Abstract

This paper deals with the Cauchy problem for a quasilinear first-order equation that includes a possibly discontinuous hysteresis operator $\mathcal{F}$:
\[
\frac{\partial}{\partial t} [u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f \quad \text{in } \mathbb{R}, \quad \text{for } t > 0.
\]
Existence of a weak solution is proved for $\mathcal{F}$ equal to a completed relay operator. In the case of $f \equiv 0$, an entropy-type condition yields Lipschitz-continuous and monotone dependence on the initial data, hence uniqueness.

Keywords: Hysteresis; Hyperbolic equations; Weak formulation; Entropy condition

Introduction

In this paper we deal with the Cauchy problem for a first-order quasilinear PDE that contains an either continuous or discontinuous hysteresis operator $\mathcal{F}$:
\[
\begin{cases}
\frac{\partial}{\partial t} [u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f & \text{in } \mathbb{R} \times [0, T], \\
(u + w)|_{t=0} = u^0 + w^0 & \text{in } \mathbb{R}.
\end{cases}
\] (1)

More specifically, we assume that $\mathcal{F}$ is equal either to a relay operator or to its regularization, cf. Figs. 1 and 2. The latter operator is continuous, whereas the former is
discontinuous; it seems then necessary to replace the relay by its (multivalued) closure with respect to appropriate topologies, i.e. the completed relay, that we formulate along the lines of [18,20]; the latter is also the limit of the regularized relays, as the regularization parameter vanishes.

We introduce a weak formulation of (1), and prove existence of a solution; for \( f \equiv 0 \) we also show continuous dependence on the data, whence uniqueness. These results might easily be extended to the larger class of either continuous or discontinuous Preisach operators (i.e., linear combinations of a possibly infinite family of relays).

This work is in the framework of a research on models of hysteresis phenomena and on related PDEs, author started several years ago; see [17] and references therein. In the last years research on mathematical aspects of hysteresis has been progressing, see, e.g., the monographs [3,4,8,11,14,17]. It seems that so far little attention has been paid to the above problem; however, for a large class of either continuous or discontinuous hysteresis operators, that includes those dealt with in this work, (1)1 can be set in the form

\[
\frac{dU}{dt} + AU \ni F \quad (U := (u, w), \ F := (f, 0)); \tag{2}
\]

here \( A \) is a multivalued m-accretive operator in \( L^1 \)-type spaces, that obviously depends on \( F \). The theory of nonlinear semigroups (see, e.g., [1,2,5]) then yields the well-posedness for a rather weak notion of solution, cf. [17, Chapter VIII]. Here accretivity is based on a fundamental inequality due to Hilpert [7], that also plays a major role in the proof of well-posedness (for \( f \equiv 0 \)) for the formulation of this work. A problem like (1), with a different hysteresis operator, was studied in [15] as a model of transport with adsorption and desorption; in that paper the reader may also find several references to engineering applications of (1).

Second-order quasilinear hyperbolic equations of the form \( \frac{d^2}{dt^2} [u + F(u)] + Au = f \), with \( A \) an elliptic operator, were studied in [18,20] using the formulation of the completed relay we also apply in this work. A different approach was used by Krejčí [9,10], see also [11, Chapters III, IV], for \( F \) equal to a Prandtl–Ishlinskiĭ operator.

The plan of this paper is as follows. In Section 1 we review the relay operator and its closure, regularize it, and provide a weak formulation. In Section 2 we formulate the Cauchy problem for Eq. (1) in the framework of Sobolev spaces. In Section 3 we derive a discrete version of Hilpert’s inequality, and in Section 4 we prove existence of a solution of the Cauchy problem, for \( F \) equal either to a completed relay operator or to its regularization. In Section 5 we assume that \( f \equiv 0 \), and prove Lipschitz-continuous and monotone dependence of the solution on the initial data (whence uniqueness), by introducing an entropy-like condition and then proceeding along the lines of the classic argument of Kružkov [12,13]. In a work apart [21] this technique is also used to prove uniqueness of the solution for a quasilinear parabolic equations with discontinuous hysteresis.

1. Discontinuous hysteresis

In this section we briefly review the definition of the (delayed) relay, and specify the functional framework. In view of inserting this operator into PDEs, we also provide a weak formulation and introduce a regularization.
Let us fix any pair \( \rho := (\rho_1, \rho_2) \in \mathbb{R}^2 \), with \( \rho_1 < \rho_2 \). For any continuous function \( u : [0, T] \rightarrow \mathbb{R} \) and any \( \xi \in \{-1, 1\} \), we set \( X_u(t) := \{ \tau \in [0, t] : u(\tau) = \rho_1 \text{ or } \rho_2 \} \) and

\[
\begin{align*}
w(0) &:= \begin{cases} -1 & \text{if } u(0) \leq \rho_1, \\ \xi & \text{if } \rho_1 < u(0) < \rho_2, \\ 1 & \text{if } u(0) \geq \rho_2, \end{cases} \\
w(t) &:= \begin{cases} w(0) & \text{if } X_u(t) = \emptyset, \\ -1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_1, \forall t \in [0, T], \\ 1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_2, \end{cases}
\end{align*}
\]

(1.1)

(1.2)

cf. Fig. 1. Any continuous function \( u : [0, T] \rightarrow \mathbb{R} \) is uniformly continuous, hence it may just have a finite number of oscillations (if any) between the thresholds \( \rho_1, \rho_2 \); therefore \( w \in BV(0, T) \). The operator \( h_\rho : C_0^0([0, T]) \times \{-1, 1\} \rightarrow BV(0, T) : (u, \xi) \mapsto w \) is thus defined. For any increasing continuous function \( \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), if \( w = h_\rho(u) \) then \( w \circ \varphi = h_\rho(u \circ \varphi) \), that is, \( h_\rho \) is rate-independent; thus it is a hysteresis operator.

**Completed relay operator**

It is easy to see that the operator \( h_\rho(\cdot, \xi) : C^0([0, T]) \rightarrow L^1(0, T) \) is not closed. We then introduce the completed relay operator, \( k_\rho : C^0([0, T]) \times [-1, 1] \rightarrow \mathcal{P}(BV(0, T)) \) (the set of the parts of \( BV(0, T) \)), we define as follows. For any \( u \in C^0([0, T]) \) and any \( \xi \in [-1, 1] \), we set \( w \in k_\rho(u, \xi) \) if and only if \( w \) is measurable in \( [0, T] \),

\[
w(0) := \begin{cases} -1 & \text{if } u(0) < \rho_1, \\ \xi & \text{if } \rho_1 \leq u(0) \leq \rho_2, \\ 1 & \text{if } u(0) > \rho_2, \end{cases}
\]

(1.3)

and, for any \( t \in [0, T] \),

\[
w(t) \in \begin{cases} -1 & \text{if } u(t) < \rho_1, \\ [-1, 1] & \text{if } \rho_1 \leq u(t) \leq \rho_2, \\ 1 & \text{if } u(t) > \rho_2, \end{cases}
\]

(1.4)
Fig. 2. The graph of the completed relay operator is outlined in part (a). Any point of the rectangle $[\rho_1, \rho_2] \times [-1, 1]$ is accessible to the pair $(u, w)$. If $u(t) = \rho_1$ ($u(t) = \rho_2$, respectively) then $w$ is locally nonincreasing (nondecreasing, respectively); if $\rho_1 < u(t) < \rho_2$ then $w$ is locally constant. The graph of the corresponding regularized relay operator is represented in part (b).

