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A K_0-avoiding dimension group with an order-unit of index two

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Abstract

We prove that there exists a dimension group G whose positive cone is not isomorphic to the dimension monoid $\operatorname{Dim} L$ of any lattice L. The dimension group G has an order-unit, and can be taken of any cardinality greater than or equal to $\aleph_2$. As to determining the positive cones of dimension groups in the range of the Dim functor, the $\aleph_2$ bound is optimal. This solves negatively the problem, raised by the author in 1998, whether any conical refinement monoid is isomorphic to the dimension monoid of some lattice. Since G has an order-unit of index 2, this also solves negatively a problem raised in 1994 by K.R. Goodearl about representability, with respect to K_0, of dimension groups with order-unit of index 2 by unit-regular rings.

Keywords: Lattice; Monoid; Dimension monoid; Dimension group; Index; V-homomorphism; Modular lattice; Von Neumann regular ring; Locally matricial

Introduction

The nonstable K-theory of a ring R studies the category of finitely generated projective right R-modules. The lattice-theoretical analogue of nonstable K-theory is encoded by the

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dimension monoid functor. The dimension monoid of a lattice L (see [16]) is the commutative monoid defined by generators $\Delta(x, y)$, for $x \leqslant y$ in L, and relations

(D0) $\Delta(x, x) = 0$, for all $x \in L$;
(D1) $\Delta(x, z) = \Delta(x, y) + \Delta(y, z)$, for all $x \leqslant y \leqslant z$ in L;
(D2) $\Delta(x \wedge y, x) = \Delta(y, x \vee y)$, for all $x, y \in L$.

The dimension monoid $\operatorname{Dim} L$ is a precursor of the semilattice $\operatorname{Con}_c L$ of compact congruences of L, in the sense that $\operatorname{Con}_c L$ is isomorphic to the maximal semilattice quotient of $\operatorname{Dim} L$, see [16, Corollary 2.3]. Furthermore, although it is still an open problem whether $\operatorname{Dim} L$ is a refinement monoid (see Section 1 for a definition) for every lattice L (see [16, Problem 3]), the answer is known for a few large classes of lattices, namely, the class of all modular lattices [16, Theorem 5.4] and the class of all lattices without infinite bounded chains (see Theorem 6.18 and Corollary 7.8 in [16]).

The question of a converse, namely whether every refinement monoid is isomorphic to the dimension monoid of some lattice, was raised by the author in [16, Problem 4]. This question is an analogue, for the Dim functor, of the Congruence Lattice Problem that asks whether every distributive $(\vee, 0)$-semilattice is isomorphic to $\operatorname{Con}_c L$, for some lattice L (see [14] for a survey). Partial positive answers were known. For example, it follows from [6, Theorem 1.5] and results in [16] (see the proof of Corollary 6.3) that for every dimension group G of cardinality at most $\aleph_1$, the positive cone G^+ is isomorphic to $\operatorname{Dim} L$ for some sectionally complemented, modular lattice L. For the cardinality $\aleph_2$ and above, the problem was still open. Different, though related, positive results about the dimension theory of complete modular lattices but also of self-injective modules or AW*-algebras, are established in [7]. In particular, the dimension monoids of complete, complemented, modular, upper continuous lattices are completely characterized.

Main theorem. *There exists a dimension group G with order-unit of index* 2 *such that for any lattice L, the positive cone G^+ of G is not the image of* $\operatorname{Dim} L$ *under any V-homomorphism. Furthermore, G may be taken of any cardinality greater than or equal to* $\aleph_2$.

(We refer to Section 1 for precise definitions.) In particular, G^+ is not isomorphic to $\operatorname{Dim} L$, for any lattice L. This solves [16, Problem 4]. Also, *G is not isomorphic to $K_0(R)$, for any unit-regular ring R* (see Corollary 6.3), which solves negatively the problem raised by K.R. Goodearl on the last page of [4]. A stronger and more precise statement of the main theorem is presented in Theorem 6.2.

