J. Math. Anal. Appl. 356 (2009) 338–345

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Finite-time stability and stabilization of nonlinear stochastic hybrid systems ☆

Ying Yang [a,b,c,*], Junmin Li [a], Guopei Chen [c]

[a] *Department of Applied Mathematics, Xidian University, Xi'an, 710071, PR China*
[b] *School of Mathematics and Statistics, Zhejiang University of Finance and Economics, Hangzhou, 310018, PR China*
[c] *Department of Mathematics, Huizhou University, Huizhou, 516007, PR China*

ARTICLE INFO

Article history:
Received 3 November 2008
Available online 10 March 2009
Submitted by H. Zwart

Keywords:
Finite-time stability
Impulsive systems
Markovian switching
Brownian motion
Stabilization

ABSTRACT

This paper deals with the problem of finite-time stability and stabilization of nonlinear Markovian switching stochastic systems which exist impulses at the switching instants. Using multiple Lyapunov function theory, a sufficient condition is established for finite-time stability of the underlying systems. Furthermore, based on the state partition of continuous parts of systems, a feedback controller is designed such that the corresponding impulsive stochastic closed-loop systems are finite-time stochastically stable. A numerical example is presented to illustrate the effectiveness of the proposed method.

1. Introduction

Nowadays, stochastic modeling, control, and optimization have played a crucial role in many applications especially in the areas of controlling science and communication technology [1,2]. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. Markovian switching stochastic system can be considered as a class of stochastic hybrid systems consisting of a family of subsystems perturbed by Brown motion, and a rule governed by a Markov process that orchestrates the switching between subsystems. Literatures considering stability analysis and design of such systems have appeared in [3–6].

Many stochastic systems exhibit impulsive and switching behaviors due to abrupt changes and switches of state at certain instants during the dynamical processes; that is, the systems switch with impulsive effects [7–10]. Moreover, impulsive and switching phenomena can be found in the fields of physics, biology, engineering and information science. Many sudden and sharp changes occur instantaneously, in the form of impulses and switches, which cannot be well described by using pure continuous or pure discrete models. Therefore, it is important and in fact, necessary to study hybrid impulsive and switching stochastic systems.

On the other hand, literatures on finite-time stability (FTS) (or short-time stability) of systems have attracted particular interests of researchers. Comparing with classical Lyapunov stability, which currently is the focus of a large and growing interdisciplinary area of research, FTS concerns the stability of a system over a finite interval of time and plays an important part in the study of the transient behavior of systems. It is important to emphasize the disconnection between classical Lyapunov stability and finite-time stability. The concept of Lyapunov asymptotic stability is largely known to the control

☆ This work was partially supported by the National Natural Science Foundations of China (60374015).
* Corresponding author at: Department of Applied Mathematics, Xidian University, Xi'an, 710071, PR China.
E-mail addresses: yy1502@sina.com (Y. Yang), jmli@mail.xidian.edu.cn (J. Li), cgpken1977@163.com (G. Chen).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.02.046

community; conversely a system is said to be finite-time stable if, once we fix a time-interval, its state starting within a specified bound does not exceed some bounds during this time-interval. The classical control theory focuses mainly on the asymptotic behavior and seldom specifies bounds on the trajectories. In fact, a system may be finite-time stable, but may become unstable after the specified interval of time. In addition, the state trajectory might exceed the given bound over a certain time interval, but asymptotically go to zero.

Some early results on FTS can be found in [11–13]. The concept of FTS has been revisited recently and discussed for linear and nonlinear systems [14–17]. A stochastic version of FTS developed in [18] and [19] for analysis of continuous and discrete stochastic system, respectively, and in [20,21] for optimal control design. But in these references, they just discussed the pure stochastic system and did not consider the impulsive and switching behaviors. It is interesting to notice the time gap between 1972 and recent papers. To the best of authors' knowledge, to date, the problems of FTS for Markovian switching stochastic systems has not been investigated. The problem is interesting but also challenging, which motivates us to study.

The organization of the paper is as follows. In Section 2, we present some preliminary materials and a formulation of problems to be considered in this paper. In Section 3, the finite-time stability of hybrid impulsive and Markovian switching stochastic systems is studied. Section 4 solves the finite-time stabilization problem by designing a state feedback controller. A numerical example is provided in Section 5. Concluding remarks are given in Section 6.

2. Problem statement and preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on the probability space. Let $\|\cdot\|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\operatorname{trace}(A^{\mathrm{T}}A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax|:\ |x| = 1\}$. If A is a symmetric matrix, denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalues, respectively.

Let $\{r(t),\ t \geqslant 0\}$ be a Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\big\{r(t+\Delta) = j \bigm| r(t) = i\big\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$. Here $\gamma_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. The Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$, it ensures that the switchings are finite in any finite-time interval of R_+ $(:= [0, \infty))$.