\[
\begin{align*}
\text{if } u(t) \neq \rho_1, \rho_2, & \quad \text{then } w \text{ is constant in a neighbourhood of } t, \\
\text{if } u(t) = \rho_1, & \quad \text{then } w \text{ is nonincreasing in a neighbourhood of } t, \\
\text{if } u(t) = \rho_2, & \quad \text{then } w \text{ is nondecreasing in a neighbourhood of } t,
\end{align*}
\]

(1.5)

cf. Fig. 2(a). The graph of $k_\rho$ in the $(u, w)$-plane invades the whole rectangle $[\rho_1, \rho_2] \times [-1, 1]$. The operator $k_\rho$ is the closure of $h_\rho(\cdot, \xi) : u \mapsto w$ with respect to the strong topology of $C^0([0, T])$ for $u$ and the weak topology of $L^1(0, T)$ for $w$, and then seems to be more prone than $h_\rho$ to be coupled with PDEs. This formulation of the completed relay operator slightly differs from that of [17, Section VI.1], where the closure of $h_\rho : (u, \xi) \mapsto w$ is derived; the present formulation may be justified via a simplification of the argument of [17].

It is not difficult to see that the conditions (1.4) and (1.5) are respectively equivalent to

\[
\begin{align*}
|w| & \leq 1, \\
\left\{ \begin{array}{ll}
(w - 1)(u - \rho_2) \geq 0, & \\
(w + 1)(u - \rho_1) \geq 0 & \text{a.e. in } ]0, T[,
\end{array} \right.
\end{align*}
\]

(1.6)

\[
\int_0^t u \, dw \geq \int_0^t \rho_2 \, dw^+ - \int_0^t \rho_1 \, dw^- =: \Psi_\rho^0(w; [0, t]) \quad \forall t \in ]0, T]
\]

(1.7)

(these are Stieltjes integrals), cf. [18]. Notice that $\Psi_\rho^0(w; [0, t])$ is finite whenever $\partial w / \partial t \in C^0(\overline{\mathbb{R}T})'$.

Regularized relay operator

Now we approximate the completed relay operator, $k_\rho$, as it is shown in Fig. 2(b). After [17, Chapter II], the dynamics that is illustrated by this graph and by the arrows defines a continuous hysteresis operator

\[
k_\rho^\varepsilon : C^0([0, T]) \times [-1, 1] \to C^0([0, T]) \cap BV(0, T) : u \mapsto w.
\]

(1.8)

Notice that

\[
w = k_\rho^\varepsilon(u, \xi) \iff w \in k_\rho(u - \varepsilon w, \xi).
\]

(1.9)
The regularized relay operator, \( k_\varepsilon \), may then also be set in weak form. By (1.6) and (1.7), the latter inclusion is equivalent to the following system of inequalities:

\[
|w| \leq 1, \quad \begin{cases}
(w - 1)(u - \varepsilon w - \rho_2) \geq 0, \\
(w + 1)(u - \varepsilon w - \rho_1) \geq 0
\end{cases} \quad \text{a.e. in }]0, T[. (1.10)
\]

\[
\int_0^t (u - \varepsilon w) \, dw \geq \Psi_\rho^0(w; [0, t]) \quad \forall t \in ]0, T]. (1.11)
\]

The latter inequality also reads

\[
\int_0^t u \, dw \geq \Psi_\rho^0(w; [0, t]) + \frac{\varepsilon}{2} [w(t)^2 - w(0)^2] =: \Psi_\rho^\varepsilon(w; [0, t]) \quad \forall t \in ]0, T]. (1.12)
\]

**Preisach models**

The large class of **Preisach models** [16] is constructed by combining a (possibly infinite) family of relay operators having different thresholds. First we define the so-called **Preisach (half-)plane** as the set of admissible thresholds of relay operators

\[
P := \{ \rho = (\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2 \},
\]

denote by \( \mathcal{R} \) the family of Borel measurable functions \( \mathcal{P} \to \{-1, 1\} \), and by \( \{\xi_\rho\} \) a generic element of \( \mathcal{R} \). For any finite (signed) Borel measure \( \mu \) over \( \mathcal{P} \), we then define the (completed) **Preisach operator**

\[
\mathcal{H}_\mu : C^0([0, T]) \times \mathcal{R} \to L^\infty(0, T),
\]

\[
[\mathcal{H}_\mu(u, \{\xi_\rho\})](t) := \int_{\mathcal{P}} [k_\rho(u, \xi_\rho)](t) \, d\mu(\rho) \quad \forall t \in [0, T]. (1.14)
\]

This operator is obviously determined by the Preisach measure \( \mu ; \mathcal{H}_\mu \) is causal and rate-independent, namely, it is a hysteresis operator. Whenever the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure, it is not difficult to see that the function \( \mathcal{H}_\mu(u, \{\xi_\rho\}) \) is continuous; under this hypothesis the operator \( \mathcal{H}_\mu(\cdot, \{\xi_\rho\}) \) is also continuous in \( C^0([0, T]) \), cf. [17, Chapter IV]. It should also be noticed that the regularized operator \( k_\varepsilon \) is an example of Preisach operator.

The above formulation is easily extended to the Preisach model: it suffices to regard \( \rho \) as a parameter, denote the output of the corresponding relay operator by \( w_\rho \), and then average (1.10) and (1.11) over all \( w_\rho \)'s with respect to the prescribed Preisach measure \( \mu \). However, in the remainder of this paper we confine ourselves to relay operators; we do so only out of simplicity, since all the results we derive might easily be extended to Preisach operators.

Detailed accounts of the Preisach model may be found, e.g., in the monographs [3,4,8, 11,14,17].
2. Weak formulation

In this section we formulate an initial-value problem in the framework of Sobolev spaces for Eq. (1) of the Introduction. We set \( R_t := \mathbb{R} \times ]0,t[ \) for any \( t > 0 \), fix any \( T > 0 \), and assume that

\[
\begin{align*}
&u^0, w^0 \in L^2(\mathbb{R}), \quad |w^0| \leq 1 \quad \text{a.e. in } \mathbb{R}, \\
&f \in L^1(R_T) \cap L^2(R_T).
\end{align*}
\]

We also assume that \( \varepsilon \geq 0 \), and provide a unified formulation of our problem for both the completed relay operator \( k_{\rho} (= k_0^0) \) and its regularization \( k_{\rho}^\varepsilon (\varepsilon > 0) \).

**Problem 2.1.** Find \( u_\varepsilon \in L^\infty(0,T;L^2(\mathbb{R})) \) and \( w_\varepsilon \in L^\infty(R_T) \) such that

\[
\begin{align*}
|w_\varepsilon| &\leq 1 \quad \text{a.e. in } R_T, \quad \frac{\partial w_\varepsilon}{\partial t} \in C_0^0(R_T)', \\
\iint_{R_T} (u_\varepsilon + w_\varepsilon - u^0 - w^0) \frac{\partial v}{\partial t} + u_\varepsilon \frac{\partial v}{\partial x} + f v \, dx \, dt &= 0, \\
\forall v \in H^1(R_T) \cap W^{1,1}(R_T), \quad v(\cdot, T) = 0, \\
\left\{ (w_\varepsilon - 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_2) \geq 0, \\
(w_\varepsilon + 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_1) \geq 0 \right\} &\quad \text{a.e. in } R_T, \\
\frac{1}{2} \int_\mathbb{R} [u_\varepsilon(x,t)^2 - u^0(x)^2] \, dx + \int_\mathbb{R} \Psi_\varepsilon^\rho(w_\varepsilon; [0,t]) \, dx &\leq \int_\mathbb{R} f u_\varepsilon \, dx \, d\tau \quad \text{for a.a. } t \in ]0,T[, \\
w_\varepsilon(\cdot, 0) &= w^0 \quad \text{a.e. in } \mathbb{R}.
\end{align*}
\]

**Interpretation**

Initial condition (2.6) makes sense because of the second part of (2.2).