The proof of our result is based on the proofs of earlier counterexamples, the first of this sort, due to the author in [15], being a dimension group with order-unit of cardinality $\aleph_2$ that is not isomorphic to $K_0(R)$, for any von Neumann regular ring R. Later counterexamples to related questions in lattice theory appeared in [12,13,17]. A common point of their proofs is that they all use the *Kuratowski Free Set Theorem*, in the form of Lemma 1.6. Also, they all express that certain distributive semilattices cannot be expressed as $\operatorname{Con}_c L$, for lattices L with *permutable congruences*.

By contrast, the proof of our main theorem does not require any assumption about permutable congruences on the lattice L. Also, unlike the construct of [15], our dimension group counterexample is not a rational vector space. This is also the case for the dimension groups considered in [4], in which the order-unit has *finite index*. However, in [4] are proven *positive* results, not from the viewpoint of the dimension theory of lattices but from the closely related (see Lemma 1.4) viewpoint of the nonstable K-theory of von Neumann regular rings. For example [4, Theorem 4.3], whenever G is an abelian lattice-ordered group with order-unit of finite index, there exists a biregular locally matricial algebra R such that $G \cong K_0(R)$; hence $G^+ \cong \operatorname{Dim} L$, where L is the lattice of all principal right ideals of R, see the proof of Corollary 6.3 (as R is unit-regular, it is sufficient to use R instead of $M_2(R)$).

1. Basic concepts

Every commutative monoid will be endowed with its *algebraic quasi-ordering*, defined by

$$x \leqslant y \quad \Leftrightarrow \quad (\exists z)(x + z = y).$$

We say that M is *conical*, if $x \leqslant 0$ implies that $x = 0$, for all $x \in M$. For commutative monoids M and N, a monoid homomorphism $\mu : M \to N$ is a *V-homomorphism*, if whenever $c \in M$ and $\bar{a}, \bar{b} \in N$ such that $\mu(c) = \bar{a} + \bar{b}$, there are $a, b \in M$ such that $c = a + b$, $\mu(a) = \bar{a}$, and $\mu(b) = \bar{b}$. An *o-ideal* of a commutative monoid M is a nonempty subset I of M such that $x + y \in I$ iff $x, y \in I$, for all $x, y \in M$. For an o-ideal I of a commutative monoid M, the least monoid congruence $\equiv_I$ that identifies all elements of I to zero is defined by

$$x \equiv_I y \quad \Leftrightarrow \quad (\exists u, v \in I)(x + u = y + v), \quad \text{for all } x, y \in M.$$

We denote by M/I the quotient monoid $M/{\equiv_I}$, and we denote by $[x]_I$ the $\equiv_I$-class of any $x \in M$. The proof of the following lemma is straightforward.

Lemma 1.1. *Let M and N be commutative monoids with N conical and let $\mu : M \to N$ be a monoid homomorphism. Then the subset $I = \{x \in M \mid \mu(x) = 0\}$ is an o-ideal of M, and there exists a unique monoid homomorphism $\overline{\mu} : M/I \to N$ such that $\overline{\mu}([x]_I) = \mu(x)$ for all $x \in M$. Furthermore, if μ is a V-homomorphism, then so is $\overline{\mu}$.*

A commutative monoid M is a *refinement monoid*, if $a_0 + a_1 = b_0 + b_1$ in M implies the existence of $c_{i,j} \in M$, for $i, j < 2$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$, for all $i < 2$. A $(\vee, 0)$-semilattice S is *distributive*, if it is a refinement monoid. Equivalently, the ideal lattice of S is distributive, see [8, Section II.5].

