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Oscillation results for Sturm–Liouville problems with an indefinite weight function

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This paper is dedicated to Professor Norrie Everitt on the occasion of his 80th birthday

Abstract

We prove oscillation results for the real eigenvalues of Sturm–Liouville problems with an indefinite weight function. An essential role is played by the signature of an eigenvalue, which is shown to be related to the signs of the corresponding leading coefficients of the Titchmarsh–Weyl m-function and of the Prüfer angle at this eigenvalue.

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1. Introduction

We consider the Sturm–Liouville problem

$$-y''(x) + q(x)\,y(x) = \lambda\, r(x)\, y(x) \quad \text{on } (0,\ell), \tag{1.1}$$

$$y'(0) = y'(\ell) = 0, \tag{1.2}$$

under the assumption that q and r are real valued functions in $L^1(0,\ell)$ with $|r| > 0$ a.e. When $r > 0$ a.e., the problem is called *right definite*, but our main focus is on the *indefinite weight case*,

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where r takes both signs on sets of positive measure. In the first case, if we denote its eigenvalues by $\lambda_1 < \lambda_2 < \cdots$ and the corresponding eigenfunctions by y_n, $n = 1, 2, \ldots$, then the number of zeros or the *oscillation count* of the eigenfunction y_n in the interval $(0, \ell)$ equals $n-1$, cf. [1, Section 8.4], [12, Theorem 13.2]. Generalizations of this result to the indefinite weight case have been considered for a long time, see [11], the more recent paper [4], and the references there. Recall that the spectrum in the indefinite weight case consists of two sequences of eigenvalues, tending to $+\infty$ and to $-\infty$, and, possibly, also of a finite number of pairs of complex conjugate nonreal eigenvalues.

In the present note we prove a formula for the oscillation count ω_n of the eigenfunction y_n corresponding to the nth positive eigenvalue λ_n of (1.1), (1.2) in the indefinite weight case. An essential ingredient is the *signature* $s(\lambda)$ of a real eigenvalue λ, see Section 2. In the generic case of an algebraically simple eigenvalue λ with eigenfunction y this signature is $s(\lambda) = \operatorname{sign}[y, y]$ with

$$[y, y] = \int_0^\ell |y(x)|^2 \, r(x) \, dx,$$

which is either positive or negative. If we assume that 0 is not an eigenvalue of problem (1.1), (1.2), that all its positive eigenvalues $\lambda_1 < \lambda_2 < \cdots$ are algebraically simple, and that the signature of λ_n is $+1$, then Theorem 4.5 gives for the oscillation count

$$\omega_n = \kappa + \sum_{j=1}^{n-1} s(\lambda_j), \tag{1.3}$$

where κ is the number of negative eigenvalues of problem (1.1), (1.2) with $r = 1$. That is, starting with κ, each eigenvalue λ_j, $1 \leqslant j \leqslant n-1$, adds its signature $+1$ or -1 to obtain the oscillation count of the eigenfunction y_n.

Our results apply to general Sturm–Liouville problems of the form

$$-(py')' + qy = \lambda r y \tag{1.4}$$

with $1/p > 0$ and in $L^1(0, \ell)$ since (1.4) can be transformed to (1.1), cf. [4, Section 4], and can be extended to general self-adjoint boundary conditions. They can also be proved for generalized second-order differential operators of Krein–Feller type, cf. [9]:

$$-dy' = \lambda y \, dM$$

with a nonmonotonic function M of bounded variation, which will be considered elsewhere.

A fundamental role in this note is played by the Titchmarsh–Weyl m-function which has occupied a central position in Sturm–Liouville theory for the best part of a century now. W.N. Everitt recognized early the importance of the Titchmarsh–Weyl function, both in the definite and the indefinite weight cases, cf. [2,7,8]. The Titchmarsh–Weyl function, and the function t which is given by $t(\lambda) = \tan \theta(\ell, \lambda)$, where θ is the Prüfer angle, are both meromorphic functions with poles at the eigenvalues of the problem (1.1), (1.2), and the signs of the leading coefficients of m and t at these poles are related to the signatures of these eigenvalues.

