Lax proper maps of locales

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Abstract

We give a proof of localic Priestley duality. Our approach is based on lax proper maps of locales, which provide a vehicle for presenting the Priestley version of full Stone duality constructively and preserve spatial intuitions.

MSC: 18B35; 18B25; 54C10

1. Introduction

The object of Townsend’s paper [5] is to give an entirely constructive version of a proof of Priestley’s duality:

\[ \text{OSToneSp} \cong \text{CohSp} \]

where \( \text{OSToneSp} \) is the category of ordered Stone spaces and \( \text{CohSp} \) are the coherent (or spectral) spaces.

Our work is motivated by Townsend’s remark in [5, Section 6]: “Clearly some spatial intuitions have been lost in this exposition in an attempt to prove the result as quickly as possible”.

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1. Introduction

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The aim of our work is to give an alternative proof of localic Priestley duality, so as to preserve spatial intuitions. Our approach is based on lax proper maps of locales. Lax proper maps between coherent locales are precisely the coherent maps, and provide an elegant vehicle for presenting the Priestley version [3] of full Stone duality [4] constructively.

We give a brief outline of the contents. First, we deal with some preliminaries. Then we introduce \( \text{lax proper maps of locales} \) in Section 3. A detailed account of the strict localic version of Priestley duality makes up Section 4, where we also extend to arbitrary sheaves over a coherent locale, the “representation” part of Priestley duality.
2. Notation and preliminaries

In this section, we establish terminology and recall various needed (mostly well-known) facts. Our basic reference is [1].

We shall find it convenient to deal with a locale \( X \) both in terms of its frame \( \mathcal{O}X \) of opens and its coframe \( \mathcal{C}X \) of “closeds”, hence with a map \( f: Y \rightarrow X \) of locales in terms of its inverse image \( f^- \) on either of these:

\[
\begin{array}{ccc}
Y & \mathcal{O}Y & \mathcal{C}Y \\
\downarrow f & \downarrow f^* & \downarrow f^* \\
X & \mathcal{O}X & \mathcal{C}X
\end{array}
\]

As usual, \( f^* \) denotes the right adjoint to \( f^- \) on opens, which becomes a left adjoint \( f^! \) to \( f^- \) on closeds.

For \( f, g: Y \rightarrow X \), put \( f \leq g \iff g^- \leq f^- \), i.e. \( g^- U \leq f^- U \) for all \( U \in \mathcal{O}X \).

A locale \( X \) becomes a topos if we extend our consideration of \( \mathcal{O}X \) to that of the total category of “generalized opens” or sheaves on \( X \); it is constructed in the standard way as the category of contravariant functors \( \mathcal{O}X \rightarrow \text{Sets} \) satisfying a patching condition on open covers. Maps \( f: Y \rightarrow X \) of locales correspond to geometric morphisms \( EY \rightarrow EX \):

\[
\begin{array}{ccc}
Y & \mathcal{O}Y & \mathcal{C}Y \\
\downarrow f & \downarrow f^* & \downarrow f^* \\
X & \mathcal{O}X & \mathcal{C}X
\end{array}
\]

The 2-categories \( \text{Loc} \) and \( \text{Top} \) (where \( \text{Loc} \) denotes the category of locales and \( \text{Top} \) the category of toposes) have indexed bilimits and -colimits; the former are preserved by \( \text{Loc} \rightarrow \text{Top} \).

Lax exact sequences

In a category \( C \) enriched over \( \text{Pos} \), the category of partially ordered sets, it makes sense to investigate the existence of lax, i.e. \( \text{Pos} \)-enriched limits and colimits. In particular:

The **lax pullback** or comma square of two morphisms \( f: P \rightarrow B \) and \( g: Q \rightarrow B \) in \( \text{Loc} \) is a universal square

\[
\begin{array}{ccc}
P & \rightarrow & P \\
\downarrow q & \downarrow f & \downarrow B \\
Q & \rightarrow & Q
\end{array}
\]

with \( gq \leq fp \).