Let us consider a nonlinear stochastic hybrid system with N modes described by $\{r(t),\ t \geqslant 0\}$ and suppose that the dynamics is described by the following:

$$\begin{cases} dx(t) = f\big(x(t), t, r(t)\big)\,dt + g\big(x(t), t, r(t)\big)\,dw(t), & \text{if } r\big(t^+\big) = r(t), \\ x\big(t^+\big) = I_{r(t^+), r(t)}\big(x(t), t\big), & \text{if } r\big(t^+\big) \neq r(t), \end{cases} \tag{1}$$

for $t \geqslant 0$ with initial value $x(0) = x_0 \in \mathrm{R}^n$, where $f : \mathrm{R}^n \times \mathrm{R}_+ \times S \to \mathrm{R}^n$, $g : \mathrm{R}^n \times \mathrm{R}_+ \times S \to \mathrm{R}^{n\times m}$, $I : \mathrm{R}^n \times \mathrm{R}_+ \to \mathrm{R}^n$, $x(t) \in \mathrm{R}^n$ is the state vector, $x(t^+) := \lim_{h\to 0^+} x(t+h)$, $x(t^-) := \lim_{h\to 0^+} x(t-h)$, $x(t^-) = x(t)$, which implies that the solution of the system (1) is left continuous. At the switching times, there exists an impulse described by the second equation of (1).

For system (1), we impose following hypotheses:

(H1) Both f and g satisfy the local Lipschitz condition and the linear growth condition with respect to x.
(H2) I satisfies the global Lipschitz condition with respect to x.

Before giving the results, we need to present the definition and useful lemma.

The general idea of finite-time stochastic stability concerns the boundedness in probability of the state of a system over a finite-time interval for given initial conditions; this concept can be formalized through the following definition.

Definition 1 *(Finite-time stochastic stability, FTSS).* (See [18].) The stochastic hybrid system (1) is FTSS with respect to $(\alpha, \beta, \lambda, T)$, if for any switching law $r(t)$,

$$P\Big\{\sup_{0\leqslant t\leqslant T}\big\|x(t)\big\| \geqslant \beta;\ \|x_0\| \leqslant \alpha\Big\} \leqslant \lambda, \quad \text{where } \alpha, \beta, \lambda, T \geqslant 0.$$

Remark 1. In [18], the research object was just a single stochastic system and did not consider the influence of switching part and impulsive effects. Therefore, our result is an extension of [18].

Let $C^{2,1}(\mathrm{R}^n \times \mathrm{R}_+ \times S; \mathrm{R}_+)$ denote the family of all nonnegative functions on $\mathrm{R}^n \times \mathrm{R}_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V(x,t,r(t)) \in C^{2,1}(\mathrm{R}^n \times \mathrm{R}_+ \times S; \mathrm{R}_+)$, define an operator LV from $\mathrm{R}^n \times \mathrm{R}_+ \times S$ to R by

$$LV\big(x,t,r(t)\big) := V_t\big(x,t,r(t)\big) + V_x\big(x,t,r(t)\big) f\big(x,t,r(t)\big) + 2^{-1}\operatorname{tr}\big\{g^{\mathrm{T}}\big(x,t,r(t)\big)V_{xx}\big(x,t,r(t)\big)g\big(x,t,r(t)\big)\big\} + \sum_{j=1}^{N}\gamma_{r(t),j}V(x,t,j) \tag{2}$$

where

$$V_t\big(x,t,r(t)\big) = \frac{\partial V(x,t,r(t))}{\partial t}, \qquad V_x\big(x,t,r(t)\big) = \left(\frac{\partial V(x,t,r(t))}{\partial x_1}, \ldots, \frac{\partial V(x,t,r(t))}{\partial x_n}\right),$$
$$V_{xx}\big(x,t,r(t)\big) = \left(\frac{\partial^2 V(x,t,r(t))}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Lemma 1 *(Generalized Ito formula). (See [22].) If $V(x,t,r(t)) \in C^{2,1}(\mathrm{R}^n \times \mathrm{R}_+ \times S; \mathrm{R}_+)$, then for any two switching instants t_1, t_2, $0 \leqslant t_1 \leqslant t_2 < +\infty$, we have*

$$EV\big(x(t_2),t_2,r(t_2)\big) = EV\big(x(t_1),t_1,r(t_1)\big) + E\int_{t_1}^{t_2} LV\big(x(s),s,r(s)\big)\,ds$$

as long as the integrations involved exist and are finite.

We will show next how FTSS can be indirectly determined by studying the probability associated with a function $V(x,t,r(t))$ defined for the stochastic hybrid system.

3. Finite time stability analysis

For given $T \geqslant 0$, we assume that the switching instants are $t_1, t_2, \ldots, t_K$, and $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_K \leqslant T$, where K denotes the number of switchings during the time interval $[0,T]$.