A brief synopsis is as follows. In the following section we associate with (1.1), (1.2) a self-adjoint operator in a Krein space, we introduce the signature of a real eigenvalue and we characterize the number κ, which appeared in (1.3), in different ways. We also introduce the fundamental system of solutions φ, ψ of Eq. (1.1) which corresponds to the left boundary condition and we prove relations between φ, Jordan chains and signatures of real eigenfunctions. In Section 3 we study the

Titchmarsh–Weyl function m, the meromorphic function t, and the Prüfer angle θ and we establish various relations between the algebraic multiplicity and signature of a real eigenvalue and the leading coefficients of the cited functions. Finally, in Section 4 we prove the main result about the oscillation count of the eigenfunctions.

2. Real eigenvalues and their signatures

1. With Eq. (1.1) we associate the inner product

$$\int_0^\ell f(x)\overline{g(x)}r(x)\,\mathrm{d}x, \qquad f,g\in\mathscr{K},$$

where $\mathscr{K}$ is the set of measurable functions f on $[0,\ell]$ such that the integral $\int_0^\ell |f(x)|^2|r(x)|\,\mathrm{d}x$ is finite. The set $\mathscr{K}$ equipped with this inner product is a Krein space, cf. [5, p. 42]. In $\mathscr{K}$ we define the operator A on the set

$$\mathscr{D}(A) := \{y\in\mathscr{K} : y, y' \text{ a.c. on } [0,\ell],\ r^{-1}(-y''+qy)\in\mathscr{K} \text{ and (1.2) holds}\}$$

by

$$Ay := r^{-1}(-y''+qy).$$

It is well known, cf. [5], that A is self-adjoint in the Krein space $\mathscr{K}$, that its spectrum is discrete and consists of two sequences of real, geometrically simple eigenvalues with limits $\pm\infty$ and, possibly, a finite number κ_0 of complex conjugate pairs of non-real eigenvalues. After a shift of the eigenvalue parameter λ in (1.1) we can assume that 0 is not an eigenvalue of A.

It follows that 0 is not an eigenvalue of the modified problems which arise if r in (1.1) is replaced by the constant function 1 or by $|r|$. The numbers of negative eigenvalues of these two modified problems, which are evidently right definite, coincide, and we denote this common number by κ. It will play an essential role in the sequel. We mention that κ is also the maximal dimension of a subspace of $\mathscr{D}(A)$ on which the quadratic form

$$\int_0^\ell (|y'(x)|^2 + q(x)|y(x)|^2)\,\mathrm{d}x, \qquad y\in\mathscr{D}(A), \tag{2.1}$$

which is independent of r, is negative. Note that the characterization of κ as the number of negative eigenvalues of a self-adjoint problem with definite weights 1 or $|r|$ makes it accessible to well known numerical evaluations such as SLEIGN2 by Bailey et al., cf. [3].

Since the eigenvalues of a Sturm–Liouville operator with separated boundary conditions are geometrically simple, the usual classification of the eigenvalues of the self-adjoint operator A in Krein space into positive, negative and neutral type becomes particularly simple: a real eigenvalue λ_0 of A is said to be of *positive* (*negative, neutral*, respectively) *type* if the corresponding eigenfunction y_0 has the property $[y_0, y_0] > 0$ (<0, $=0$, respectively). There is only a finite number κ_+ of positive eigenvalues of A which are not of positive type, and only a finite number κ_- of negative eigenvalues of A which are not of negative type, and, cf. [5,6],

$$\kappa_+ + \kappa_- + \kappa_0 \leqslant \kappa. \tag{2.2}$$

Moreover, a real eigenvalue is algebraically simple if and only if it is not of neutral type.