The **co-inserter** of a pair of morphisms \( P \xrightarrow{f} B \) is a morphism \( q: B \rightarrow Q \) which universally forces \( qf \leq qg \).

Consider a diagram of locales

\[
\begin{array}{ccc}
\iff & \iff & \iff \\
\downarrow i & \downarrow q & \downarrow C \\
\iff & \iff & \iff \\
\downarrow b & \downarrow \iff & \downarrow C \\
\iff & \iff & \iff \\
X & \xrightarrow{q} & C
\end{array}
\]

where \( (b, t): \iff \rightarrow X \times X \) is a preorder on \( X \), \( q \) surjective and \( qb \leq qt \).

We say (1) is **lax exact on the left** (resp. right) if \( (b, t) \) is the lax kernel pair of \( q \) (resp. \( q \) universally forces \( qb \leq qt \)); (1) is **lax exact** if it is so both on the left and on the right.

Note that lax exactness follows from that on the left (resp. right) if \( q \) is lax regular (resp. \( (b, t) \) is effective). The inclusion \( q^-: \mathcal{O}C \rightarrow \mathcal{O}X \) picks those \( U \) of \( X \) satisfying \( b^- U \leq t^- U \), i.e. which are upclosed in \( \iff \) (we also say \( \iff \) acts on \( U \)).

Similar to the “non-lax” case – see [7] – we have:
Lemma 2.1. (1) is lax exact on the right iff \( q^* q_* \) is the largest among \( j : \mathcal{O} X \to \mathcal{O} X \) satisfying
\[
j \leq id, \quad j \leq b_s t^* j. \quad \square
\]

3. Definition of lax properness

Proper maps

We briefly recall the definitions of \( C \)-lattices and proper maps of locales (see [7]).

A \( C \)-lattice is a partially ordered set \( P \) which has finite joins and meets of filtered (i.e. down-directed) subsets, satisfying the distributive law
\[
x \lor \bigwedge F = \bigwedge \{ x \lor f \mid f \in F \}
\]
for all \( x \in P \) and filtered \( F \subseteq P \).

\( C \)-lattice homomorphisms preserve finite joins and filtered meets. The category \( Cl \) of \( C \)-lattices is a complete and cocomplete, symmetric closed Pos-category in which coframes are the meet-semi-lattices. Limits of \( C \)-lattices are calculated in Pos.

A map \( f : Y \to X \) of locales is proper if

(i) \( f_* : \mathcal{O} Y \to \mathcal{O} X \) preserves directed joins;
(ii) \( f_* (f^* U \lor V) = U \lor f_* V \) for all \( U \in \mathcal{O} X, \ V \in \mathcal{O} Y \).

The second part states that \( f \) is a closed map and ensures that \( f_* \) preserves all directed joins internal to the topos \( \mathcal{E} X \), i.e. \( f \) is proper precisely when it is compact considered as a locale in \( \mathcal{E} X \). Apart from the obvious composition stability properties, proper maps are preserved under pullbacks, the coequalizer of an equivalence relation with proper projections is proper and stable, and all proper surjections arise in this way.

Lax proper maps

We say a map \( f : Y \to X \) is lax proper if it satisfies (i) but is not necessarily closed (equivalently, if \( f! : C Y \to C X \) preserves infima over down-directed sets).

Theorem 3.1. If \( f \) is lax proper, then in every comma square of the form

\[
\begin{array}{ccc}
C & \overset{k}{\longrightarrow} & Y \\
\downarrow h & & \downarrow f \\
Z & \overset{g}{\longrightarrow} & X
\end{array}
\]

the map \( h \) is proper. Moreover, the diagram

\[
\begin{array}{ccc}
CC & \overset{k}{\leftarrow} & CY \\
\downarrow h & & \downarrow f_! \\
CZ & \overset{g}{\leftarrow} & CX
\end{array}
\]

commutes.