Theorem 1. *Consider the system* (1), *if there exist a function $V(x,t,r(t)) \in C^{2,1}(\mathrm{R}^n \times \mathrm{R}_+ \times S; \mathrm{R}_+)$, Lebesgue integrable bounded positive functions $\varphi_{r(t)}(x,t)$, $r(t) \in S$, positive constants $C_1, C_2, M, \mu_{r(t)} > 0$, $r(t) \in S$, and $0 < a < 1$, such that the following conditions hold*:

(1) $C_1\|x\| \leqslant V\big(x,t,r(t)\big) \leqslant C_2\|x\|, \quad t \geqslant 0,$

(2) $LV\big(x,t,r(t)\big) \leqslant \varphi_{r(t)}(x,t), \quad \text{if } r\big(t^+\big) = r(t),$

(3) $V\big(x\big(t^+\big),t^+,r\big(t^+\big)\big) \leqslant \mu_{r(t)}V\big(x(t),t,r(t)\big), \quad r\big(t^+\big) \neq r(t),$

(4) $\dfrac{(1+\sum_{j=k+1}^{K}\prod_{i=k+1}^{j}\mu_{r(t_i)})(\Phi_{r(t_{k+1})}(x(t_{k+1}),t_{k+1}) - \Phi_{r(t_{k+1})}(x(t_k),t_k))}{(1+\sum_{j=1}^{K}\prod_{i=1}^{j}\mu_{r(t_i)})(V_0 + \Phi_{r_1}(x(t_1),t_1))} \leqslant a^k,$

where $k = 1,\ldots,K-1$, $\Phi_{r(t)}(x,t)$ denotes the primitive function of $\varphi_{r(t)}(x,t)$, i.e., $\Phi_{r(t)}(x(t),t) = \int_0^t \varphi_{r(s)}(x(s),s)\,ds$,

(5) $\displaystyle\sum_{j=1}^{K}\prod_{i=1}^{j}\mu_{r(t_i)} \leqslant M.$

Then, for given $\alpha, \beta, T \geqslant 0$,

$$P\Big\{\sup_{0\leqslant t\leqslant T}\big\|x(t)\big\| \geqslant \beta;\ \|x_0\| \leqslant \alpha\Big\} \leqslant \frac{(1+M)C_2\alpha + (2+M-a)\Phi_{\sup}(T)}{C_1\beta(1-a)}, \tag{3}$$

where $\Phi_{\sup}(T) = \sup_{0\leqslant t\leqslant T}\Phi_{r(t)}(x(t),t)$.

Proof. From condition (1), we have

$$P\Big\{\sup_{0\leqslant t\leqslant T}\big\|x(t)\big\| \geqslant \beta;\ \|x_0\| \leqslant \alpha\Big\} \leqslant P\Big\{\sup_{0\leqslant t\leqslant T}V\big(x,t,r(t)\big) \geqslant r;\ V_0 \leqslant r_0\Big\},$$

where $r = C_1\beta$, $r_0 = C_2\alpha$.

Then, to prove (3) holds, it suffices to show the following inequality holds:

$$P\Big\{\sup_{0\leqslant t\leqslant T} V\big(x,t,r(t)\big)\geqslant r;\ V_0\leqslant r_0\Big\}\leqslant \frac{(1+M)C_2\alpha+(2+M-a)\Phi_{\sup}(T)}{C_1\beta(1-a)}.$$

Noting that the switching instants are $t_1, t_2, \dots, t_K$, and $0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_K\leqslant T$,

$$\begin{aligned}&P\Big\{\sup_{0\leqslant t\leqslant T} V\big(x,t,r(t)\big)\geqslant r\Big\}\\&\quad\leqslant P\Big\{\sup_{0\leqslant t\leqslant t_1} V\big(x,t,r(t)\big)\geqslant r\Big\}+P\Big\{\sup_{t_1^+\leqslant t\leqslant t_2} V\big(x,t,r(t)\big)\geqslant r\Big\}+\cdots+P\Big\{\sup_{t_K^+\leqslant t\leqslant T} V\big(x,t,r(t)\big)\geqslant r\Big\}\\&\quad\triangleq P_0+P_1+\cdots+P_K,\end{aligned}$$

$$\begin{aligned}EV\big(x(t_k),t_k,r(t_k)\big)&=EV\big(x(t_{k-1}^+),t_{k-1}^+,r(t_{k-1}^+)\big)+E\int_{t_{k-1}^+}^{t_k} LV\big(x(s),s,r(s)\big)\,ds\\&\leqslant EV\big(x(t_{k-1}^+),t_{k-1}^+,r(t_{k-1}^+)\big)+E\int_{t_{k-1}^+}^{t_k} \varphi_{r(s)}\big(x(s),s\big)\,ds,\quad k=1,\dots,K+1\ \big(t_0^+=0,\ t_{K+1}=T\big).\end{aligned}$$

From the definition of $\Phi_{r(t)}(x,t)$, we know that

$$\Phi_{r(b)}\big(x(b),b\big)-\Phi_{r(b)}\big(x(a),a\big)=\int_a^b \varphi_{r(s)}\big(x(s),s\big)\,ds,$$

and also noticing that $P\{\sup_{t_{k-1}^+\leqslant t\leqslant t_k} V(x,t,r(t))\geqslant r\}\leqslant \frac{EV(x(t_k),t_k,r(t_k))}{r}$, we have

$$\begin{aligned}rP\Big\{\sup_{t_{k-1}^+\leqslant t\leqslant t_k} V\big(x,t,r(t)\big)\geqslant r\Big\}&\leqslant EV\big(x(t_k),t_k,r(t_k)\big)\leqslant EV\big(x(t_{k-1}^+),t_{k-1}^+,r(t_{k-1}^+)\big)+E\int_{t_{k-1}^+}^{t_k}\varphi_{r(s)}\big(x(s),s\big)\,ds\\&=EV\big(x(t_{k-1}^+),t_{k-1}^+,r(t_{k-1}^+)\big)+\Phi_{r(t_k)}\big(x(t_k),t_k\big)-\Phi_{r(t_k)}\big(x(t_{k-1}),t_{k-1}\big).\end{aligned}$$