We use the notation, terminology, and results of [2] for partially ordered abelian groups. For partially ordered abelian groups G and H, a group homomorphism $f : G \to H$ is a *positive homomorphism*, if $f[G^+] \subseteq H^+$. For a partially ordered abelian group G and a positive integer n, we say that an element $e \in G^+$ has *index at most n*, if $(n + 1)x \leqslant e$

implies that $x = 0$, for all $x \in G^+$. We say that $e \in G^+$ is an *order-unit* of G, if for all $x \in G$, there exists a natural number n such that $x \leqslant ne$.

We say that a partially ordered abelian group G is

- an *interpolation group*, if for all $x, x', y, y' \in G$, if $x, x' \leqslant y, y'$, then there exists $z \in G$ such that $x, x' \leqslant z \leqslant y, y'$;
- *unperforated*, if $mx \geqslant 0$ implies that $x \geqslant 0$, for every $x \in G$ and every positive integer m;
- *directed*, if $G = G^+ + (-G^+)$;
- a *dimension group*, if G is a directed, unperforated interpolation group.

Particular cases of dimension groups are the *simplicial groups*, that is, the partially ordered abelian groups isomorphic to finite powers of the additive group $\mathbb{Z}$ of all integers, ordered componentwise. A theorem of Effros, Handelman, and Shen states that dimension groups are exactly the direct limits of simplicial groups, but we shall not need this result in the present paper.

A *pointed partially ordered abelian group* is a pair (G, e_G), where G is a partially ordered abelian group and $e_G \in G^+$. We shall call e_G the *distinguished element* of (G, e_G). For pointed partially ordered abelian groups (G, e_G) and (H, e_H), a positive homomorphism $f : G \to H$ is *normalized*, if $f(e_G) = e_H$. We shall write pointed partially ordered abelian groups either in the form (G, e_G) in case the distinguished element e_G needs to be specified, or simply G otherwise.

For any lattice L, the symbol $\Delta(\_, \_)$ is extended to any pair of elements of L, by defining $\Delta(x, y) = \Delta(x \wedge y, x \vee y)$, for all $x, y \in L$. The map Δ thus extended satisfies all the basic properties defining distances, see [16, Proposition 1.9].

Lemma 1.2. *The following statements hold, for all $x, y, z \in L$:*

(i) $\Delta(x, y) = 0$ *iff* $x = y$;
(ii) $\Delta(x, y) = \Delta(y, x)$;
(iii) $\Delta(x, z) \leqslant \Delta(x, y) + \Delta(y, z)$.

Of course, in (iii) above, the commutative monoid $\operatorname{Dim} L$ is endowed with its algebraic quasi-ordering.

The following result is an immediate consequence of [16, Lemma 4.11], applied to the partial semigroup of closed intervals of L endowed with projectivity as in [16, Section 5]. It concentrates most of the nontrivial information that we will need about the dimension monoid.

Lemma 1.3. *Let L be a modular lattice, let $u \leqslant v$ in L, and let $\boldsymbol{a}, \boldsymbol{b} \in \operatorname{Dim} L$. If $\boldsymbol{a} + \boldsymbol{b} = \Delta(u, v)$, then there are a positive integer n and a decomposition $u = w_0 \leqslant w_1 \leqslant \cdots \leqslant w_{2n} = v$ such that*

$$\boldsymbol{a} = \sum \big(\Delta(w_{2i}, w_{2i+1}) \mid i < n\big) \quad \textit{and} \quad \boldsymbol{b} = \sum \big(\Delta(w_{2i+1}, w_{2i+2}) \mid i < n\big).$$

For a unital ring R, we denote by $\mathrm{FP}(R)$ the category of all finitely generated projective right R-modules, and by $V(R)$ the monoid of all isomorphism classes of members of $\mathrm{FP}(R)$, see [5]. The monoid $V(R)$ encodes the so-called *nonstable K-theory* of R. If $[X]$ denotes the isomorphism class of a member X of $\mathrm{FP}(R)$, then the addition of $V(R)$ is defined by $[X] + [Y] = [X \oplus Y]$, for all $X, Y \in \mathrm{FP}(R)$. The monoid $V(R)$ is, of course, always conical. In case R is von Neumann regular (that is, for all $x \in R$ there exists $y \in R$ such that $xyx = x$), $V(R)$ is a refinement monoid, see [3, Theorem 2.8].