Proof. First note that, since \( f \) is lax proper, \( f_! \) is a map of \( C \)-lattices and, by decomposing (2) as

\[
\begin{array}{ccc}
C & \overset{r}{\longrightarrow} & L & \overset{q}{\longrightarrow} & Y \\
\downarrow h & & \downarrow p & & \downarrow f \\
Z & \overset{g}{\longrightarrow} & X & \overset{f}{\longrightarrow} & X
\end{array}
\]
where the left hand square is a pullback and the right hand is a comma square, it is, by [7, Proposition 3.4], enough to show that \( p \) is proper and \( f_! = pq^- \). For the first note, the inverse image \( i^- : \mathcal{C}(X \times Y) \to \mathcal{CL} \) of closeds along the inclusion \( i = (p, q): L \hookrightarrow X \times Y \) satisfies

\[
i^- (C \times (f^- E \land D)) \leq i^- ((C \land E) \times D)
\]

(3)

and is universal amongst morphism from \( \mathcal{C}(X \times Y) \) in \( \mathbf{CL} \) with this property. Let \( \sigma : \mathcal{C}(X \times Y) \to CX \) be the unique morphism of \( \mathcal{C} \)-lattices which maps \( C \times D \) to \( C \land f_! D \); \( \sigma \) is well-defined, since \( (C, D) \mapsto C \land f_! D \) is a bimorphism of \( \mathcal{C} \)-lattices. It is easy to check that \( \sigma \) satisfies (3), and hence factors through \( i^- \), say \( \sigma = \phi i^- \); \( \mathcal{CL} \to CX \). Then

\[
\phi p^- C = \phi i^- (C \times Y) = \sigma (C \times Y) = C \land f_! Y \leq C
\]

and so \( \phi p^- \leq id \). Also,

\[
\begin{align*}
p^- \phi (p^- C \land q^- D) &= p^- \phi i^- (C \times D) \\
&= p^- (C \land f_! D) \\
&= p^- C \land p^- f_! D \\
&\geq p^- C \land q^- D.
\end{align*}
\]

Since elements of the form \( p^- C \land q^- D \) generate \( \mathcal{C}(X \times Y) \) as a \( \mathcal{C} \)-lattice, \( p^- \phi \geq id \). Hence, \( \phi = p_! \) and \( \phi = p_! \) is \( CX \)-linear (i.e. \( p_! \) is a map of \( \mathcal{C} \)-lattices and preserves the actions of \( CX \)), since it is so on generators:

\[
\begin{align*}
\phi (p^- C \land p^- E \land q^- D) &= \phi (p^- (C \land E) \land q^- D) \\
&= C \land E \land f_! D \\
&= \phi (p^- C \land q^- D) \land E.
\end{align*}
\]

We have shown that \( p \) is proper ([7, Lemma 3.3]); the identity \( pq^- = f_! \) is immediate from the definition of \( \phi \). \( \square \)

The following lemma can be proved in a similar way as in [7, Lemma 5.2].

Lemma 3.2. For a preorder

\[
\begin{array}{c}
\leq \\
\overset{i}{\rightarrow} \\
\overset{b}{\rightarrow} \\
\rightarrow
\end{array}
\]

with \( b \) proper, \( b_! t^- \) is a coclosure operator. \( \square \)

Lemma 3.3. Suppose in (1) that \( b \) is proper. Then (1) is lax exact on the right iff \( q^- q_! = b_! t^- \).

Proof. The fact that \( b_! t^- \) is a coclosure operator is easily seen to imply that it is the largest among \( j : \mathcal{OX} \to \mathcal{OX} \) satisfying \( j \leq id, j \leq b_! t^- j \); the statement thus follows from Lemma 2.1. \( \square \)

We are now able to state the following:

Theorem 3.4. Suppose that (1) is lax exact on the right, with \( b \) proper. Then \( q \) is lax proper. Conversely, any lax proper surjection arises in this way.

