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)[1 - p(\sigma(s))]\mathrm{d}s > (n-1)2^{(n-1)(n-2)}. \tag{15}$$

In the case of (15), $\sigma(t)$ is nondecreasing. Since $(n-1)! < (n-1)2^{(n-1)(n-2)}$ when $n \ge 3$, it follows that Corollary 1 improves and generalizes Theorem 2 in [6].

Remark 2. Recently, the following sufficient conditions for (3) to be oscillatory were obtained (see Theorems 1 and 2 in [10]): either

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)\mathrm{d}s > \frac{1}{e}2^{(n-1)^2}(n-1)!$$

or $\sigma(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\sigma^{n-1}(s)\mathrm{d}s > 2^{(n-1)^2}(n-1)!.$$

Clearly, Corollary 1 improves and generalizes the above results.

Remark 3. When $p(t)\equiv 0$ and $n=2$, Corollary 1 reduces to being if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\sigma(s)\mathrm{d}s > \frac{1}{e}, \tag{16}$$

or $\sigma(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\sigma(s)\mathrm{d}s > 1, \tag{17}$$

then Eq. (4) is oscillatory. These results have been established in [7,8]; also see [9].

3. Further results

Let $\delta(t) = \max_{t_0\le s\le t}\sigma(s)$ and $\delta^{-1}(t) = \sup\{s \ge t_0 : \delta(s) = t\}$, $\delta^{-(k+1)}(t) = \sup\{s \ge \delta^{-k}(t_0) : \delta^{-k}(s) = t\}$. Set $Q(t) = \frac{1}{(n-1)!}q(t)\sigma^{n-1}(t)[1-p(\sigma(t))]$.

Define a sequence $\{Q_k(t)\}$ of functions as follows:

$$Q_1(t) = \int_{\delta(t)}^{t} Q(s)\mathrm{d}s, \quad t \ge \delta^{-1}(t_0),$$

$$Q_{k+1}(t) = \int_{\delta(t)}^{t} Q(s)Q_k(s)\mathrm{d}s, \quad t \ge \delta^{-(k+1)}(t_0), k = 1, 2, \ldots. \tag{18}$$

Theorem 2. *Assume that there exists a positive integer K such that*

$$\liminf_{t\to\infty} Q_K(t) > \frac{1}{\mathrm{e}^K}. \tag{19}$$

Then (1) is oscillatory.

Proof. Let $\liminf_{t\to\infty} Q_K(t) > \alpha > 0$. In view of (19), we can choose a constant $0<\lambda_0<1$ such that $\alpha\lambda_0^K > \frac{1}{\mathrm{e}^K}$; that is,

$$\liminf_{t\to\infty} \lambda_0^K Q_K(t) > \frac{1}{\mathrm{e}^K}. \tag{20}$$

Suppose, for the sake of contradiction, that (1) has an eventually positive solution $x(t)$. Let $y(t) = x(t)+p(t)x(\tau(t))$. We can proceed as in the proof of Theorem 1 and show that for λ_0 the equation

$$z'(t) + \frac{\lambda_0}{(n-1)!}q(t)\sigma^{n-1}(t)[1-p(\sigma(t))]z(\sigma(t)) = 0 \tag{21}$$

has an eventually positive solution. On the other hand, by [11, Theorem 1] and (20), all the solutions of (21) are oscillatory. This is a contradiction. The proof of Theorem 2 is complete. □

Remark 4. Theorem 2 is a new result, which improves and generalizes Theorem 2 in [6]; even for second-order neutral differential equations, Theorem 2 still holds. In particular, when $p(t)\equiv 0$ and $n=2$, for the delay differential equation (4), we have the following result.

Corollary 2. *Assume that there exists a K such that*

$$\liminf_{t\to\infty} Q_K(t) > \frac{1}{\mathrm{e}^K}. \tag{22}$$

Then the Eq. (4) is oscillatory.

Remark 5. Corollary 2 improves and generalizes the relative results from [7,8]. When $K = 1$, (22) reduces to (16).

Example. Consider the equation

$$\left[x(t) + \frac{1}{2}x(t-\pi)\right]^{(n)} + \frac{(n-1)!(1+\cos t)}{e(t-\pi)^{n-1}}x(t-\pi)\ln[e + x^2(t-\pi)] = 0, \quad t \geq t_0 > \pi, n \text{ is even.} \tag{23}$$

From (18), we have

$$Q(t) = \frac{1}{2e}(1+\cos t), t \geq t_0 \quad \text{and} \quad \liminf_{t\to\infty} \int_{t-\pi}^{t} Q(s)\mathrm{d}s = \frac{1}{2e}(\pi - 2) < \frac{1}{e}.$$

It is easy to show that (see [12])

$$Q_1(t) = \frac{1}{2e}(\pi + 2\sin t), \quad t \geq t_0,$$

$$Q_2(t) = \frac{1}{4e^2}(\pi^2 + 2\pi\sin t - 4\cos t), \quad t \geq 2\pi,$$

$$Q_3(t) = \frac{1}{8e^3}\left[\pi^3 - 2\pi + (2\pi^2 - 8)\sin t - 4\pi\cos t\right], \quad t \geq 3\pi,$$

$$Q_4(t) = \frac{1}{16e^4}\left[\pi^4 - 4\pi^2 + 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4)\cos t\right], \quad t \geq 4\pi.$$

$$\liminf_{t\to\infty} Q_k(t) < \frac{1}{e^k}, \quad k = 2, 3. \text{ But } \liminf_{t\to\infty} Q_4(t) > \frac{1}{e^4}.$$

Thus, by Theorem 2, (23) is oscillatory. The known results in the literature are not applicable to (23).

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