In the following, we denote the algebraic multiplicity for a real eigenvalue λ_0 of A by $\mu(\lambda_0)$, and we define the *signature* $s(\lambda_0)$ as follows: If $\mu := \mu(\lambda_0)$, and if the elements $y_0, y_1, \ldots, y_{\mu-1}$ form a Jordan chain for A at λ_0, then the Gram matrix $(c_{ij})_{i,j=0}^{\mu-1}$ with $c_{ij} := [y_i, y_j]$, $i,j = 0,1,\ldots,\mu-1$, is a Hankel matrix: $c_{ij} = c_{i+j}$, $c_j = 0$ if $j = 0,1,\ldots,\mu-2$, and $c_{\mu-1} \neq 0$, cf. [10, Theorem 3.2]. The sign of $c_{\mu-1} =: c_{\mu-1}(\lambda_0)$ is independent of the choice of the Jordan chain, and we define it as the signature of λ_0: $s(\lambda_0) = \operatorname{sign} c_{\mu-1}(\lambda_0)$. If $\mu = 1$, i.e., λ_0 is algebraically simple, then this definition reduces to $s(\lambda_0) = 1$ (resp. -1) if λ_0 is of positive (resp. negative) type.

2. We consider the fundamental system $\varphi(x,\lambda)$, $\psi(x,\lambda)$ of solutions of the differential equation (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = 0,$$

$$\psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1.$$

For $x \in [0,\ell]$ the functions $\varphi(x,\cdot)$, $\psi(x,\cdot)$ are entire and

$$\begin{vmatrix} \varphi(x,\lambda) & \varphi'(x,\lambda) \\ \psi(x,\lambda) & \psi'(x,\lambda) \end{vmatrix} = 1, \qquad 0 \leqslant x \leqslant \ell,\ \lambda \in \mathbb{C}. \tag{2.3}$$

Evidently, λ_0 is an eigenvalue of problem (1.1), (1.2) if and only if $\varphi'(\ell,\lambda_0) = 0$, and in this case the function $\varphi(\cdot,\lambda_0)$ is a corresponding eigenfunction. The order of the zero λ_0 of the entire (characteristic) function $\varphi'(\ell,\cdot)$ is denoted by $\mu_c(\lambda_0)$.

Lemma 2.1. *Let λ_0 be a real eigenvalue of A and set*

$$y_j = \frac{1}{j!}\frac{\partial^j \varphi}{\partial \lambda^j}(\cdot,\lambda_0), \quad j = 0,1,\ldots.$$

Then $\mu_c(\lambda_0) = \mu(\lambda_0)$, the elements $y_0, y_1, \ldots, y_{\mu(\lambda_0)-1}$ form a Jordan chain of A at λ_0, the element $c := c_{\mu(\lambda_0)-1}(\lambda_0)$ of the corresponding Hankel matrix is given by $c = -y'_{\mu(\lambda_0)}(\ell)y_0(\ell)$, and the signature of λ_0 is $s(\lambda_0) = \operatorname{sign} c = -\operatorname{sign} y'_{\mu(\lambda_0)}(\ell)y_0(\ell)$.

Proof. Since $y_j \in \mathscr{D}(A)$, we have $y_j'(l) = 0$ for $j = 0,\ldots,\mu(\lambda_0)-1$, whence $\mu_c(\lambda_0) \geqslant \mu(\lambda_0)$. Now let

$$d(x,\lambda) = \varphi'(x,\lambda)\frac{\partial\varphi}{\partial\lambda}(x,\lambda) - \varphi(x,\lambda)\frac{\partial\varphi'}{\partial\lambda}(x,\lambda).$$

Noting that $d'(x,\lambda) = r(x)\varphi(x,\lambda)^2$ and $d(0,\lambda_0) = 0$, we have

$$d(\ell,\lambda) := \int_0^\ell \varphi(x,\lambda)^2 r(x)\,\mathrm{d}x \tag{2.4}$$

at $\lambda = \lambda_0$. Differentiating (2.4) $\mu(\lambda_0) - 1$ times with respect to λ and using properties of the Gram matrix stated above we obtain the second statement, which immediately leads to the identity for the signature. Finally, $c \neq 0$ gives $y'_{\mu(\lambda_0)}(\ell) \neq 0$, whence $\mu_c(\lambda_0) \leqslant \mu(\lambda_0)$. □