Proof. (1) lax exact on the right with \( b \) proper means \( q^- q_! = b_! t^- \), which implies that \( q_! \) preserves directed joins \( (q^- 1 \circ 1) \); \( b \) proper implies that \( b_! \) preserves directed joins; \( t^- \) preserves all joins, i.e. \( q \) is a lax proper. Conversely, suppose that \( q \) is a lax proper surjection. Let \( (b, t) \) be the lax-kernel pair of \( q \). Then \( (b, t) \) is a pre-order on \( X \) and, by Theorem 3.1, \( b \) is proper and \( q^- q_! = b_! t^- \). So, (1) is lax exact on the right, with \( b \) proper. \( \square \)

4. Priestley duality

Coherent locales

Recall from [1] that a locale \( X \) is said to be coherent or spectral if its compact opens are preserved under finite intersection and form a base; indeed, any base of compact opens closed under finite meets would do — closing under
finite joins in $OX$ will produce the distributive lattice $KX$ of all compact opens. In particular, a coherent locale is compact. We shall call any map of locales coherent if it preserves compact opens under inverse image. It is easy to verify that lax proper maps are coherent. For coherent locales, the converse holds, moreover, by the following lemma:

**Lemma 4.1.** Suppose that $f : Y \to X$ is coherent, with $X$ a coherent locale. Then $f$ is lax proper.

**Proof.** Given an updirected collection $\mathcal{D}$ of opens of $Y$, it will, by the coherence of $X$, be enough if we can show, for a compact open $U$ in $X$, that $U \leq f_* \bigvee D \Rightarrow U \leq f_* D$, for some $D \in \mathcal{D}$. But using the adjunction $f^* \dashv f_*$, this is immediate once we know that $f^* U$ is compact. □

We denote the category of coherent locales and coherent $\equiv$ lax proper maps between them by $\text{Coh}$. It is not hard to see that coherence is inherited by both compact open and closed sublocales, and such that the insertion maps are coherent.

**The duality with distributive lattices**

The category $\text{Coh}$ is just a faithful picture in the category of locales of the dual of the category $\text{DLat}$ of distributive lattices ([1]); consider the following diagram of categories, with the obvious free and forgetful functors:

![Diagram](image)

The (algebraic) lifting $\text{DLat} \to \text{Frm}$ of the free functor $\text{Set} \to \text{Frm}$ is faithful — it is in fact monadic in a strong $\text{Pos}$ sense, which means in particular that the adjunction respects the order on maps. Explicitly, a distributive lattice $D$ is extended freely to a frame by embedding it as principal ideals ($\equiv$ compact elements) into its lattice $\mathcal{I} D$ of ideals. The functor $\mathcal{I}$ preserves all colimits and lax colimits; in addition, it preserves cotensoring with a finite partially ordered set $P$, that is, for any distributive lattice $D$, $\mathcal{I}(D^P) \cong (\mathcal{I}D)^P$ ($D^P \equiv$ order-preserving maps $P \to D$).

It is useful to translate the properties of the free frame on one generator into localic terms. The corresponding locale — the Sierpinski locale $\mathcal{S}$ — has an open point $t : 1 \hookrightarrow \mathcal{S}$ given by the generator; $t$ is the universal open sublocale, that is, there is a natural order-preserving correspondence between opens $U$ of a locale $X$ and “characteristic maps” $\chi_U : X \to \mathcal{S}$, induced by pulling back $t$. It follows that $\mathcal{S}$ classifies truth values over the (base) topos $\text{Set}$; in particular, $\text{pt} \mathcal{S} \equiv \Omega \cong \mathcal{P} 1$. We may now reinterpret (4) as

![Diagram](image)

where the functor $\text{Set}(\cdot, \Omega)^{op} : \text{Set}^{op} \to \text{Loc}$ assigns to a set its locale of (characteristic functions of) subsets and is adjoint on the right to $\text{DLat}(\cdot, \Omega)^{op} : \text{Loc} \to \text{DLat}$, which assigns to a distributive lattice its locale of (characteristic functions of) prime filters, that is, its spectrum.