Equation (2.3) entails

\[
\frac{\partial}{\partial t} (u_\varepsilon + w_\varepsilon) + \frac{\partial u_\varepsilon}{\partial x} = f \quad \text{in } D'(R_T).
\]

As \( f - \partial u_\varepsilon/\partial x \in L^2(0,T;H^{-1}(\mathbb{R})) \), this equation also holds in the latter space. We then get the initial condition in the sense of traces:

\[
(u_\varepsilon + w_\varepsilon)|_{t=0} = u^0 + w^0 \quad \text{in } H^{-1}(\mathbb{R}).
\]

Let us denote by \( \langle \cdot, \cdot \rangle \) the duality pairing between the spaces \( H^{-1}(\mathbb{R}) \) and \( H^1(\mathbb{R}) \). If \( u_\varepsilon \in L^2(0,T;H^1(\mathbb{R})) \), we can multiply (2.7) by \( u_\varepsilon \), getting

\[
\int_0^t \left\langle \frac{\partial}{\partial \tau} (u_\varepsilon + w_\varepsilon), u_\varepsilon \right\rangle d\tau = \int_\mathbb{R} f u_\varepsilon \, dx \, d\tau.
\]
Inequality (2.5) then reads
\[
\int_0^t \left( \frac{\partial}{\partial \tau} (u_\varepsilon + w_\varepsilon), u_\varepsilon \right) d\tau \geq \frac{1}{2} \int_0^t [u_\varepsilon(x,t)^2 - u^0(x)^2] dx + \int_{\mathbb{R}} \Psi_\varepsilon(w_\varepsilon; [0,t]) dx
\]
for a.a. \( t \in ]0, T[, \) (2.9)
which may be regarded as a reformulation of (1.12). By (1.8)–(1.12) the conditions (2.4)–(2.6) then stand for the relation
\[
w_\varepsilon \in k_\varepsilon^\rho(u_\varepsilon, w^0) \quad \text{a.e. in } \mathbb{R}. \) (2.10)
This argument is rigorously justified only if \( u_\varepsilon \in L^2(0, T; H^1(\mathbb{R})) \), and this property is far from being obvious for the solutions of Problem 2.1_\varepsilon. This problem may then be interpreted as a weak formulation of an initial-value problem for the system (2.7), (2.10).
Henceforth we shall write \((u, w)\) in place of \((u_\varepsilon, w_\varepsilon)\).

3. Hilpert-type inequalities

In view of proving the well-posedness of Problem 2.1_\varepsilon (for \( f \equiv 0 \)), we review a fundamental property of a class of continuous hysteresis operators, and provide a discretized version. First let us set
\[
s_0(\eta) := -1 \quad \text{if } \eta < 0, \quad s_0(0) := 0, \quad s_0(\eta) := 1 \quad \text{if } \eta > 0.
\]

Lemma 3.1 (Hilpert’s inequality [7]). Let \( \varepsilon > 0 \). Then for any \((u_i, \xi_i) \in W^{1,1}(0, T) \times \mathbb{R} \) \((i = 1, 2)\), setting \( \bar{w} := k_\varepsilon^\rho(u_1, \xi_1) - k_\varepsilon^\rho(u_2, \xi_2) \) we have
\[
\frac{d}{dt} s_0(u_1 - u_2) \geq \frac{d}{dt} |\bar{w}| \quad \text{a.e. in } ]0, T[. \) (3.1)
For the proof of this statement we refer to [7] and [17, Section III.2]. Let us now set
\[
G_\rho^0(u, \xi) := \begin{cases} 
-1 & \text{if } u < \rho_1, \\
[-1, \xi) & \text{if } u = \rho_1, \\
\{\xi\} & \text{if } \rho_1 < u < \rho_2, \quad \forall (u, \xi) \in \mathbb{R} \times [-1, 1], \\
[\xi, 1] & \text{if } u = \rho_2, \\
\{1\} & \text{if } u > \rho_2,
\end{cases}
\]
cf. Fig. 3(a). Notice that for any \( \varepsilon > 0 \) the relation \( w \in G_\rho^0(u - \varepsilon w, \xi) \) defines a single-valued maximal monotone function \( G_\rho^\varepsilon \), cf. Fig. 3(b):
\[
w = G_\rho^\varepsilon(u, \xi) \iff w \in G_\rho^0(u - \varepsilon w, \xi).
\]
It is not difficult to see that the recursive equation
\[
w^n = G_\rho^\varepsilon(u^n, w^{n-1}) \quad \forall n
\]
defines a time-discretized version of the relay relation \( w = k_\varepsilon^\rho(u, w^0) \); indeed if \( u \) is the piecewise-linear interpolate of the \( u^n \)'s with time step \( h \), then \( \bar{w}(nh) = w^n \) for any \( n \).
Fig. 3. (a) Graphs of the multi-valued function $G(\cdot, \xi)$; (b) the corresponding single-valued function $G_{\varepsilon}(\cdot, \xi)$, for any $\xi \in [-1, 1]$.

**Lemma 3.2** (Time-discretized Hilpert’s inequality). For $i = 1, 2$, let $\varepsilon > 0$ and $\{u_i^n\}$ and $\{w_i^n\}$ be two real sequences such that

$$w_i^n = G_{\varepsilon}(u_i^n, w_i^{n-1}) \quad \forall n, \text{ for } i = 1, 2. \quad (3.4)$$

Then, setting $\tilde{u}_i^n := u_i^n - u_2^n$ and $\tilde{w}_i^n := w_i^n - w_2^n$,

$$\left(\tilde{w}_i^n - \tilde{w}_i^{n-1}\right)s_0(\tilde{u}_i^n) \geq |\tilde{w}_i^n| - |\tilde{w}_i^{n-1}| \quad \forall n \in \mathbb{N}. \quad (3.5)$$

**Proof.** First let us set $u_i^{\ast n} := u_i^n - \varepsilon w_i^n$, $\tilde{u}_i^{\ast n} := u_1^{\ast n} - u_2^{\ast n}$, and notice that for $i = 1, 2$

$$\rho_1 < u_i^{\ast n} < \rho_2 \quad \Rightarrow \quad w_i^n = w_i^{n-1},$$
$$u_i^{\ast n} \leq \rho_1 \quad \Rightarrow \quad w_i^n \leq w_i^{n-1},$$
$$u_i^{\ast n} \geq \rho_2 \quad \Rightarrow \quad w_i^n \geq w_i^{n-1}. \quad (3.6)$$

To prove (3.5) it suffices to show that, by a suitable choice of $\sigma^n \in \text{sign}(\tilde{w}_i^n)$,

$$\left(\tilde{w}_i^n - \tilde{w}_i^{n-1}\right)s_0(\tilde{u}_i^n) - \sigma^n \geq 0 \quad \forall n \in \mathbb{N}. \quad (3.7)$$

This inequality is here checked by distinguishing the different cases that may occur at any $n$:

(i) if either $\tilde{w}_i^n > 0$ and $\tilde{u}_i^n > 0$, or $\tilde{w}_i^n < 0$ and $\tilde{u}_i^n < 0$, then $\sigma^n = s_0(\tilde{u}_i^n)$ and (3.7) is fulfilled;
(ii) if $\tilde{w}_i^n = 0$, then we can take $\sigma^n = s_0(\tilde{u}_i^n)$ and (3.7) is fulfilled;
(iii) if $\tilde{w}_i^n > 0$ and $\tilde{u}_i^n \leq 0$, then $\tilde{u}_i^{\ast n} < 0$. By Fig. 2(a) it is clear that $\rho_1 \leq u_1^{\ast n} < u_2^{\ast n} < \rho_2$, whence $u_i^{\ast n} < \rho_2$ and $\rho_1 < u_i^{\ast n}$. By (3.6), $u_i^{\ast n} < \rho_2$ ($\rho_1 < u_i^{\ast n}$, respectively) entails that $w_i^n \leq w_i^{n-1}$ ($w_i^n \geq w_i^{n-1}$, respectively); hence $\tilde{w}_i^n \leq \tilde{w}_i^{n-1}$. As $s_0(\tilde{u}_i^n) = -1$, (3.7) follows;
(iv) if $\tilde{w}_i^n < 0$ and $\tilde{u}_i^n \geq 0$, then exchanging the indices 1 and 2 we are reduced to the case (iii). \qed

**Remark.** Let us denote the Heaviside function by $H$. As $s_0 + 1 = 2H$, (3.1) and (3.5) are respectively equivalent to
\[
\begin{align*}
\frac{d\tilde{w}}{dt} H(u_1 - u_2) & \geq \frac{d}{dt} (\tilde{w})^+ \quad \text{a.e. in }]0, T[,} \\
(\tilde{w}^n - \tilde{w}^{n-1}) H(\tilde{u}^n) & \geq (\tilde{u}^n)^+ - (\tilde{u}^{n-1})^+ \quad \forall n \in \mathbb{N}.}
\end{align*}
\tag{3.8}
\]

\[
\begin{align*}
\frac{d\tilde{w}}{dt} (\tilde{w}) & \geq \frac{d}{dt} (\tilde{w})^+ \quad \text{a.e. in }]0, T[,} \\
(\tilde{w}^n - \tilde{w}^{n-1}) H(\tilde{u}^n) & \geq (\tilde{u}^n)^+ - (\tilde{u}^{n-1})^+ \quad \forall n \in \mathbb{N}.}
\end{align*}
\tag{3.9}
\]

4. Existence

In this section we prove existence of a solution of Problem \(2.1_\varepsilon\) for any \(\varepsilon \geq 0\) via time-discretization, derivation of a priori estimates, and passage to the limit.

**Theorem 4.1.** Let \(\varepsilon \geq 0\). If (2.1) is fulfilled then Problem \(2.1_\varepsilon\) has a solution. Moreover, for any \(\ell \in [1, +\infty[\),

\[
\begin{align*}
u^0, w^0 & \in BV(\mathbb{R}), \quad f \in L^\ell(0, T; BV(\mathbb{R})) \quad \Rightarrow \\
u, w & \in L^\infty(0, T; BV(\mathbb{R})) \cap W^{1,\ell}(0, T; C^0(\mathbb{R}')).}
\end{align*}
\tag{4.1}
\]

**Proof.** (i) **Approximation.** Let us fix any \(m \in \mathbb{N}\), and set

\[
\begin{align*}
u^0_m := \nu^0, \quad w^0_m := w^0, \quad f^n_m := \frac{1}{h} \int_{(n-1)h}^{nh} f(\cdot, t) dt \quad \text{for } n = 1, \ldots, m.
\end{align*}
\]

We now approximate our problem via an implicit time-discretization scheme.

**Problem 2.1_{\varepsilon,m}.** For \(n = 1, \ldots, m\), find \(u^n_m \in H^1(\mathbb{R})\) and \(w^n_m \in L^2(\mathbb{R})\) such that

\[
\begin{align*}
u^n_m & \in G^\varepsilon(\nu^n_m, w^n_m - 1) \quad \text{a.e. in } \mathbb{R}, \text{ for } n = 1, \ldots, m,} \\
\frac{w^n_m - w^{n-1}_m}{h} + \frac{w^n_m - w^{n-1}_m}{h} + \frac{d u^n_m}{dx} & = f^n_m \quad \text{a.e. in } \mathbb{R}, \text{ for } n = 1, \ldots, m.
\end{align*}
\tag{4.2}
\]

We note that if \(\varepsilon > 0\), (4.2) is an equality.

As \(G^\varepsilon\) is maximal monotone with respect to the first argument, existence of an approximate solution can easily be proved step by step.

(ii) **A priori estimates.** For any family \(\{v^n_m\}_{n=1,\ldots,m}\) of functions \(\mathbb{R} \to \mathbb{R}\), let us first set

\[
\begin{align*}
u_m := \text{piecewise-linear interpolate of } v^0_m, \ldots, v^n_m \text{ in } [0, T], \quad \text{a.e. in } \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
\tilde{\nu}_m(\cdot, t) := v^n_m \quad \text{a.e. in } \mathbb{R} \forall t \in ](n-1)h, nh[, \text{ for } n = 1, \ldots, m.}
\end{align*}
\tag{4.4}
\]

Let us multiply Eq. (4.3) by \(h u^n_m\), and sum for \(n = 1, \ldots, \ell\), for any \(\ell \in [1, \ldots, m]\). As \(\mathcal{D}(\mathbb{R})\) is dense in \(H^1(\mathbb{R})\), it is easy to see that

\[
\int_{\mathbb{R}} \frac{d u^n_m}{dx} u^n_m dx = \frac{1}{2} \int_{\mathbb{R}} \frac{d}{dx} (u^n_m)^2 dx = 0 \quad \forall n.
\]