That is

$$P_{k-1}\leqslant\frac{1}{r}\big[EV\big(x(t_{k-1}^+),t_{k-1}^+,r(t_{k-1}^+)\big)+\Phi_{r(t_k)}\big(x(t_k),t_k\big)-\Phi_{r(t_k)}\big(x(t_{k-1}),t_{k-1}\big)\big],\quad k=1,\dots,K+1,$$

and

$$\begin{aligned}P_0+P_1+\cdots+P_K\leqslant\frac{1}{r}\big[&V(x_0,t_0,r_0)+\Phi_{r(t_1)}\big(x(t_1),t_1\big)+EV\big(x(t_1^+),t_1^+,r(t_1^+)\big)+\Phi_{r(t_2)}\big(x(t_2),t_2\big)\\&-\Phi_{r(t_2)}\big(x(t_1),t_1\big)+\cdots+EV\big(x(t_K^+),t_K^+,r(t_K^+)\big)+\Phi_{r(T)}\big(x(T),T\big)-\Phi_{r(T)}\big(x(t_K),t_K\big)\big].\end{aligned}\tag{4}$$

Since V satisfies condition (3), we have

$$EV\big(x(t_k^+),t_k,r(t_k^+)\big)\leqslant\mu_{r(t_k)}EV\big(x(t_k),t_k,r(t_k)\big),\quad k=1,\dots,K.\tag{5}$$

For convenience, we write $r(t_k)$ as k in the following proof.

Combining (4) with (5), we obtain

$$\begin{aligned}P_0+P_1+\cdots+P_K\leqslant\frac{1}{r}\Bigg[&\Bigg(1+\mu_1+\mu_2\mu_1+\mu_3\mu_2\mu_1+\prod_{k=1}^{K}\mu_k\Bigg)\big(V_0+\Phi_1\big(x(t_1),t_1\big)\big)\\&+\Bigg(1+\mu_2+\mu_3\mu_2+\prod_{k=2}^{K}\mu_k\Bigg)\big(\Phi_2\big(x(t_2),t_2\big)-\Phi_2\big(x(t_1),t_1\big)\big)+\cdots+(1+\mu_K)\big(\Phi_K\big(x(t_K),t_K\big)\\&-\Phi_K\big(x(t_{K-1}),t_{K-1}\big)\big)+\Phi_T\big(x(T),T\big)-\Phi_T\big(x(t_K),t_K\big)\Bigg].\end{aligned}$$

Consider conditions (4) and (5), we have

$$P_0 + P_1 + \cdots + P_K \leqslant \frac{1}{r}\left[\frac{(1+\sum_{j=1}^{K}\prod_{i=1}^{j}\mu_i)(V_0 + \Phi_1(x(t_1),t_1))}{1-a} + \Phi_T\big(x(T),T\big) - \Phi_T\big(x(t_K),t_K\big)\right]$$
$$\leqslant \frac{1}{r}\left[\frac{(1+M)(V_0+\Phi_{\sup}(T))}{1-a} + \Phi_{\sup}(T)\right].$$

That is

$$P\Big\{\sup_{0\leqslant t\leqslant T} V\big(x,t,r(t)\big) \geqslant r\Big\} \leqslant \frac{1}{r}\left[\frac{(1+M)(V_0+\Phi_{\sup}(T))}{1-a} + \Phi_{\sup}(T)\right] \leqslant \frac{1}{r}\left[\frac{(1+M)(r_0+\Phi_{\sup}(T))}{1-a} + \Phi_{\sup}(T)\right]$$
$$\leqslant \frac{(1+M)C_2\alpha + (2+M-a)\Phi_{\sup}(T)}{C_1\beta(1-a)}. \quad \square$$

For convenience in application, we often use the functions of the form

$$V\big(x,t,r(t)\big) = x^{\mathrm{T}} P_{r(t)} x \tag{6}$$

for some symmetric positive-definite matrices $P_{r(t)}$.

Note that for $x \neq 0$, $V_x(x,t,r(t)) = 2x^{\mathrm{T}}P_{r(t)}$ and $V_{xx}(x,t,r(t)) = 2P_{r(t)}$.