The functor $\text{Spec}$ clearly factors through the category of coherent locales and the generic open insertion $t : 1 \hookrightarrow \mathcal{S}$ lies in $\text{Coh}$ ($\mathcal{K} \mathcal{S} \equiv \{0 \leq t \leq \mathcal{S}\}$). The picture (5) thus restricts, to become

![Diagram](image)
It is virtually immediate that the adjoint factors Spec and \( \mathcal{K} \) define a duality between the categories of distributive lattices and coherent locales.

The functor Spec: \( \text{DLat}^{\text{op}} \to \text{Loc} \) transforms colimits – both ordinary and lax – of distributive lattices into corresponding localic limits, which means that the subcategory of coherent locales are closed under these.

**Stone locales**

Stone duality is obtained by restricting to Boolean algebras on the distributive lattice side. The corresponding locales are said to be Stone; it is not hard to see that a locale is Stone precisely when it is compact and zero-dimensional, i.e. has a base of clopen sublocales. We denote the category of Stone locales by \( \text{St} \).

We then have that the following are equivalent for a locale \( X \):

1. \( X \) is Stone;
2. \( X \) is Hausdorff and coherent;
3. \( X \) is compact and totally separated.

**Proof.** (i) ⇒ (ii): A zero-dimensional locale is regular, hence Hausdorff.
(ii) ⇒ (iii): For sublocales of Hausdorff locales, compact ⇒ closed.
(iii) ⇒ (i): Suppose that \( X \) is compact and totally separated and let \( U \in \mathcal{O}X \) be given; we want to show that 
\[
U \leq \bigvee \{C \leq U \mid C \text{ clopen}\}.
\]
Then
\[
K \times U = K \times U \land -\Delta = \bigvee \{P \times Q \mid t(P, Q), P \in \mathcal{O}K, Q \in \mathcal{O}X\}
\]
and \( t \) satisfies the conditions of [6, Lemma 2.2]. We conclude that
\[
U \leq \bigvee \{Q \in \mathcal{O}X \mid t(K, Q)\} \leq \bigvee \{C \leq U \mid C \text{ clopen}\}.
\]

The category \( \text{St} \) of Stone locales is thus the intersection of those of coherent and compact Hausdorff; it is full in \( \text{Loc} \), since all maps between compact Hausdorff locales are proper.

**The constructible cover of a coherent locale**

The category of Boolean algebras is both a (full) reflective and coreflective subcategory of that of distributive lattices; the coreflection associates with a distributive lattice \( D \) its complemented elements, whereas the reflection embeds \( D \) in its Boolean hull \( B \) — the embedding \( D \hookrightarrow B \) is epimorphic in \( \text{DLat} \), since \( D \) and the complements of its elements together generate \( B \). In terms of Stone locales, the reflection becomes a coreflection which supplies every coherent locale \( C \) with a universal, monomorphic cover \( q: X \hookrightarrow C \) by a Stone locale \( X \). Using the fact that \( \text{Coh} \) is reflective in \( \text{Loc} \), the universal property of the lax proper covering map \( q \) can be extended to all locales: a map \( f: Y \to C \) lifts through \( q \) precisely when \( f^{-1}U \) is complemented in \( \mathcal{O}Y \) for all \( U \in \mathcal{K}X \).
Lax exact sequences of coherent locales

We refer to a preordered locale
\[ \sqsubseteq \xrightarrow{t} b \xrightarrow{q} C \]
as coherent if \( X, \sqsubseteq, b \) and \( t \) are coherent and, moreover, \( b \) is closed and the following interpolation property is satisfied:

For all \( U \in KX, V \in OX: b^{-}U \leq t^{-}V \)
\[ \implies \text{there exists upclosed } D \in KX \text{ such that } U \leq D \leq V. \] (7)

Lemma 4.2. Suppose that (1) is a lax exact sequence of coherent locales. Then the interpolation property (7) holds iff \( q^{-}q_{\ast} = b_{\ast}t^{-} \).