By (4.2) we then get
\[
\frac{1}{2} \int \left[ (u^\ell_m)^2 - (u^0)^2 \right] dx + \int \Psi^\varepsilon(\varphi_m; [0, \ell h]) dx \leq \sum_{n=0}^{\ell} \int f^n_m u^n_m dx
\]
\[
\leq \|f\|_{L^1(0,T;L^2(\mathbb{R}))} \max_{n=0,\ldots,\ell} \|u^n_m\|_{L^2(\mathbb{R})} \quad \text{for } \ell = 1, \ldots, m. \tag{4.5}
\]
A simple calculation then yields
\[
\|u_m\|_{L^\infty(0,T;L^2(\mathbb{R}))}, \quad \|\Psi^\varepsilon(\varphi_m; [0, t])\|_{L^\infty(0,T;L^1(\mathbb{R}))} \leq C_1. \tag{4.6}
\]
(By \(C_1, C_2, \ldots\) we denote suitable positive constants independent of \(m, \varepsilon\).) Hence
\[
\left\| \frac{\partial u_m}{\partial t} \right\|_{L^1(\mathbb{R}_T)} \leq C_2, \tag{4.7}
\]
and by comparing the terms of (4.3) we get
\[
\left\| \frac{\partial u_m}{\partial t} + \frac{\partial u_m}{\partial x} \right\|_{L^1(\mathbb{R}_T)} \leq C_3. \tag{4.8}
\]
(iii) A contraction property. First let us assume that \(\varepsilon > 0\), and set
\[
s_j(\zeta) := \max\{\min\{j \zeta, 1\}, -1\} \quad \forall \zeta \in \mathbb{R}, \forall j \in \mathbb{N},
\]
\[
\delta_k v(x) := v(x + k) - v(x) \quad \forall x \in \mathbb{R}, \forall k > 0, \forall v : \mathbb{R} \to \mathbb{R}.
\]
Notice that \(s_j \to s_0\) pointwise in \(\mathbb{R}\). By applying \(\delta_k\) to (4.3) we have
\[
\frac{\delta_k u^n_m - \delta_k u^{n-1}_m}{h} + \frac{\delta_k u^n_m - \delta_k u^{n-1}_m}{h} + \frac{d\delta_k u^n_m}{dx} = \delta_k f^n_m \quad \text{a.e. in } \mathbb{R}, \forall n. \tag{4.9}
\]
Let us multiply this equation by \(hs_j(\delta_k u^n_m)\) and integrate over \(\mathbb{R}\); as
\[
\int \frac{d\delta_k u^n_m}{dx} s_j(\delta_k u^n_m) dx = \int \left( \frac{d}{dx} \int_0^{\delta_k u^n_m} s_j(\zeta) d\zeta \right) dx = 0 \quad \forall n,
\]
we get
\[
\int (\delta_k u^n_m - \delta_k u^{n-1}_m) s_j(\delta_k u^n_m) dx + \int (\delta_k u^n_m - \delta_k u^{n-1}_m) s_j(\delta_k u^n_m) dx
\]
\[
\leq h \int |\delta_k f^n_m| dx \quad \forall n.
\]
By passing to the limit as \(j \to \infty\) we get the same inequality with \(s_0\) in place of \(s_j\). Note that
\[
(\delta_k u^n_m - \delta_k u^{n-1}_m)s_0(\delta_k u^n_m) \geq |\delta_k u^n_m| - |\delta_k u^{n-1}_m|;
\]
moreover, by the discretized Hilpert inequality (3.5),
\[
(\delta_k u^n_m - \delta_k u^{n-1}_m)s_0(\delta_k u^n_m) \geq |\delta_k u^n_m| - |\delta_k u^{n-1}_m|.
\]
We thus get
\[ \int \left( |\delta_k u_m^n| - |\delta_k u_m^{n-1}| \right) dx + \int \left( |\delta_k w_m^n| - |\delta_k w_m^{n-1}| \right) dx \leq h \int |\delta_k f_m^n| dx \quad \forall n, \]
whence, summing with respect to \( n \),
\[ \int \left( |\delta_k u_\ell| + |\delta_k w_\ell| \right) dx \leq \int \left( |\delta_k u^0| + |\delta_k w^0| \right) dx + h \sum_{n=0}^{\ell} \int |\delta_k f_m^n| dx \]
for \( \ell = 1, \ldots, m \). \hfill (4.10)

(iv) **Limit procedure.** By the above estimates, there exist \( u, w \) such that, as \( m \to \infty \) along a suitable sequence,
\[ \tilde{u}_m, u_m \rightharpoonup u \quad \text{weakly star in } L^\infty (0, T; L^2 (\mathbb{R})), \]
\[ \tilde{w}_m, w_m \rightharpoonup w \quad \text{weakly star in } L^\infty (\mathbb{R}_T), \]
\[ \frac{\partial w}{\partial t} \rightharpoonup \frac{\partial w}{\partial t} \quad \text{weakly star in } C^0 (\overline{\mathbb{R}_T})'. \hfill (4.11) \]
Moreover, by (4.7) and (4.10), the classic Riesz compactness criterion yields
\[ \tilde{w}_m, w_m \rightharpoonup w \quad \text{strongly in } L^1_{\text{loc}} (\mathbb{R}_T); \hfill (4.12) \]

hence
\[ \iint_{\mathbb{R}_T} \tilde{w}_m \tilde{u}_m \varphi \, dx \, dt \to \iint_{\mathbb{R}_T} w u \varphi \, dx \, dt \quad \forall \varphi \in \mathcal{D}(\mathbb{R}_T). \hfill (4.13) \]

The formulae (4.3) and (4.5) also read
\[ \frac{\partial}{\partial t} (u_m + w_m) + \frac{\partial \tilde{u}_m}{\partial x} = \tilde{f}_m \quad \text{in } H^{-1} (\mathbb{R}), \text{ a.e. in } [0, T[, \hfill (4.14) \]
\[ \frac{1}{2} \int \left[ \tilde{u}_m^2 (t) - (u^0)^2 \right] dx + \int \Psi^\varepsilon (\tilde{w}_m; [0, t]) dx \leq \iint \tilde{f}_m \tilde{u}_m \, dx \, d\tau \]
\[ \forall t \in [0, T]; \hfill (4.15) \]
by passing to the limit in (4.14) and to the inferior limit in (4.15) as \( m \to \infty \), we then get (2.7) and (2.5). For any nonnegative \( \varphi \in \mathcal{D}(\mathbb{R}_T) \) the inclusion (4.2) entails
\[ \iint_{\mathbb{R}_T} (\tilde{w}_m - 1)(\tilde{u}_m - \rho_2) \varphi \, dx \, dt \geq \varepsilon \iint_{\mathbb{R}_T} \left( \frac{\tilde{w}_m^2}{2} - \tilde{w}_m \right) \, dx \, dt, \]
\[ \iint_{\mathbb{R}_T} (\tilde{w}_m + 1)(\tilde{u}_m - \rho_1) \varphi \, dx \, dt \geq \varepsilon \iint_{\mathbb{R}_T} \left( \frac{\tilde{w}_m^2}{2} + \tilde{w}_m \right) \, dx \, dt; \hfill (4.16) \]
by (4.13), passing to the inferior limit as \( m \to \infty \) we then get (2.4).
If $u^0_0, w^0_0 \in BV(R)$ and $f \in L^p(0, T; BV(R))$ for some $p \in [1, +\infty]$, then by (4.10) $u_m$ and $w_m$ are uniformly bounded in $L^\infty(0, T; BV(R))$. By comparison in (4.14) we then get that $\partial (u_m + w_m)/\partial t$ is uniformly bounded in $L^p(0, T; C^0(\bar{R}'))$. By (4.2)

$$\left| \frac{\partial u_m}{\partial t} + \frac{\partial w_m}{\partial t} \right| \leq \left| \frac{\partial}{\partial t} (u_m + w_m) \right| \quad \text{a.e. in } R_T;$$

we then infer that $\partial u_m/\partial t$ and $\partial w_m/\partial t$ are both uniformly bounded in $L^p(0, T; C^0(\bar{R}'))$, and (4.1) follows. □

**Remark.** The estimates (4.6)–(4.8) and (4.10) are uniform with respect to $\varepsilon \geq 0$. Therefore apparently there is no gain of regularity in replacing the completed relay operator by its regularization.

**Proposition 4.2 (Robustness).** Let $\varepsilon \geq 0$, and for any $n \in N$ let $\rho_{1n} < \rho_{2n}$ and $(u_n, w_n)$ be a corresponding solution of Problem 2.1. If

$$\rho_{1n} \rightarrow \rho_1, \quad \rho_{2n} \rightarrow \rho_2, \quad \rho_1 < \rho_2,$$

then there exists $(u, w)$ such that, as $n \rightarrow \infty$ along a subsequence,

$$u_n \rightarrow u \quad \text{weakly star in } L^\infty(0, T; L^2(R));$$

$$w_n \rightarrow w \quad \text{weakly star in } L^\infty(R_T) \text{ and strongly in } L^1_{loc}(R_T),$$

$$\frac{\partial w_n}{\partial t} \rightarrow \frac{\partial w}{\partial t} \quad \text{weakly star in } C^0(\bar{R}_T'),$$

$$u_n + w_n \rightarrow u + w \quad \text{weakly star in } L^\infty(0, T; L^2(R)) \cap H^1(0, T; H^{-1}(R)).$$

Moreover, $(u, w)$ is a solution of Problem 2.1 corresponding to the pair $(\rho_1, \rho_2)$.