Hence, applying operator (2), we have

$$LV\big(x,t,r(t)\big) := 2x^{\mathrm{T}}P_{r(t)} f\big(x,t,r(t)\big) + \mathrm{tr}\big\{g^{\mathrm{T}}\big(x,t,r(t)\big)P_{r(t)}g\big(x,t,r(t)\big)\big\} + \sum_{j=1}^{N}\gamma_{r(t),j}\big(x^{\mathrm{T}}P_j x\big). \tag{7}$$

Based on Theorem 1, the following useful corollary can be easily established:

Corollary 1. *Assume that there exist N symmetric positive-definite matrices $P_{r(t)}$, Lebesgue integrable bounded positive functions $\varphi_{r(t)}(x,t)$, $r(t) \in S$, and positive constants M, $\mu_{r(t)}$, $r(t) \in S$, $0 < a < 1$, such that for given $\alpha, \beta, T \geqslant 0$, the following conditions hold*:

(1) $2x^{\mathrm{T}}P_{r(t)} f(x,t,r(t)) + \mathrm{tr}\{g^{\mathrm{T}}(x,t,r(t))P_{r(t)}g(x,t,r(t))\} + \sum_{j=1}^{N}\gamma_{r(t),j}(x^{\mathrm{T}}P_j x) \leqslant \varphi_{r(t)}(x,t)$,

(2) $I^{\mathrm{T}}_{r(t^+),r(t)}(x(t),t)P_{r(t^+)}I_{r(t^+),r(t)}(x(t),t) \leqslant \mu_{r(t)}x^{\mathrm{T}}P_{r(t)}x$,

(3) $\varphi_{\sup}(T) = \frac{C_1\beta\lambda(1-a)-(1+M)C_2\alpha}{(2+M-a)T}$,

(4) *conditions* (4), (5) *of Theorem* 1 *are satisfied, where* $\varphi_{\sup}(T) = \sup_{0\leqslant t\leqslant T}\varphi_{r(t)}(x,t)$, $C_1 = \min_{1\leqslant r(t)\leqslant N}\lambda_{\min}(P_{r(t)})$, $C_2 = \max_{1\leqslant r(t)\leqslant N}\lambda_{\max}(P_{r(t)})$.

Then the system (1) *is finite-time stochastically stable w.r.t.* $(\alpha, \beta, \lambda, T)$.

Proof. Let $V(x,t,r(t))$ be defined by Eq. (6). The conditions of Theorem 1 can be easily checked.

Therefore, the assertion of Theorem 1 follows:

$$P\Big\{\sup_{0\leqslant t\leqslant T}\|x(t)\| \geqslant \beta;\ \|x_0\| \leqslant \alpha\Big\} \leqslant \frac{(1+M)C_2\alpha + (2+M-a)\Phi_{\sup}(T)}{C_1\beta(1-a)}. \tag{8}$$

Using condition (3), we have

$$\Phi_{\sup}(T) \leqslant \frac{C_1\beta\lambda(1-a) - (1+M)C_2\alpha}{(2+M-a)}.$$

Substituting it into Eq. (8), we have

$$P\Big\{\sup_{0\leqslant t\leqslant T}\big\|x(t)\big\| \geqslant \beta;\ \|x_0\| \leqslant \alpha\Big\} \leqslant \lambda.$$

Thus, we get the desired result. $\square$

4. Finite time stabilization

The previous section focuses on FTSS analysis, and the result may extended to design controllers that stochastically stabilize a system over a finite-time. Next, based on the result of Corollary 1, we aim to design a state-feedback control law $u(t)$ which consists of two parts $u_1(t)$ and $u_2(t)$ such that the closed-loop system

$$\begin{cases} dx(t) = \big[f\big(x(t), t, r(t)\big) + h_1\big(x(t), t, r(t)\big)u_1(t)\big]\,dt \\ \qquad\quad + \big[g\big(x(t), t, r(t)\big) + h_2\big(x(t), t, r(t)\big)u_2(t)\big]\,dw(t), \quad \text{if } r\big(t^+\big) = r(t), \\ x\big(t^+\big) = I_{r(t^+), r(t)}\big(x(t), t\big), \quad \text{if } r\big(t^+\big) \neq r(t), \end{cases} \tag{9}$$

is FTSS with respect to the parameter $(\alpha, \beta, \lambda, T)$. Here $h_1 : \mathrm{R}^n \times \mathrm{R}_+ \times S \to \mathrm{R}^{n\times n}$, $h_2 : \mathrm{R}^n \times \mathrm{R}_+ \times S \to \mathrm{R}^{n\times 1}$, $u_1(t) \in \mathrm{R}^{n\times l}$, $u_2(t) \in \mathrm{R}^{1\times m}$, note that different control $u_1(t)$ and $u_2(t)$ appears in shift parts and diffusion parts of the underlying stochastic subsystems.

For the system (9), we define $V(x, t, r(t)) = \eta_{r(t)}|x|^2$ for $(x, t, r(t)) \in \mathrm{R}^n \times \mathrm{R}_+ \times S$, where $\eta_{r(t)} > 0$, $r(t) \in S$. Using Eq. (7) with $P_{r(t)} =$ the identity matrix and condition (1) of Corollary 1, we can derive that

$$\begin{aligned} &2\eta_{r(t)}x^{\mathrm{T}}h_1\big(x, t, r(t)\big)u_1(t) + \eta_{r(t)}h_2^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\big|u_2(t)\big|^2 + 2\eta_{r(t)}\big|g^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\big|\big|u_2(t)\big| \\ &\quad + 2\eta_{r(t)}x^{\mathrm{T}}f\big(x, t, r(t)\big) + \eta_{r(t)}\big|g\big(x, t, r(t)\big)\big|^2 + \sum_{j=1}^{N}\gamma_{r(t), j}\eta_j|x|^2 \leqslant \varphi_{r(t)}(x, t). \end{aligned} \tag{10}$$

And the other conditions of Corollary 1 can be easily checked. Thus, we can choose the set of possible control laws of $u_1(t)$ and $u_2(t)$ such that Eq. (10) holds.