It is well known that for a von Neumann regular ring R, the matrix ring $M_2(R)$ is von Neumann regular [3, Theorem 1.7]. Denote by $\mathcal{L}(R)$ the (complemented, modular) lattice of principal right ideals of R. The nonstable K-theory of von Neumann regular rings and the dimension theory of lattices are related by the following result, which is an immediate consequence of [16, Proposition 10.31].

Lemma 1.4. *Let R be a von Neumann regular ring, and put $L = \mathcal{L}(M_2(R))$. Then $V(R) \cong \operatorname{Dim} L$.*

An example due to G.M. Bergman, see [3, Example 4.26], shows that $\mathcal{L}(M_2(R))$ cannot be replaced by $\mathcal{L}(R)$ in the statement of Lemma 1.4.

For a set X and a natural number n, we denote by $[X]^n$ (respectively $[X]^{\leqslant n}$) the set of all subsets Y of X such that $|Y| = n$ (respectively, $|Y| \leqslant n$). Furthermore, we denote by $[X]^{<\omega}$ the set of all finite subsets of X. The set-theoretical core of the proof of the main theorem consists of the following two results.

Lemma 1.5. *Let X be a set of cardinality at least $\aleph_2$ and let $\Phi : X \to [X]^{<\omega}$. Then there exists a subset Y of X of cardinality $\aleph_2$ such that $\eta \notin \Phi(\xi)$, for all distinct $\xi, \eta \in Y$.*

Proof. This is a particular case of a result proved by D. Lázár [10]. See also [1, Corollary 44.2]. $\square$

Lemma 1.6. *Let X be a set of cardinality at least $\aleph_2$, let $\Psi : [X]^2 \to [X]^{<\omega}$. Then there are distinct $\alpha, \beta, \gamma \in X$ such that $\alpha \notin \Psi(\{\beta, \gamma\})$, $\beta \notin \Psi(\{\alpha, \gamma\})$, and $\gamma \notin \Psi(\{\alpha, \beta\})$.*

Proof. This is a particular case of a result proved by C. Kuratowski [9]. See also [1, Theorem 46.1]. $\square$

We denote by $\mathbb{Z}^{(X)}$ the additive group of all maps $f : X \to \mathbb{Z}$ such that the *support* of f, namely $\{x \in X \mid f(x) \neq 0\}$, is finite. A subset X in a partially ordered set P is *cofinal*, if every element of P lies below some element of X. We identify n with the set $\{0, 1, \dots, n-1\}$, for every natural number n.

2. The functor I on partially ordered abelian groups

We shall denote by $\mathcal{L} = (-, 0, \mathbf{e}, \leqslant, \bowtie)$ the first-order signature consisting of one binary operation $-$ (interpreted as the 'difference'), one binary relation $\leqslant$, two constants 0 and $\mathbf{e}$, and one 4-ary operation $\bowtie$. Let $\mathcal{D}$ denote the class of models of the following axiom system (Σ), written in $\mathcal{L}$:

$$(\Sigma):\quad \begin{cases} \text{(POAG)} & \text{All axioms of partially ordered abelian groups in } (-, 0, \leqslant).\\ \text{(POINT)} & 0 \leqslant \mathbf{e}.\\ \text{(UNPERF)} & \text{Unperforation.}\\ \text{(INDEX)} & (\forall x)(0 \leqslant 3x \leqslant \mathbf{e} \Rightarrow x = 0).\\ \text{(INTERP)} & (\forall x, x', y, y')(x, x' \leqslant y, y' \Rightarrow x, x' \leqslant \bowtie(x, x', y, y') \leqslant y, y').\\ \text{(SYMM)} & (\forall x, x', y, y')(\bowtie(x, x', y, y') = \bowtie(x', x, y, y') = \bowtie(x, x', y', y)). \end{cases}$$