3. The Titchmarsh–Weyl function and the Prüfer angle

In the following, the Titchmarsh–Weyl function $m(\lambda)$ of problem (1.1), (1.2) and a second function $t(\lambda)$, which is closely related to the Prüfer angle θ, will play an important role. They are defined as follows:

$$m(\lambda) := \frac{\psi'(\ell,\lambda)}{\varphi'(\ell,\lambda)}, \qquad t(\lambda) := \frac{\varphi(\ell,\lambda)}{\varphi'(\ell,\lambda)}.$$

The poles of the meromorphic functions t and m obviously coincide and they also coincide with the zeros of $\varphi'(\ell,\cdot)$ (even as to multiplicities); here we have to observe that $\varphi'(\ell,\lambda)=0$ implies $\varphi(\ell,\lambda)\neq 0$ and $\psi'(\ell,\lambda)\neq 0$ by (2.3).

We next define the Prüfer angle θ as the solution of the initial value problem

$$\theta'(x,\lambda) = \cos^2\theta(x,\lambda) + (r(x)\lambda - q(x))\sin^2\theta(x,\lambda), \qquad \theta(0,\lambda)=\pi/2. \tag{3.1}$$

It is standard that

$$\tan\theta(x,\lambda) = \frac{\varphi(x,\lambda)}{\varphi'(x,\lambda)}, \qquad x\in[0,\ell],\ \lambda\in\mathbb{C},$$

so, in particular,

$$\tan\theta(\ell,\lambda) = t(\lambda). \tag{3.2}$$

In the sequel, formula (3.2) will be essential. If the coefficients q and r are continuous and $\theta(x,\lambda)=k\pi$ for some integer k, then (3.1) implies that $\theta'(x,\lambda)=1$, so the function $\theta(\cdot,\lambda)$ is increasing at x. It is well known that $\theta(\cdot,\lambda)$ is also increasing through $k\pi$ for general q and r as considered here, cf. [4, Lemma 3.1]. Thus $\theta(\ell,\lambda)$, for both definite and indefinite weight functions r, can be used to count the number $\omega(\lambda)$ of zeros of $\varphi(\cdot,\lambda)$ in $(0,\ell)$. Hence, if λ_0 is a real eigenvalue of (1.1), (1.2), then

$$\theta(\ell,\lambda_0) = \left(\omega(\lambda_0)+\tfrac{1}{2}\right)\pi. \tag{3.3}$$

For any integer ω, we introduce the function

$$\theta_\omega := \theta(\ell,\cdot) - \left(\omega+\tfrac{1}{2}\right)\pi. \tag{3.4}$$

In the following, if h is a function which is meromorphic in a neighborhood of λ_0, then the leading coefficient of its Laurent or Taylor expansion is denoted by $c(h,\lambda_0)$, that is, if

$$h(\lambda) = \sum_{v=n}^{\infty} c_v(\lambda-\lambda_0)^v$$

for some integer n with $c_n\neq 0$, then

$$c(h,\lambda_0) := c_n.$$

Lemma 3.1. *If λ_0 is a real eigenvalue of A, then the orders of the poles of the function m and of the function t at λ_0 and the order of the zero of the function $\theta_{\omega(\lambda_0)}$ at λ_0 coincide and are equal to $\mu(\lambda_0)$. For the leading coefficients of these functions at λ_0 we have*

$$c(m,\lambda_0) = -c_{\mu(\lambda_0)-1}(\lambda_0)^{-1}, \quad c(t,\lambda_0) = \varphi(\ell,\lambda_0)^2 c(m,\lambda_0) = -c(\theta_{\omega(\lambda_0)},\lambda_0)^{-1}.$$