Case 1. $x^{\mathrm{T}}h_1(x, t, r(t)) = 0$, $u_1(t) = u_2(t) = 0$, for $h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t)) = |g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| = 0$.

$u_1(t) = 0$, $0 \leqslant |u_2(t)| \leqslant u'(t)$, for $h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t)) \neq 0$, $|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| \neq 0$, and

$$\begin{aligned} &h_2^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\left(2\eta_{r(t)}x^{\mathrm{T}}f\big(x, t, r(t)\big) + \eta_{r(t)}\big|g\big(x, t, r(t)\big)\big|^2 + \sum_{j=1}^{N}\gamma_{r(t), j}\eta_j|x|^2\right) \\ &\quad \leqslant \eta_{r(t)}\big|g^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\big|^2 + h_2^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\varphi_{r(t)}(x, t). \end{aligned}$$

Let $A_1 = 2\eta_{r(t)}x^{\mathrm{T}}f(x, t, r(t)) + \eta_{r(t)}|g(x, t, r(t))|^2 + \sum_{j=1}^{N}\gamma_{r(t), j}\eta_j|x|^2 - \varphi_{r(t)}(x, t)$,

$$u'(t) = \frac{-\eta_{r(t)}|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| + B_1}{\eta_{r(t)}h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))}$$

where $B_1 = \sqrt{\eta_{r(t)}^2|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))|^2 - \eta_{r(t)}h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))A_1}$.

Case 2. $x^{\mathrm{T}}h_1(x, t, r(t)) \neq 0$,

$$u_1(t) \leqslant \frac{-A_1}{2\eta_{r(t)}x^{\mathrm{T}}h_1(x, t, r(t))}, \qquad u_2(t) = 0,$$

for $h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t)) = |g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| = 0$.

$$u_1(t) \leqslant \frac{\lambda_1(t)}{2\eta_{r(t)}x^{\mathrm{T}}h_1(x, t, r(t))}, \qquad 0 \leqslant \big|u_2(t)\big| \leqslant u'(t),$$

for $h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t)) \neq 0$, $|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| \neq 0$.

Let $A_2 = 2\eta_{r(t)}x^{\mathrm{T}}f(x, t, r(t)) + \eta_{r(t)}|g(x, t, r(t))|^2 + \sum_{j=1}^{N}\gamma_{r(t), j}\eta_j|x|^2 - \lambda_2(t)$,

$$u'(t) = \frac{-\eta_{r(t)}|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| + B_2}{\eta_{r(t)}h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))}$$

where

$$\begin{aligned} &B_2 = \sqrt{\eta_{r(t)}^2\big|g^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)\big|^2 - \eta_{r(t)}h_2^{\mathrm{T}}\big(x, t, r(t)\big)h_2\big(x, t, r(t)\big)A_2}, \\ &0 < \lambda_1(t) + \lambda_2(t) \leqslant \varphi_{r(t)}(x, t). \end{aligned}$$

Remark 2. The set of possible control laws under the situation that $h_2^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t)) \neq 0$, $|g^{\mathrm{T}}(x, t, r(t))h_2(x, t, r(t))| = 0$ can be deduced by similar discussion.