As all axioms of (Σ) are universal Horn sentences, it follows from basic results of the algebraic theory of quasivarieties (see [11, Section V.11]) that every model G for a subsignature $\mathcal{L}'$ of $\mathcal{L}$ has a unique (up to isomorphism) $\mathcal{L}'$-homomorphism $j_G : G \to \mathbf{I}(G)$ which is universal among $\mathcal{L}'$-homomorphisms from G to some member of $\mathcal{D}$. This means that $\mathbf{I}(G)$ is a member of $\mathcal{D}$, and for every $\mathcal{L}'$-homomorphism $f : G \to H$ with H a member of $\mathcal{D}$, there exists a unique $\mathcal{L}$-homomorphism $h : \mathbf{I}(G) \to H$ such that $f = h \circ j_G$.

Applying the universality to the $\mathcal{L}$-substructure of $\mathbf{I}(G)$ generated by the image of j_G yields immediately, in the particular case of pointed partially ordered abelian groups, the following lemma.

Lemma 2.1. *For any pointed partially ordered abelian group G, the structure $\mathbf{I}(G)$ is the closure, under the operations $(x, y) \mapsto x - y$ and $(x, x', y, y') \mapsto \bowtie(x, x', y, y')$, of the image of j_G.*

The operation $\bowtie$ on $\mathbf{I}(G)$ is a particular instance of the following notion.

Definition 2.2. An *interpolator* on a partially ordered abelian group G is a map $\imath : G^4 \to G$ that satisfies the axioms (INTERP) and (SYMM) of the axiom system (Σ). That is,

$$(\forall x, x', y, y' \in G)\big(x, x' \leqslant y, y' \Rightarrow x, x' \leqslant \imath(x, x', y, y') \leqslant y, y'\big),$$
$$(\forall x, x', y, y' \in G)\big(\imath(x, x', y, y') = \imath(x', x, y, y') = \imath(x, x', y', y)\big).$$

It is obvious that a partially ordered abelian group has an interpolator iff it is an interpolation group. We shall naturally view each member of $\mathcal{D}$ as an ordered pair $(G, \imath)$, where G is an unperforated partially ordered abelian group and $\imath$ is an interpolator on G.

For pointed partially ordered abelian groups, the meaning of $\mathbf{I}$ takes the following form: $\mathbf{I}(G)$ is a member of $\mathcal{D}$, the map j_G is a positive homomorphism from G to $\mathbf{I}(G)$, and for every $(H, \imath) \in \mathcal{D}$ and every normalized positive homomorphism $f : G \to H$, there exists a unique $\mathcal{L}$-homomorphism $h : (\mathbf{I}(G), \bowtie) \to (H, \imath)$ such that $f = h \circ j_G$. We shall denote this h by $f_{[\imath]}$, see the left-hand side diagram of Fig. 1. In case both G and H are partially ordered abelian groups and $f : G \to H$ is a normalized positive homomorphism, the map $\mathbf{I}(f) = (j_H \circ f)_{[\bowtie]}$ is the unique $\mathcal{L}$-homomorphism $h : \mathbf{I}(G) \to \mathbf{I}(H)$ such that $h \circ j_G = j_H \circ f$, see the middle diagram of Fig. 1.