**Outline of the proof.** The argument follows the lines of the above estimate and limit procedure. In particular, $\rho_1 < \rho_2$, (4.19) 2 and (4.19) 3 entail

$$\liminf_{n \rightarrow \infty} \Psi^{\varepsilon}_{\rho_n}(w_n; [0, t]) \geq \Psi^{\varepsilon}_{\rho}(w; [0, t]).$$

The latter result applies also if $\rho_1 = \rho_2$; however, in that case the convergence (4.19) 3 drops.

5. Uniqueness

In this section we study the uniqueness of the solution of Problem 2.1 for any $\varepsilon \geq 0$. Apparently the low regularity of the solution does not allow one to apply Hilpert’s argument based on the inequality (3.1), cf. [7]. Indeed, even under the regularity of (4.1), in general, Eq. (2.7) does not hold pointwise, and thus it cannot be multiplied by a discontinuous function.

We then use a different technique. In order to select a unique solution, we append an entropy-type condition; we show that any limit of solutions of the above time-discretized
problems fulfills this additional condition, and use it to prove that the solution depends Lipschitz-continuously and monotonically on the initial data; of course this entails the uniqueness of the solution, too. This mimics the classic procedure that Kružkov introduced for quasilinear first-order equations without hysteresis [12,13].

For technical reasons, we are able to perform this program only assuming that the source term identically vanishes (i.e., \( f \equiv 0 \)), although the well-posedness of the semigroup solution of problem (2) suggests that the solution might be unique in general.

Let us denote by \( \mathcal{L}_\rho \) the hysteresis region, namely, the subset of \( \mathbb{R}^2 \) that represents admissible pairs \((u,w)\); this set consists of the rectangle \( [\rho_1, \rho_2] \times [-1, 1] \) and of the two half-lines \( ]-\infty, \rho_1[ \times \{-1\} \) and \( ]\rho_2, +\infty[ \times \{1\} \). Notice that, trivially,

\[
\hat{\theta} \in k^0_\rho(\theta, \hat{\theta}) \quad \forall (\theta, \hat{\theta}) \in \mathcal{L}_\rho.
\]

**Theorem 5.1.** Let \( \varepsilon \geq 0 \) and assume that

\[
u^0, w^0 \in L^2(\mathbb{R}), \quad \exists \alpha > 0: u^0, w^0 \in W^{\alpha,1}(\mathbb{R}),
\]

\[
|w^0| \leq 1 \quad a.e. \text{ in } \mathbb{R}, \quad f \equiv 0 \quad a.e. \text{ in } \mathbb{R}_T.
\]

Then there exists a solution of Problem 2.1_\varepsilon such that

\[
\iint_{\mathbb{R}_T} \left( |u - \theta| + |w - \hat{\theta}| \right) \frac{\partial v}{\partial t} + |u - \theta| \frac{\partial v}{\partial x} \, dx \, dt \geq 0
\]

\[
\forall v \in D(\mathbb{R}_T), \quad v \geq 0, \quad \forall (\theta, \hat{\theta}) \in \mathcal{L}_\rho.
\]

If \( u \in L^\infty(\mathbb{R}_T) \), taking \( \theta = \pm \|u\|_{L^\infty(\mathbb{R}_T)} \) one easily sees that (5.4) entails the PDE (2.7), for \( f \equiv 0 \).

**Proof.** (i) First we improve the convergences (4.11). Let \( \{u^m_n\}, \{w^m_n\}, u \) and \( w \) be constructed via the approximation procedure of the previous section. By (4.10) and (5.2)

\[
\|u^m\|_{L^\infty(0,T;W^{\alpha,1}(\mathbb{R}))}, \quad \|u^m\|_{L^\infty(0,T;W^{\alpha,1}(\mathbb{R}))} \leq C_4.
\]

By the approximate equation (4.14) and by (4.11)_3 we have

\[
\left\| \frac{\partial u_m}{\partial t} \right\|_{L^1(\mathbb{R}_T) + L^2(0,T;H^{-1}(\mathbb{R}))} \leq \left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(\mathbb{R}_T)} + \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(0,T;H^{-1}(\mathbb{R}))}
\]

\[
\leq \left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(\mathbb{R}_T)} + \|u_m\|_{L^2(\mathbb{R}_T)} \leq C_5.
\]

By the classic Aubin’s lemma, the two latter estimates and (4.11)_3 yield

\[
u_m \rightarrow u, \quad w_m \rightarrow w \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}_T).
\]

(ii) Now we come to the main part of the argument. Let us assume that \( \theta \neq \rho_1, \rho_2 \), so that there exists \( \varepsilon > 0 \) such that \( k^\varepsilon_\rho \) maps \( \theta \) to \( \hat{\theta} \). Once we prove our statement for any pair \((\theta, \hat{\theta})\) like this, an obvious approximation procedure will then provide it for any \((\theta, \hat{\theta}) \in \mathcal{L}_\rho\).
Let us fix any nonnegative \( v \in \mathcal{D}(\mathbb{R}_T) \), set
\[
v^n_m := \frac{1}{h} \int_{(n-1)h}^{nh} v(\cdot, t) \, dt \quad \text{a.e. in } \mathbb{R},
\]
multiply Eq. (4.3) by \( h\bar{\theta}(u^n_m - \theta)v^n_m \), and sum for \( n = 1, \ldots, m \). Notice that
\[
(u^n_m - u^{n-1}_m)s_0(u^n_m - \theta) \geq |u^n_m - \theta| - |u^{n-1}_m - \theta|,
\]
\[
\frac{d}{dx} s_0(u^n_m - \theta) \geq \frac{d}{dx} |u^n_m - \theta|;
\]
as \( \varepsilon > 0 \) we can also apply the discretized Hilbert inequality (3.5), that yields
\[
(u^n_m - w^{n-1}_m)s_0(u^n_m - \theta) \geq (u^n_m - w^{n-1}_m)s_0(w^n_m - \hat{\theta}) \geq |w^n_m - \hat{\theta}| - |w^{n-1}_m - \hat{\theta}|.
\]
Thus we get
\[
\sum_{n=0}^{m} \int_{\mathbb{R}} \left[ (|u^n_m - \theta| - |u^{n-1}_m - \theta| + |w^n_m - \hat{\theta}| - |w^{n-1}_m - \hat{\theta}|) v^n_m + \frac{d}{dx} |u^n_m - \theta| v^n_m \right] dx \leq 0 \quad \forall v \in \mathcal{D}(\mathbb{R}_T), \ v \geq 0,
\]
(5.8)
or also, by continuous and discrete partial integration,
\[
h \sum_{n=0}^{m} \int_{\mathbb{R}} \left[ (|u^n_m - \theta| + |w^n_m - \hat{\theta}|) v^n_m - v^{n-1}_m + |u^n_m - \theta| \frac{d}{dx} v^n_m \right] dx \geq 0.
\]
Passing to the limit as \( m \to \infty \), by (5.7) we get (5.4) for any \( \varepsilon > 0 \). We now pass to the limit as \( \varepsilon \to 0 \); as all the estimates we derived are uniform with respect to \( \varepsilon \), we then get (5.4) also for \( \varepsilon = 0 \). \( \Box \)