Proof. As was mentioned already, the poles of the meromorphic functions t and m coincide and they also coincide with the zeros of $\varphi'(\ell,\cdot)$ (even as to multiplicities). Moreover,

$$\frac{1}{m(\lambda)} = \frac{\varphi'(\ell,\lambda)}{\psi'(\ell,\lambda)},$$

and $c(\varphi'(\ell,\cdot),\lambda_0)=y'_{\mu(\lambda_0)}(\ell)$ in the notation in Lemma 2.1. Thus,

$$c(m,\lambda_0)^{-1}=\frac{y'_{\mu(\lambda_0)}(\ell)}{\psi'(\ell,\lambda_0)}=\varphi(\ell,\lambda_0)y'_{\mu(\lambda_0)}(\ell)=y_0(\ell)y'_{\mu(\lambda_0)}(\ell)=-c_{\mu(\lambda_0)}(\lambda_0)$$

by (2.3) and Lemma 2.1. Similarly,

$$c(t,\lambda_0)^{-1}=\varphi(\ell,\lambda_0)^{-1}c(\varphi'(\ell,\cdot),\lambda_0)=y_0(\ell)^{-2}c(m,\lambda_0)^{-1}.$$

It remains to establish the contentions about θ. Since $\theta_{\omega(\lambda_0)}(\lambda_0)=0$ and $\tan'(0)=1$, we conclude from differentiating

$$\tan\theta_{\omega(\lambda_0)}(\lambda)=-\cot\theta(\ell,\lambda)=-\frac{1}{t(\lambda)}$$

$\mu(\lambda_0)$ times with respect to λ that the orders of the zero at λ_0 of the function $\theta_{\omega(\lambda_0)}$ and the pole of m at λ_0 coincide and that $c(\theta_{\omega(\lambda_0)},\lambda_0)=-c(t,\lambda_0)^{-1}$. □

We summarize some of the results of this and the foregoing section in the following theorem. Recall that A is the self-adjoint operator associated with problem (1.1), (1.2) in the Krein space $\mathscr{K}$.

Theorem 3.2. *For $\lambda_0\in\mathbb{R}$ the following are equivalent*:

(i) *λ_0 is an eigenvalue of the operator A of algebraic multiplicity $\mu(\lambda_0)$.*
(ii) *λ_0 is a pole of the Titchmarsh–Weyl coefficient m of order $\mu(\lambda_0)$.*
(iii) *λ_0 is a pole of the function t of order $\mu(\lambda_0)$.*
(iv) *$\theta(\ell,\lambda_0)=\left(\omega+\frac{1}{2}\right)\pi$ for some nonnegative integer ω and the order of the zero λ_0 of the function θ_ω in (3.4) is $\mu(\lambda_0)$.*
(v) *λ_0 is a zero of the function $\varphi'(\ell,\cdot)$ of order $\mu(\lambda_0)$.*

For the corresponding leading coefficients we have

$$\operatorname{sign}\ c(m,\lambda_0)=\operatorname{sign}\ c(t,\lambda_0)=-s(\lambda_0),\qquad \operatorname{sign}\ c(\theta_\omega,\lambda_0)=s(\lambda_0),$$

$$\operatorname{sign}c(\varphi'(\ell,\cdot),\lambda_0)=-s(\lambda_0)\operatorname{sign}\ \varphi(\ell,\lambda_0)=(-1)^{\omega+1}s(\lambda_0).$$

Proof. We only have to prove the last identity. But this follows immediately from the proof of Lemma 3.1 and the fact that $\operatorname{sign}\sin\theta(\ell,\lambda)=\operatorname{sign}\varphi(\ell,\lambda)$. □

Theorem 3.2 implies in particular that the signs of the leading coefficients of m, t, and θ_ω are all determined by the signature of λ_0.

4. Oscillation results

Now we assume that r is positive on a set of positive measure, and also negative on a set of positive measure. The latter assumption is mainly for Remark 4.8, although if it is not satisfied, then the results are well known. Recall that, by assumption, 0 is not an eigenvalue of A, and denote the eigenvalues of A by

$$\cdots<\lambda_{-2}<\lambda_{-1}<0<\lambda_1<\lambda_2<\cdots.$$

We write for short $\mu_n := \mu(\lambda_n)$, $\omega_n := \omega(\lambda_n)$, $s_n := s(\lambda_n)$, and we also introduce the numbers

$$\delta_n := (-1)^{\mu_n} s_n, \quad n = \pm 1, \pm 2, \ldots. \tag{4.1}$$

For any integer j, we denote by I_j the open interval

$$I_j := \left(\left(j - \tfrac{1}{2}\right)\pi, \left(j + \tfrac{1}{2}\right)\pi\right),$$

and for a function h which is defined and continuous in a neighborhood of $\lambda_0 \in \mathbb{R}$ the notation $h(\lambda_0-) \in I_j$ ($h(\lambda_0+) \in I_j$, respectively) means that there is an $\varepsilon > 0$ such that $h(\lambda) \in I_j$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ (for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, respectively).