Standard categorical arguments give the following two lemmas.

$$\begin{tikzcd} & (H,\iota) \\ G \arrow[ur, "f"] \arrow[r, "j_G"'] & (\mathbf{I}(G),\bowtie) \arrow[u, "f_{[\iota]}"'] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "j_H"] & \mathbf{I}(H) \\ G \arrow[u, "f"] \arrow[r, "j_G"'] & \mathbf{I}(G) \arrow[u, "\mathbf{I}(f)"'] \end{tikzcd} \qquad \begin{tikzcd} & & (G,\iota) \\ \mathbf{I}(E) \arrow[r, "\mathbf{I}(\varphi)"'] \arrow[urr, "(f\circ\varphi)_{[\iota]}"] & \mathbf{I}(F) \arrow[ur, "f_{[\iota]}"] & \\ E \arrow[u, "j_E"] \arrow[r, "\varphi"'] & F \arrow[u, "j_F"] \arrow[uur, "f"'] & \end{tikzcd}$$

Fig. 1. Illustrating $f_{[\iota]}$, $\mathbf{I}(f)$, and Lemma 2.4.

Lemma 2.3. *The correspondences $G \mapsto \mathbf{I}(G)$, $f \mapsto \mathbf{I}(f)$ define a functor from the category of pointed partially ordered abelian groups with normalized positive homomorphisms to the category $\mathcal{D}$ with $\mathcal{L}$-homomorphisms. This functor preserves direct limits.*

Lemma 2.4. *Let E, F, G be pointed partially ordered abelian groups, let $\varphi : E \to F$ and $f : F \to G$ be normalized positive homomorphisms, and let ι be an interpolator on G. Then $(f \circ \varphi)_{[\iota]} = f_{[\iota]} \circ \mathbf{I}(\varphi)$ (see the right-hand side diagram of Fig.* 1).

The following lemma expresses that $f_{[\iota]}$ is not 'too far' from f.

Lemma 2.5. *Let G and H be pointed partially ordered abelian groups, let $f : G \to H$ be a normalized positive homomorphism, and let ι be an interpolator on H. Then the image of $f_{[\iota]}$ is the least ι-closed subgroup of H containing the image of f.*

Proof. Denote by H' the least ι-closed subgroup of H containing $\operatorname{im} f$. The subset $G' = \{x \in \mathbf{I}(G) \mid f_{[\iota]}(x) \in H'\}$ is a subgroup of $\mathbf{I}(G)$, closed under the interpolator $\bowtie$ as H' is closed under ι and $f_{[\iota]}$ is a $\mathcal{L}$-homomorphism. Since G' contains $\operatorname{im} j_G$, it follows from Lemma 2.1 that $G' = \mathbf{I}(G)$. $\square$

The following lemma is even more specific to pointed partially ordered abelian groups.

Lemma 2.6. *Let G be a pointed partially ordered abelian group. Then the following statements hold.*

(i) $(\mathbf{I}(G), j_G(e_G))$ *is an unperforated pointed interpolation group with $j_G(e_G)$ of index at most* 2.
(ii) *The subset $j_G[G]$ is cofinal in* $\mathbf{I}(G)$.
(iii) *If G is directed, then $\mathbf{I}(G)$ is a dimension group.*
(iv) *If e_G is an order-unit of G, then $j_G(e_G)$ is an order-unit of* $\mathbf{I}(G)$.

Proof. (i) is trivial.

(ii) Denote by H the convex subgroup of $\mathbf{I}(G)$ generated by the image of j_G. Observe that H is closed under the canonical interpolator $\bowtie$ of $\mathbf{I}(G)$, so it is naturally equipped with a structure of model for $\mathcal{L}$. Denote by f the restriction of j_G from G to H, and by

$e' : H \hookrightarrow \mathbf{I}(G)$ the inclusion map. Denote by h the unique $\mathcal{L}$-homomorphism from $\mathbf{I}(G)$ to H such that $h \circ j_G = f$. From $e' \circ h \circ j_G = e' \circ f = j_G$ and the universal property of j_G, it follows that $e' \circ h = \mathrm{id}_{\mathbf{I}(G)}$, and so $h(x) = x$, for all $x \in \mathbf{I}(G)$. Therefore, $H = \mathbf{I}(G)$.

(iii) follows immediately from (i) and (ii), while (iv) follows immediately from (ii). □

3. The functors E and F

It follows from [11, Theorem V.11.2.4] that in any quasivariety, one can form the "object defined by a given set of generators and relations." The following definition uses this general construction in the case of pointed partially ordered abelian groups.