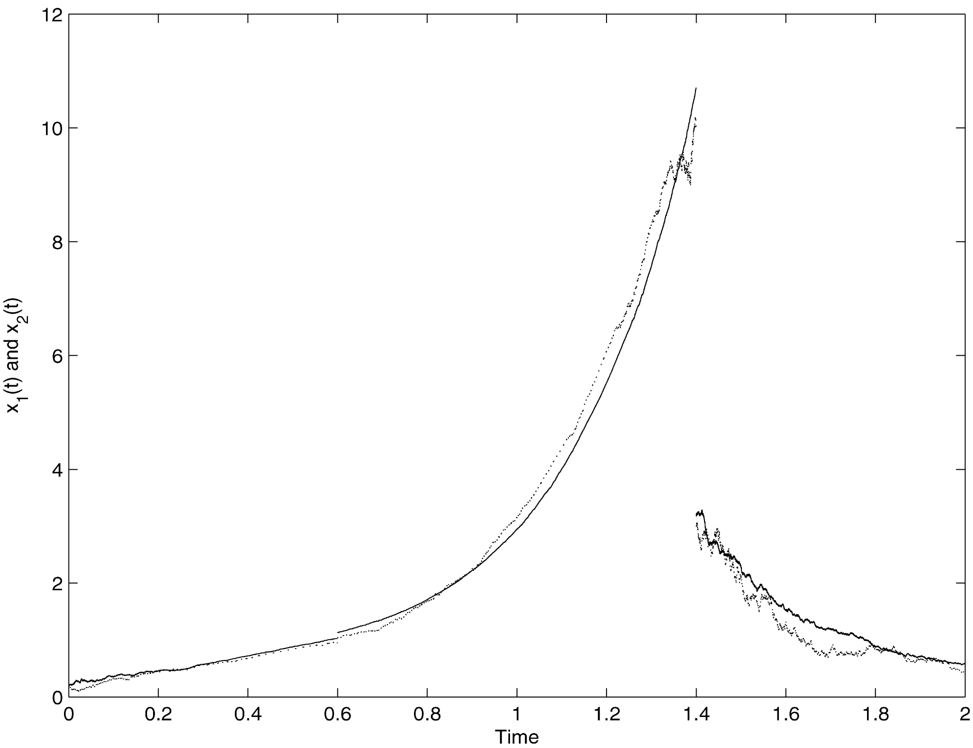

Fig. 1. $x_1(t)$ (solid curve) and $x_2(t)$ (dashed curve) of the closed-loop system.

5. Numerical example

In this section, we present an example to illustrate the effectiveness of the proposed methods. Let $\omega(t)$ be a one-dimensional Brownian motion and $r(t)$ be a Markov chain taking values in $S = \{1, 2\}$ with generator $\Gamma = (\gamma_{ij})_{2\times 2} = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Consider the Markovian switching stochastic controlled systems with impulsive effects of the form

$$\begin{cases} dx(t) = \big[f\big(x(t), t, r(t)\big) + h_1\big(x(t), t, r(t)\big)u_1(t)\big]\,dt \\ \qquad + \big[g\big(x(t), t, r(t)\big) + h_2\big(x(t), t, r(t)\big)u_2(t)\big]\,dw(t), \quad \text{if } r\big(t^+\big) = r(t), \\ x\big(t^+\big) = I_{r(t^+), r(t)}\big(x(t), t\big), \quad \text{if } r\big(t^+\big) \neq r(t), \end{cases} \tag{11}$$

for $t \geqslant 0$, where

$$f(x,t,1) = \begin{bmatrix} x_1 \cos 2t \\ x_2 \cos 2t \end{bmatrix}, \qquad f(x,t,2) = \begin{bmatrix} x_1 \sin t \\ x_2 \sin t \end{bmatrix}, \qquad g(x,t,1) = g(x,t,2) = \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 \end{bmatrix},$$

$$h_1(x,t,1) = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{4}x_2 \end{bmatrix}, \qquad h_1(x,t,2) = \begin{bmatrix} \frac{1}{4}x_1 \\ \frac{1}{2}x_2 \end{bmatrix}, \qquad h_2(x,t,1) = h_2(x,t,2) = \begin{bmatrix} x_1 \\ \frac{1}{3}x_2 \end{bmatrix},$$

$$I_{1,2}(x(t),t) = \begin{bmatrix} 1.1x_1 \\ 1.1x_2 \end{bmatrix}, \qquad I_{2,1}(x(t),t) = \begin{bmatrix} 0.3x_1 \\ 0.3x_2 \end{bmatrix}.$$

We would like to choose $u_1(t)$ and $u_2(t)$ in such a way that the closed-loop system (11) is finite-time stable with respect to $\alpha = 0.25$, $\beta = 3$, $\lambda = 0.3$, $T = 2$. For that we fix the number of switching times $K = 2$. By applying Theorem 2 with $\eta_1 = 2$, $\eta_2 = 2.01$, $M = 1.3$, $a = 0.3$, we can compute that $\varphi_{\sup}(2) = 0.053$. Therefore, we can choose $\varphi_1(x,t)$ as $\cos\frac{\pi}{2}(1+t) + 0.053\sin\frac{\pi}{2}t$, and $\varphi_2(x,t)$ as $0.053\cos\frac{\pi}{2}(t-1) + \sin\frac{\pi}{2}(t+2)$ and choose $u_1(t)$ and $u_2(t)$ as following:

$$u_1(t) = \begin{cases} \dfrac{\cos\frac{\pi}{2}(1+t)}{2x_1^2 + x_2^2}, & r(t) = 1, \\[2ex] \dfrac{\sin\frac{\pi}{2}(2+t)}{1.005x_1^2 + 2.01x_2^2}, & r(t) = 2, \end{cases} \tag{12}$$

$$u_2(t) = \begin{cases} \dfrac{-2(x_1^2 + \frac{1}{6}x_2^2) + \sqrt{\Omega_1}}{2(x_1^2 + \frac{1}{9}x_2^2)}, & r(t) = 1, \\ \dfrac{-2.01(x_1^2 + \frac{1}{6}x_2^2) + \sqrt{\Omega_2}}{2.01(x_1^2 + \frac{1}{9}x_2^2)}, & r(t) = 2, \end{cases} \tag{13}$$

where $\Omega_1 = (8\cos 2t - 0.02)(x_1^2 + \frac{1}{9}x_2^2)(x_1^2 + x_2^2) + 0.053 \sin \frac{\pi}{2} t - \frac{1}{9}x_1^2 x_2^2$, $\Omega_2 = (8\cos 2t + 0.02)(x_1^2 + \frac{1}{9}x_2^2)(x_1^2 + x_2^2) + 0.053 \cos \frac{\pi}{2}(t-1) - \frac{1}{9}x_1^2 x_2^2$.