Lemma 4.1. *If, for some $n \neq 0$ and j, $\theta(\ell, \lambda_n-) \in I_j$, then $\omega_n = j - \frac{1}{2}(1 + \delta_n)$.*

Proof. By (3.3) and Theorem 3.2, $\theta(\ell, \cdot) - \left(\omega_n + \frac{1}{2}\right)$ has a zero of order μ_n at λ_n, so there are two cases: either $\omega_n = j - 1$ or $\omega_n = j$.

In the first case, $\theta(\ell, \cdot)$ is decreasing on some interval $(\lambda_n - \varepsilon, \lambda_n)$, and so either $s_n = -1$ and μ_n is odd, or $s_n = 1$ and μ_n is even, and both possibilities lead to $\delta_n = 1$. Then $1 + \delta_n = 2$ as required. In the second case, $\theta(\ell, \cdot)$ is increasing on $(\lambda_n - \varepsilon, \lambda_n)$, and so either $s_n = 1$ and μ_n is odd, or $s_n = -1$ and μ_n is even. Both possibilities give $\delta_n = -1$, whence $1 + \delta_n = 0$ as required. □

Corollary 4.2. *If $n \neq 0, 1$ then $\omega_n = \omega_{n-1} + \frac{1}{2}(s_{n-1} - \delta_n)$.*

Proof. Since $\theta(\ell, \lambda_{n-1}) = (\omega_{n-1} + \frac{1}{2})\pi$,

$$\theta(\ell, \lambda_{n-1}+) \in I_k,$$

where $k = \omega_{n-1}$ if $s_{n-1} = -1$ and $k = \omega_{n-1} + 1$ if $s_{n-1} = 1$. This means that $k = \omega_{n-1} + \frac{1}{2}(s_{n-1} + 1)$. Since $\theta(\ell, \lambda) \neq \pi/2 \bmod \pi$ for $\lambda \in (\lambda_{n-1}, \lambda_n)$, we also have $\theta(\ell, \lambda_n-) \in I_k$, and the result follows from Lemma 4.1. □

Corollary 4.3. *If $n \neq 0, 1$ then $|\omega_n - \omega_{n-1}| = 0$ or 1.*

Similar results hold for $n = 1$ but with index $n - 1$ replaced by -1.

Recall that κ was defined as the number of negative eigenvalues of the problem (1.1), (1.2) with the weight function r replaced by the function $r = 1$, or as the maximal dimension of a subspace of $\mathscr{D}(A)$ on which form (2.1) is negative.

Lemma 4.4. *κ is the greatest integer below $\pi^{-1}\theta(\ell, 0) + \frac{1}{2}$. Moreover, for the number $\omega(0)$ of zeros of the function $\varphi(\cdot, 0)$ we have*

$$\omega(0) = \begin{cases} \kappa & \text{if } \theta(\ell, 0) > \kappa\pi, \\ \kappa - 1 & \text{otherwise.} \end{cases}$$

Proof. Let the Prüfer angle and eigenvalues corresponding to the case $r = 1$ of (1.1), (1.2) be denoted by θ^1 and λ_n^1, $n = 1, 2, \ldots$, respectively. We can assume that $\lambda_1^1 < \lambda_2^1 < \cdots$, and since (3.1) shows that $\theta^1(\ell,\lambda)$ increases with λ, we have $\theta^1(\ell,\lambda_n^1)=\left(n-\frac{1}{2}\right)\pi$. By the definition of κ, $\lambda_\kappa^1<0<\lambda_{\kappa+1}^1$, so in fact

$$\left(\kappa-\tfrac{1}{2}\right)\pi<\theta^1(\ell,0)<\left(\kappa+\tfrac{1}{2}\right)\pi. \tag{4.2}$$

Since $\theta^1(\ell,0)=\theta(\ell,0)$, the first contention is established.