Definition 3.1. For a set X, we denote by $(\mathbf{E}(X), \boldsymbol{e}^X)$ the pointed partially ordered abelian group defined by generators $\boldsymbol{a}_\xi^X$, for $\xi \in X$, and relations $0 \leqslant \boldsymbol{a}_\xi^X \leqslant \boldsymbol{e}^X$, for $\xi \in X$. We put $\boldsymbol{b}_\xi^X = \boldsymbol{e}^X - \boldsymbol{a}_\xi^X$, for all $\xi \in X$.

For $Y \subseteq X$, there are unique positive homomorphisms $e_{Y,X} : \mathbf{E}(Y) \to \mathbf{E}(X)$ and $r_{X,Y} : \mathbf{E}(X) \twoheadrightarrow \mathbf{E}(Y)$ such that

$$e_{Y,X}\big(\boldsymbol{e}^Y\big) = \boldsymbol{e}^X, \qquad e_{Y,X}\big(\boldsymbol{a}_\eta^Y\big) = \boldsymbol{a}_\eta^X, \quad \text{for all } \eta \in Y, \tag{3.1}$$

$$r_{X,Y}\big(\boldsymbol{e}^X\big) = \boldsymbol{e}^Y, \qquad r_{X,Y}\big(\boldsymbol{a}_\xi^X\big) = \begin{cases} \boldsymbol{a}_\xi^Y, & \text{for all } \xi \in Y, \\ 0, & \text{for all } \xi \in X \setminus Y. \end{cases} \tag{3.2}$$

Hence $r_{X,Y} \circ e_{Y,X} = \mathrm{id}_{\mathbf{E}(Y)}$, and hence $\mathbf{E}(Y)$ is a *retract* of $\mathbf{E}(X)$. Therefore, we shall identify $\mathbf{E}(Y)$ with its image $e_{Y,X}[\mathbf{E}(Y)]$ in $\mathbf{E}(X)$, so that $e_{Y,X}$ becomes the inclusion map from $\mathbf{E}(Y)$ into $\mathbf{E}(X)$. Similarly, we shall from now on write $\boldsymbol{e}$ instead of $\boldsymbol{e}^X$, $\boldsymbol{a}_\xi$ instead of $\boldsymbol{a}_\xi^X$, and $\boldsymbol{b}_\xi$ instead of $\boldsymbol{b}_\xi^X$.

Definition 3.2. For sets X and Y and a map $f : X \to Y$, we denote by $\mathbf{E}(f)$ the unique positive homomorphism from $\mathbf{E}(X)$ to $\mathbf{E}(Y)$ such that $\mathbf{E}(f)(\boldsymbol{e}) = \boldsymbol{e}$ and $\mathbf{E}(f)(\boldsymbol{a}_\xi) = \boldsymbol{a}_{f(\xi)}$, for all $\xi \in X$.

The proof of the following lemma will introduce a useful explicit description of the pointed partially ordered abelian group $\mathbf{E}(X)$.

Lemma 3.3. *The correspondences $X \mapsto \mathbf{E}(X)$, $f \mapsto \mathbf{E}(f)$ define a functor from the category of sets to the category of all unperforated partially ordered abelian groups with order-unit. This functor preserves direct limits.*

Proof. All items are established by standard categorical arguments, except the statements about order-unit and, especially, unperforation, that require an explicit description of $\mathbf{E}(X)$. Denote by $\mathfrak{P}(X)$ the powerset of X, and by $\bar{e}$ the constant function on $\mathfrak{P}(X)$ with value 1. Furthermore, for all $\xi \in X$, we denote by $\bar{a}_\xi$ the characteristic function of $\{Y \in \mathfrak{P}(X) \mid \xi \in Y\}$. Finally, we let F_X be the additive subgroup of $\mathbb{Z}^{\mathfrak{P}(X)}$ generated by

$\{\bar{a}_\xi \mid \xi \in X\} \cup \{\bar{e}\}$, endowed with its componentwise ordering. The proof of the following claim is immediate.