Proof. Suppose that \( q^{-}q_{\ast} = b_{\ast}t^{-} \) and let \( U \in KX, V \in OX \) such that
\[ b^{-}U \leq t^{-}V. \]
Then \( U \leq b_{\ast}t^{-}V = q^{-}q_{\ast}V \leq V. \) Now
\[ q_{\ast}V = \bigvee \{ D \in KC \mid q^{-}D \leq V \} \]
and so,
\[ U \leq q^{-} \left( \bigvee \{ D \in KC \mid q^{-}D \leq V \} \right) = \bigvee \{ q^{-}D \mid q^{-}D \leq V, \ D \text{ compact} \}. \]
But \( U \) is compact, hence \( U \leq q^{-}D \) for some \( D \). So, we have
\[ U \leq q^{-}D \leq V. \]

For the converse, we need to show that \( b_{\ast}t^{-}V \leq q^{-}q_{\ast}V \) for all \( V \in KX \). It suffices to show, for \( U \in KX \), that
\[ U \leq b_{\ast}t^{-}V \implies U \leq q^{-}q_{\ast}V. \]
Let \( U, V \in KX \) and suppose that \( U \leq b_{\ast}t^{-}V \), i.e. \( b^{-}U \leq t^{-}V \). By the interpolation property, there exists an upclosed \( D \in KX \) such that \( U \leq D \leq V \), i.e. \( U \leq q^{-}W \leq V \) for some \( W \in OC \). So, \( W \leq q_{\ast}V \), which implies that \( q^{-}W \leq q^{-}q_{\ast}V \). Hence \( U \leq q^{-}q_{\ast}V. \) \( \square \)

Theorem 4.3. In (1) suppose that \( X \) is coherent. Then the following are equivalent:

(i) (1) is a lax exact sequence of coherent locales;
(ii) \( \langle X, \sqsubseteq \rangle \) coherent and (1) is lax exact on the right;
(iii) \( C \) and \( q \) coherent and (1) is lax exact on the left.

Moreover, if this is the case, then \( q \) is mono iff \( \sqsubseteq \) is a partial order.

Proof. (i) \( \Rightarrow \) (ii): (1) a lax exact sequence implies that it is lax exact on the right and lax exact on the left, i.e.
\[ \begin{array}{ccc}
\sqsubseteq & \xrightarrow{t} & X \\
\downarrow{b} & \leq & \downarrow{q} \\
X & \xrightarrow{q} & C
\end{array} \]
is a comma square with \( q \) coherent \( \equiv \) lax proper. By Theorem 3.1, \( b \) is proper (and hence closed) and \( q^{-}q_{\ast} = b_{\ast}t^{-} \), which implies the interpolation property (Lemma 4.2).