**Theorem 5.2** (Lipschitz-continuous and monotone dependence on the initial data). Assume that \( \varepsilon \geq 0 \) and \( f \equiv 0 \). For \( i = 1, 2 \) let
\[
u^0_i \in L^2 \cap L^1(\mathbb{R}), \quad w^0_i \in L^1(\mathbb{R}), \quad |w^0_i| \leq 1 \ a.e. \ in \ \mathbb{R},
\]
(5.9)
and \((u_1, w_1) \in L^\infty(0, T; L^2(\mathbb{R})) \times L^\infty(\mathbb{R}_T)\) be a corresponding solution of Problem 2.1\( \varepsilon \) that fulfils (5.4). Then
\[
\int_{\mathbb{R}} (|u_1 - u_2| + |w_1 - w_2|)(x, t) \, dx \leq \int_{\mathbb{R}} (|u^0_1 - u^0_2| + |w^0_1 - w^0_2|) \, dx
\]
for a.a. \( t \in ]0, T[ \),
(5.11)
\[
\int_{\mathbb{R}} [(u_1 - u_2)^+(x, t) + (w_1 - w_2)^+(x, t)] \, dx \leq \int_{\mathbb{R}} [(u^0_1 - u^0_2)^+ + (w^0_1 - w^0_2)^+] \, dx
\]
for a.a. \( t \in ]0, T[ \).
(5.12)
Proof. This argument is based on Kružkov’s technique of doubling the variables, cf. [12, 13].

By writing the inequality (5.4) for \((u_1(x, t), w_1(x, t))\) and \((\theta, \hat{\theta}) = (u_2(\xi, \tau), w_2(\xi, \tau))\) for almost any fixed \((\xi, \tau) \in \mathbb{R}_T\), we get

\[
\int_{\mathbb{R}_T} \left( |u_1(x, t) - u_2(\xi, \tau)| + |w_1(x, t) - w_2(\xi, \tau)| \right) \frac{\partial v}{\partial t}(x, t) \, dx \, dt \\
+ \int_{\mathbb{R}_T} \left( |u_1(x, t) - u_2(\xi, \tau)| \frac{\partial v}{\partial x}(x, t) \right) \, dx \, dt \geq 0 \quad \forall v \in \mathcal{D}(\mathbb{R}_T), \ v \geq 0;
\]

(5.13)

by writing the same inequality for \((u_2(\xi, \tau), w_2(\xi, \tau))\) and \((\theta, \hat{\theta}) = (u_1(x, t), w_1(x, t))\) for almost any fixed \((x, t) \in \mathbb{R}_T\), we similarly get

\[
\int_{\mathbb{R}_T} \left( |u_2(\xi, \tau) - u_1(x, t)| + |w_2(\xi, \tau) - w_1(x, t)| \right) \frac{\partial v}{\partial \tau}(\xi, \tau) \, d\xi \, d\tau \\
+ \int_{\mathbb{R}_T} \left( |u_2(\xi, \tau) - u_1(x, t)| \frac{\partial v}{\partial \xi}(\xi, \tau) \right) \, d\xi \, d\tau \geq 0 \quad \forall v \in \mathcal{D}(\mathbb{R}_T), \ v \geq 0.
\]

(5.14)

In both of these inequalities let us now take any nonnegative \(v = v(x, t, \xi, \tau) \in \mathcal{D}((\mathbb{R}_T)^2)\), and then integrate (5.13) ((5.14), respectively) with respect to \((\xi, \tau)\) to \((x, t)\), respectively) over \(\mathbb{R}_T\). By summing these two inequalities we get

\[
\iiint_{(\mathbb{R}_T)^2} \left[ \left( |u_1(x, t) - u_2(\xi, \tau)| + |w_1(x, t) - w_2(\xi, \tau)| \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \tau} \right) \right) \right. \\

+ \left. \left( |u_1(x, t) - u_2(\xi, \tau)| \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial \xi} \right) \right) \right] \, dx \, dt \, d\xi \, d\tau \geq 0.
\]

(5.15)

Let us now fix a mollifier \(\psi \in \mathcal{D}(\mathbb{R})\) such that

\[\psi \geq 0, \quad \psi(s) = 0 \quad \text{if } |s| \geq 1, \quad \int_{\mathbb{R}} \psi(s) \, ds = 1,\]

and set \(\psi_\eta(s) := \psi(s/\eta)/\eta\) for any \(s \in \mathbb{R}\) and any \(\eta > 0\). For any nonnegative \(z \in \mathcal{D}(\mathbb{R}_T)\), let us then take

\[v_\eta(x, t, \xi, \tau) = \psi_\eta(x - \xi) \psi_\eta(t - \tau) z \left( \frac{x + \xi}{2}, \frac{t + \tau}{2} \right)\]

in (5.15), and pass to the limit as \(\eta \to 0\). Denoting by \(\delta_0\) the Dirac measure in \(\mathbb{R}\) concentrated at the origin, for any fixed \((x, t) \in \mathbb{R}_T\), \(\delta_0(x - \xi)\delta_0(t - \tau)\) equals the Dirac measure in \(\mathbb{R}^2\) concentrated at the point \((x, t)\). Denoting by \(D_1z\) and \(D_2z\) the two partial derivatives of the function \(z\), we have

\[
\frac{\partial v_\eta}{\partial t} + \frac{\partial v_\eta}{\partial \tau} = \psi_\eta(x - \xi) \psi_\eta(t - \tau) D_2z \left( \frac{x + \xi}{2}, \frac{t + \tau}{2} \right) \\
\to \delta_0(x - \xi)\delta_0(t - \tau)D_2z(x, t) \quad \text{in } \mathcal{D}'(\mathbb{R}_T), \quad \text{as } \eta \to 0,
\]

in (5.15), as \(\eta \to 0\).
\begin{align*}
\frac{\partial v_\eta}{\partial x} + \frac{\partial v_\eta}{\partial \xi} &= \psi_\eta(x - \xi) \psi_\eta(t - \tau) D_1 z \left( \frac{x + \xi}{2}, \frac{t + \tau}{2} \right) \\
&\rightarrow \delta_0(x - \xi) \delta_0(t - \tau) D_1 z(x, t) \quad \text{in } \mathcal{D}'(\mathbb{R}_T), \text{ as } \eta \to 0.
\end{align*}

We then get
\begin{equation}
\int_\mathbb{R}_T \left| (u_1(x, t) - u_2(x, t)) + |w_1(x, t) - w_2(x, t)| \right| D_2 z(x, t) \ dx \ dt \geq 0 \quad \forall z \in \mathcal{D}(\mathbb{R}_T), \ z \geq 0. \tag{5.16}
\end{equation}

For any \( \tilde{t} \in ]0, T[ \) and any \( \eta \in ]0, \tilde{t}[ \), let us denote by \( g_\eta \) the indicator function of the rectangle \( ]-1/\eta, 1/\eta[ \times ]\eta, \tilde{t}[ \), and set
\begin{equation}
z_\eta(x, t) = \int \int_{\mathbb{R}_T} g_\eta(x - \xi, t - \tau) \psi_\eta(\xi) \psi_\eta(\tau) \, d\xi \, d\tau \quad \forall (x, t) \in \mathbb{R} \times ]0, \tilde{t}[. \tag{5.19}
\end{equation}