Now $\omega(0)=n$ if $\theta(\ell,0)\in(n\pi,(n+1)\pi]$ and since $n=\kappa-1$ or κ are the only possibilities allowed by (4.2), the proof is complete. □

We turn now to the calculation of ω_n, the number of zeros of the eigenfunction $\varphi(\cdot,\lambda_n)$ at positive λ_n, in terms of the algebraic multiplicities μ_j and the signatures s_j of the eigenvalues λ_j, $1\leqslant j\leqslant n$. Define numbers $\hat{s}_n$ at λ_n by

$$\hat{s}_n:=\begin{cases} s_n & \text{if } \mu_n \text{ is odd,}\\ 0 & \text{otherwise,}\end{cases} \qquad n=\pm 1,\pm 2,\ldots.$$

Recalling δ_n from (4.1), we see that

$$s_n-\delta_n=2\hat{s}_n. \tag{4.3}$$

Theorem 4.5. *For positive integers n the oscillation count ω_n of the eigenfunction $\varphi(\cdot,\lambda_n)$ at the positive eigenvalue λ_n of A satisfies*

$$\omega_n=\kappa+\sum_{j=1}^{n-1}\hat{s}_j-\tfrac{1}{2}(1+\delta_n).$$

Proof. With θ^1 as in the proof of Lemma 4.4 we have $\theta^1(\ell,0)=\theta(\ell,0)\in I_\kappa$ by (4.2), so $\theta(\ell,\lambda_1-)\in I_\kappa$. Application of Corollary 4.2, Lemma 4.1, and (4.3) in turn leads to

$$\begin{aligned}\omega_n&=\omega_1+\sum_{j=1}^{n-1}\tfrac{1}{2}(s_j-\delta_{j+1})\\ &=\kappa-\tfrac{1}{2}(1+\delta_1)+\sum_{j=1}^{n-1}\tfrac{1}{2}(s_j-\delta_j)+\tfrac{1}{2}(\delta_1-\delta_n)\\ &=\kappa-\tfrac{1}{2}(1+\delta_n)+\sum_{j=1}^{n-1}\hat{s}_j. \qquad \square\end{aligned}$$

Remark 4.6. The number on the right-hand side of the formula in Theorem 4.5 is a nonnegative integer since it is an oscillation count. This can be seen also a priori, since the number of signatures -1 on the right-hand side and also the number of eigenvalues λ_n with $\mu_n>1$ is $\leqslant\kappa$—see (2.2).

Remark 4.7. The formula in Theorem 4.5 admits the following interpretation. The interval I_k containing $\theta(\ell,\lambda_n)$ can be obtained by first moving along the x-axis to estimate $\theta(\ell,0)$. This gives the κ term, and corresponds to Lemma 4.4. Then we move along the line $x=\ell$ from $\lambda=0$ to $\lambda=\lambda_n$.

This involves changing k by $\hat{s}_j$ at each $\lambda_j, 1 \leqslant j < n$, and the final term $-\frac{1}{2}(1 + \delta_n)$ selects either the sup or the inf of I_k.

Remark 4.8. For the negative eigenvalues λ_{-n}, $n = 1, 2, \ldots$, the oscillation count ω_{-n} of the eigenfunction $\varphi(\cdot, \lambda_{-n})$ of A satisfies

$$\omega_{-n} = \kappa - \sum_{j=1}^{n-1} \hat{s}_{-j} - \frac{1}{2}(1 + s_{-n}), \qquad n = 1, 2, \ldots.$$

This can be proved by amending the argument for Theorem 4.5, or by making the substitution $r \mapsto -r$, $\lambda \mapsto -\lambda$ and applying Theorems 3.2 and 4.5.

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