(ii) \( \Rightarrow \) (iii): By Theorem 3.4, \( q \) is lax proper, and hence coherent. Now
\[ OC = \{ U \in OX : b^{-}U \leq t^{-}U \}. \]
C is coherent, since \( K_1, K_2 \in \mathcal{K}C \Rightarrow K_1 \wedge K_2 \in \mathcal{K}C \). Moreover, compact opens form a base: let \( P \in \mathcal{O}C \), then, since \( X \) is coherent, 
\[
P = \bigvee \{ V \leq P : V \in \mathcal{K}X \}.
\]
Consider \( V \leq P \), where \( V \in \mathcal{K}X \). Then 
\[
b^{-}V \leq b^{-}P \leq t^{-}P = \bigvee \{ t^{-}W \mid W \leq P, W \in \mathcal{K}X \}.
\]
But \( V \) compact implies that \( b^{-}V \leq t^{-}W \) for some \( W \in \mathcal{K}X \). By (7), there exists an upclosed \( D \in \mathcal{K}X \) such that 
\[
V \leq D \leq W \leq P.
\]
So, 
\[
P = \bigvee \{ D \leq P \mid D \text{ is upclosed}, D \in \mathcal{K}X \}.
\]
Further, we need to show that \( \sqsubseteq \xrightarrow{\, b \,} X \xrightarrow{\, q \,} C \) is lax exact on the left, i.e. \( \langle b, t \rangle \) is the lax kernel pair of \( q \). Suppose that \( \langle b', t' \rangle : \sqsubseteq' \xrightarrow{\, b \,} X \times X \) is the lax kernel pair of \( q \). Then, since \( q \) is lax proper, we have \( q^{-}q_{*} = b'_{*}t'^{-} \) (Theorem 3.1). But then, for \( D \in \mathcal{O}X \), 
\[
D \text{ upclosed in } \sqsubseteq \text{ iff } D \text{ upclosed in } \sqsubseteq'.
\]

So, the pre-orders \( \sqsubseteq \xrightarrow{\, b \,} X \) and \( \sqsubseteq' \xrightarrow{\, b \,} X \) are the same.

(iii) \( \Rightarrow \) (i): (1) lax exact on the left with \( q \) coherent \( \equiv \) lax proper (for coherent locales) implies \( b \) proper and \( q^{-}q_{*} = b_{*}t^{-} \) (Theorem 3.1). By Lemma 3.3, (1) is lax exact on the right and hence an exact sequence of coherent locales \( (X, q, C) \) coherent implies \( \sqsubseteq, b, t \) coherent.

Note that \( q \) mono iff \( q^{-}q_{*} = id : \mathcal{O}X \rightarrow \mathcal{O}X \). But, in this case, 
\[
q^{-}q_{*} = b_{*}t^{-}
\]
and equivalence with anti-symmetry is then readily verified. \( \square \)

**Totally order-separated locales**

Let \( \langle X, \sqsubseteq \rangle \) be an order-Hausdorff locale. We say that \( U, V \in \mathcal{O}X \) are **totally order-separated** in \( \langle X, \sqsubseteq \rangle \) if there exists a clopen \( C \) of \( X \) which is also upclosed in \( \sqsubseteq \), such that \( U \leq C \) and \( V \leq -C \). Call \( \langle X, \sqsubseteq \rangle \) **totally order-separated** if
\[
X \times X - \sqsubseteq = \bigvee \{ U \times V \mid U, V \text{ totally order-separated} \}.
\]

Note that totally order-separated implies totally separated.

**Lemma 4.4.** Suppose that \( X \) is compact. Then the following are equivalent:

(i) \( \langle X, \sqsubseteq \rangle \) is totally order-separated;

(ii) for all \( U, V \in \mathcal{O}X \),
\[
b^{-}U \leq t^{-}V \Rightarrow U \leq \bigvee \{ C \leq V \mid C \text{ clopen and upclosed} \}.
\]

**Proof.** Suppose that (i) holds, and \( U, V \in \mathcal{O}X \) satisfy \( b^{-}U \leq t^{-}V \). Let \( K = X - V \). The relation 
\[
\langle t(P, Q) \iff \exists C \text{ clopen in } X \text{ such that } P \leq C \leq Q
\]
between opens \( P \) of \( X \) and \( Q \) of \( K \) is easily seen to be admissible. Also, \( U \times K \leq \sqsubseteq \), which gives \( U \times K = U \times K \wedge \sqsubseteq = \{ P \times Q \mid t(P, Q) \} \).

Since \( K \) is compact, [6, Lemma 2.2] applies, and we can conclude that \( U \leq \bigvee \{ C \in \mathcal{O}X \mid C \text{ closed, } C \leq V \} \).