Taking \( z = z_\eta \in \mathcal{D}(\mathbb{R}) \) in (5.16) and passing to the limit as \( \eta \to 0 \), we get (5.11), whence (5.10). Finally, integrating Eq. (2.7) in time we also have
\begin{equation}
\int \mathbb{R} (u_1 - u_2 + w_1 - w_2)(x, t) \, dx = \int \mathbb{R} (u_1^0 - u_2^0 + w_1^0 - w_2^0) \, dx
\end{equation}

for a.a. \( t \in ]0, T[ \), and by adding this equality to (5.11) we get (5.12). \( \Box \)

\section*{Extension to the Preisach model}

Let us prescribe a positive, finite Borel measure \( \mu \) over the set of admissible thresholds, \( \mathcal{P} \), cf. (1.13); denoting the ordinary \( N \)-dimensional Lebesgue measure by \( \lambda_N \), let us then equip \( \mathbb{R}^N \times \mathcal{P} \) with the product measure \( \lambda_N \times \mu \) (here we are interested to the case of \( N = 2 \) or 3). Let us assume that
\begin{equation}
\begin{aligned}
&u^0 \in L^2(\mathbb{R}), \quad f \in L^1(\mathbb{R}_T) \cap L^2(\mathbb{R}_T), \quad w^0 \in L^2(\mathbb{R}; L^1(\mathcal{P})), \\
&|w^0| \leq 1 \quad \text{a.e. in } \mathbb{R} \times \mathcal{P},
\end{aligned} \tag{5.17}
\end{equation}

and set
\begin{equation}
\tilde{w}^0(x) := \int_{\mathcal{P}} w^0(x, \rho) \, d\mu(\rho) \quad \text{for a.a. } x \in \mathbb{R}. \tag{5.18}
\end{equation}

For any \( \varepsilon \geq 0 \) we can now provide a weak formulation of the Cauchy problem (1) of the Introduction, for \( F \) equal to the Preisach operator associated to the measure \( \mu \).

\section*{Problem 2.2} Find \( u_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R})) \) and \( w_\varepsilon \in L^\infty(\mathbb{R}_T \times \mathcal{P}) \) such that, setting
\begin{equation}
\tilde{w}_\varepsilon(x, t) := \int_{\mathcal{P}} w_\varepsilon(x, t, \rho) \, d\mu(\rho) \quad \text{for a.a. } (x, t) \in \mathbb{R}_T, \tag{5.19}
\end{equation}
one has

\[ |w_\varepsilon| \leq 1 \quad \text{a.e. in } \mathbb{R}_T \times \mathcal{P}, \quad \frac{\partial w_\varepsilon}{\partial t} \in L^\infty(\mathcal{P}; C^0(\mathbb{R}_T)'), \quad (5.20) \]

\[ \iint_{\mathbb{R}_T} \left( (u_\varepsilon + \bar{w}_\varepsilon - u^0 - \bar{w}^0) \frac{\partial v}{\partial t} + u_\varepsilon \frac{\partial v}{\partial x} + f v \right) dx dt = 0 \quad \forall v \in H^1(\mathbb{R}_T) \cap W^{1,1}(\mathbb{R}_T), \quad v(\cdot, T) = 0, \quad (5.21) \]

\[ \left\{ \begin{array}{l} (w_\varepsilon - 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_2) \geq 0, \\
(w_\varepsilon + 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_1) \geq 0 \end{array} \right\} \quad \text{a.e. in } \mathbb{R}_T \times \mathcal{P}, \quad (5.22) \]

\[ \frac{1}{2} \int_{\mathbb{R}} \left[ u_\varepsilon(x,t)^2 - u^0(x)^2 \right] dx + \iint_{\mathbb{R} \times \mathcal{P}} \Psi_{\varepsilon}^\prime(w_\varepsilon; [0, t]) dx d\mu(\rho) \]

\[ \leq \iint_{\mathbb{R}_T} f u_\varepsilon dx d\tau \quad \text{for a.a. } t \in ]0, T[, \quad (5.23) \]

\[ w_\varepsilon(\cdot, 0, \cdot) = w^0 \quad \text{a.e. in } \mathbb{R} \times \mathcal{P}. \quad (5.24) \]

(Thus \( w_\varepsilon \) depends on \( (x, t, \rho) \), whereas \( u_\varepsilon \) and \( \bar{w}_\varepsilon \) only depend on \( (x, t) \).)

The results of two latter sections can be extended to this problem. For instance, it is easy to see that here the estimate procedure is unchanged. We do not develop this somehow routinely procedure, that the reader can find detailed for a different equation, e.g., in [18]. We just mention the extension of the entropy-type condition (5.4): setting

\[ \tilde{\mathcal{L}} := \{(\theta, \hat{\theta}): \theta \in \mathbb{R}, \hat{\theta}: \mathcal{P} \to [-1, 1], (\theta, \hat{\theta}(\rho)) \in \mathcal{L}_\rho \text{ for } \mu\text{-a.a. } \rho \in \mathcal{P} \}, \]

one can show that for any \( \varepsilon \geq 0 \) Problem 2.2\( \varepsilon \) has one and only one solution such that

\[ \iint_{\mathbb{R}_T} \left[ \left( |u_\varepsilon - \theta| + \int_\mathcal{P} |w_\varepsilon(\cdot, \cdot, \rho) - \hat{\theta}(\rho)| d\mu(\rho) \right) \frac{\partial v}{\partial t} + |u_\varepsilon - \theta| \frac{\partial v}{\partial x} \right] dx dt \geq 0 \quad \forall v \in D(\mathbb{R}_T), \quad v \geq 0, \quad (\theta, \hat{\theta}) \in \tilde{\mathcal{L}}. \quad (5.25) \]

**Open questions**

(i) In Theorem 5.2 we assumed that the source term vanishes: of course it would be of interest to remove or at least to relax this hypothesis.

A related question concerns the connection between the present notion of solution and that based on the theory of nonlinear semigroups of [17, Chapter VIII].

(ii) Further investigation on Eq. (1) might concern the study of the associated time-periodic problem, under a time-periodic source term \( f \).

(iii) For any nondecreasing, continuous real function \( \alpha \), it seems possible to extend the well-posedness to the Cauchy problem for the equation

\[ \frac{\partial}{\partial t} \left[ u + \alpha(u) + F(u) \right] + \frac{\partial u}{\partial x} = 0 \quad \text{in } \mathbb{R} \times ]0, T[, \quad (5.26) \]
Existence of a solution can here be proved after deriving a property like (4.10) for time-increments.

(iv) The following system arises in modelling traffic flow, see, e.g., [6]:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial w}{\partial x} &= f, \\
&\text{in } \mathbb{R}_T.
\end{aligned}
\]  

(5.27)

Here \( \varphi \) is a nonmonotone real function; in a very simplified setting its graph might be \( N \)-shaped, without vertical parts. In alternative, one might also consider a system of the form

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial w}{\partial x} &= f, \\
w &\in cu + k_{\rho}(u), \\
&\text{in } \mathbb{R}_T,
\end{aligned}
\]  

(5.28)

c being a positive constant and \( k_{\rho} \) a completed relay operator. The passage from the system (5.27) to (5.28) might be supported by a similar argument to that of [19]; however, a rigorous derivation is not completely clear, and for either system it is not obvious that the associated Cauchy problem has a solution. This final issue looks as relevant as challenging.

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