Under the control law (12), (13), let $x_0 = (0.15, 0.2)^{\mathrm{T}}$, the system trajectory $x(t)$ is shown in Fig. 1. During the interval $[0, 2]$, although the system is not asymptotically stable, we can compute that the probability when $\|x\|$ exceeds the given bound $\beta = 3.5$ is 0.137, which is less than $\lambda = 0.3$. (Considering the effect of Brown motion, we have done several simulations. In each of simulation, the probability is always less than 0.3. What we have showed in Fig. 1 is just a typical one.) Therefore, the closed-loop system is finite-time stable w.r.t. $(0.25, 3.5, 0.3, 2)$. Thus, our design goals have achieved.

6. Conclusion

The issues of finite-time stability and stabilization for nonlinear stochastic hybrid systems have been studied and corresponding results have been presented. Using multiple Lyapunov functions theory, a sufficient condition for finite-time stability has been given. Furthermore, based on the state partition of continuous parts of systems, a feedback controller has been designed such that the corresponding impulsive stochastic closed-loop systems are finite-time stochastically stable.

More research effects will be devoted to more relaxed conditions of FTSS for stochastic hybrid systems and the applications of the results presented here to packet-dropping problems in network control systems and time-delayed systems.

References

[1] H.I. Kushner, P. Dupuis, Numerical Methods for Stochastic Control Problems in Continuous Time, Springer-Verlag, New York, 2001.
[2] J.P. Hespanha, A model for stochastic hybrid systems with application to communication networks, Nonlinear Anal. 62 (2005) 1353–1383.
[3] Y. Ji, H.J. Chizeck, Controllability, stability and continuous-time Markovian jump linear quadratic control, IEEE Trans. Automat. Control 35 (1990) 777–788.
[4] X. Mao, Stability of stochastic differential equations with Markovian switching, Stochastic Process. Appl. 79 (1999) 45–67.
[5] L. Hu, P. Shi, B. Huang, Stochastic stability and robust control for sampled-data systems with Markovian jump parameters, J. Math. Anal. Appl. 313 (2006) 504–517.
[6] V. Dragan, T. Morozan, Stability and robust stabilization to linear stochastic systems described by differential equations with Markovian jumping and multiplicative white noise, Stochastic Process. Appl. 20 (2002) 33–92.
[7] H. Ye, A.N. Michel, L. Hou, Stability analysis of systems with impulse effect, IEEE Trans. Automat. Control 43 (1998) 1719–1723.
[8] G. Xie, L. Wang, Necessary and sufficient conditions for controllability and observability of switched impulsive control systems, IEEE Trans. Automat. Control 49 (2004) 960–966.
[9] Z.H. Guan, D.J. Hill, X. Shen, On hybrid impulsive and switching systems and application to nonlinear control, IEEE Trans. Automat. Control 50 (2005) 1058–1062.
[10] Z.G. Li, C.Y. Wen, Y.C. Soh, Analysis and design of impulsive control systems, IEEE Trans. Automat. Control 46 (2001) 894–897.
[11] P. Dorato, Short time stability in linear time-varying systems, in: Proceedings of IRE Int. Convention Record Part 4, 1961, pp. 83–87.
[12] L. Weiss, E.F. Infante, Finite time stability under perturbing forces and on product spaces, IEEE Trans. Automat. Control 12 (1967) 54–59.
[13] H.D. Angelo, Linear Time-Varying Systems: Analysis and Synthesis, Allyn and Bacon, Boston, 1970.
[14] F. Amato, M. Ariola, P. Dorato, Finite-time control of linear systems subject to parametric uncertainties and disturbances, Automatica 37 (2001) 1459–1463.
[15] F. Amato, M. Ariola, Finite-time control of discrete-time linear systems, IEEE Trans. Automat. Control 50 (2005) 724–729.
[16] Y.G. Hong, J.K. Wang, Nonsmooth finite-time stabilization of a class of nonlinear systems, Sci. China Ser. E 35 (2005) 663–672.
[17] X.Q. Huang, W. Lin, B. Yang, Global finite-time stabilization of a class of uncertain nonlinear systems, Automatica 41 (2005) 881–888.
[18] H.J. Kushner, Finite-time stochastic stability and the analysis of tracking systems, IEEE Trans. Automat. Control 11 (1966) 219–227.
[19] S. Mastellone, P. Dorato, C.T. Abdallah, Finite-time stability of discrete-time nonlinear systems: Analysis and design, in: Proceedings of the 43rd IEEE Conference on Decision and Control, 2004, pp. 2572–2577.
[20] L.J. Van Mellaert, P. Dorato, Numerical solution of an optimal control problem with a probability criterion, IEEE Trans. Automat. Control 17 (1972) 543–546.
[21] M.A. Rami, X. Chen, J.B. Moore, X.Y. Zhou, Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls, IEEE Trans. Automat. Control 46 (2001) 428–440.
[22] A.V. Skorohod, Symptotic Methods in the Theory of Stochastic Differential Equations, American Mathematical Society, Providence, RI, 2004.