The converse (ii) \( \Rightarrow \) (i) is immediate. \( \square \)

Call \( \langle X, \sqsubseteq \rangle \) a **Priestley locale** if \( X \) is compact and \( \langle b, t \rangle \) \( X \times X \) a closed partial order with \( \langle X, \sqsubseteq \rangle \) totally order-separated.
Proposition 4.5. \( \langle X, \sqsubseteq \rangle \) is a Priestley locale iff \( X \) is Stone and \( \sqsubseteq \xrightarrow{t} X \) can be embedded in a lax exact diagram \((1)\) with \( X \xrightarrow{q} C \) a monomorphism of coherent locales; conversely, any monomorphic cover by \( X \) of a coherent locale arises in this way.

Proof. If \( \langle X, \sqsubseteq \rangle \) is Priestley, then \( X \) is compact and totally separated, hence Stone. Let \( (q, C) \) be the co-inserter of \( \langle b, t \rangle : \sqsubseteq \xhookrightarrow{t} X \times X \). Then \((1)\): \( \sqsubseteq \xrightarrow{t} X \xrightarrow{q} C \) is lax right exact with \( C \) and \( q \) coherent. Since \( \langle X, \sqsubseteq \rangle \) is totally order-separated, \( \sqsubseteq \) is effective, i.e. \( \langle b, t \rangle \) is the lax kernel pair of \( q \), and so \((1)\) is lax left exact. Hence \((1)\) is a lax exact diagram of coherent locales and, by Theorem 4.3, \( q \) is mono as \( \sqsubseteq \) is a partial order. Conversely, \( X \) Stone implies that \( X \) is compact and \( q \) mono implies that \( \sqsubseteq \) is a partial order. \( \langle X, \sqsubseteq \rangle \) is totally order-separated by Lemma 4.4: let \( U, V \in \mathcal{O}X, b^{-}U \leq t^{-}V \). For any \( W \leq U, W \in \mathcal{K}X \), we have \( b^{-}W \leq b^{-}U \leq t^{-}V \) and so, by the interpolation property \((7)\), there exists an upclosed clopen \( C \) such that \( W \leq C \leq V \); since \( U \) is covered by such \( W \) (by coherence of \( X \)), the result follows. So, \( \langle X, \sqsubseteq \rangle \) is a Priestley locale.

The converse is readily verified. \( \square \)

Every coherent locale \( C \) has a universal monomorphic cover, the patch cover, \( q : X \rightarrow C \) by a Stone locale \( X \); via Proposition 4.5, this produces an equivalence

\[ \langle X, \sqsubseteq \rangle \leftrightarrow C \]

of the category of Priestley locales and order preserving maps with the category \textbf{Coh}.

We now extend to arbitrary sheaves, over a coherent locale, the “representation” part of Priestley duality, viz.

The opens of a coherent locale \( C \) can be recovered as the opens of a partially order Stone locale \( \langle X, \sqsubseteq \rangle \) on which the order \( \sqsubseteq \) acts.

The full Beck–Chevalley condition for sheaves holds for comma squares of coherent toposes, as was proved in [2, Theorem 2]:

For a lax pullback

\[ L \xrightarrow{k} Y \]
\[ h \downarrow \quad \leq \quad \downarrow f \]
\[ Z \xrightarrow{r} X \]

of coherent locales, the diagram (of toposes of sheaves and functors)

\[ \mathcal{E}L \xrightarrow{k^*} \mathcal{E}Y \]
\[ h_* \downarrow \quad f_* \downarrow \]
\[ \mathcal{E}Z \xrightarrow{g^*} \mathcal{E}X \]

commutes up to (canonical) isomorphism. \( \square \)

Choosing for \( f \) in \((8)\) the patch cover \( q : X \rightarrow C \) of the coherent locale \( C \), we obtain:

Theorem 4.6. The sheaves (étale maps) over a coherent locale \( C \) can be recovered as the sheaves over a partially ordered Stone locale \( \langle X, \sqsubseteq \rangle \) (namely, the Priestley locale associated with \( C \)) on which the order acts. \( \square \)

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