Claim 1. *For all $m \in \mathbb{Z}$ and all $(n_\xi \mid \xi \in X) \in \mathbb{Z}^{(X)}$, $m\bar{e} + \sum(n_\xi \bar{a}_\xi \mid \xi \in X) \geqslant 0$ in F_X iff $m + \sum(n_\xi \mid \xi \in Y) \geqslant 0$ in $\mathbb{Z}$ for every $Y \in \mathfrak{P}(X)$.*

Claim 2. *There exists an isomorphism from $\mathbf{E}(X)$ onto F_X that sends $\boldsymbol{e}$ to $\bar{e}$ and each $\boldsymbol{a}_\xi$ to the corresponding $\bar{a}_\xi$.*

Proof. It suffices to verify that F_X satisfies the universal property defining $\mathbf{E}(X)$, that is, for every pointed partially ordered abelian group (G, e) with elements $a_\xi \in G$ such that $0 \leqslant a_\xi \leqslant e$, for $\xi \in X$, there exists a (necessarily unique) positive homomorphism from F_X to G that sends $\bar{e}$ to e and each $\bar{a}_\xi$ to the corresponding a_ξ. This, in turn, amounts to verifying the following statement:

$$m\bar{e} + \sum(n_\xi \bar{a}_\xi \mid \xi \in X) \geqslant 0 \quad \Rightarrow \quad me + \sum(n_\xi a_\xi \mid \xi \in X) \geqslant 0, \tag{3.3}$$

for all $m \in \mathbb{Z}$ and all $(n_\xi \mid \xi \in X) \in \mathbb{Z}^{(X)}$. As $(n_\xi \mid \xi \in X)$ has finite support, we may assume without loss of generality that X is finite. By Claim 1, the premise of (3.3) means that $m + \sum(n_\xi \mid \xi \in Y) \geqslant 0$ in $\mathbb{Z}$ for every $Y \in \mathfrak{P}(X)$. We shall conclude the proof by induction on $|X|$. For $|X| = 0$ it is immediate. For $X = \{\xi\}$, $m \geqslant 0$, and $m + n \geqslant 0$, we compute

$$me + na_\xi \geqslant me + (-m)a_\xi = m(e - a_\xi) \geqslant 0.$$

Now the induction step. Pick $\eta \in X$, and set $k = \max\{0, -n_\eta\}$. Hence

$$-n_\eta \leqslant k \leqslant m + \sum(n_\xi \mid \xi \in Y), \quad \text{for all } Y \subseteq X \setminus \{\eta\}. \tag{3.4}$$

Therefore, the element

$$me + \sum(n_\xi a_\xi \mid \xi \in X) = (ke + n_\eta a_\eta) + \Bigl((m - k)e + \sum \bigl(n_\xi a_\xi \mid \xi \in X \setminus \{\eta\}\bigr)\Bigr)$$

is, by the induction hypothesis, expressed as the sum of two elements of G^+, thus it belongs to G^+. □

It follows from Claim 2 that

$$m\boldsymbol{e} + \sum(n_\xi \boldsymbol{a}_\xi \mid \xi \in X) \geqslant 0 \quad \text{iff} \quad m + \sum(n_\xi \mid \xi \in Y) \geqslant 0 \quad \text{for all } Y \subseteq X, \tag{3.5}$$

for all $m \in \mathbb{Z}$ and all $(n_\xi \mid \xi \in X) \in \mathbb{Z}^{(X)}$. Both statements about unperforation and order-unit follow immediately. □

Notation 3.4. We put $\mathbf{F} = \mathbf{I} \circ \mathbf{E}$, the composition of the two functors $\mathbf{I}$ and $\mathbf{E}$.

By using Lemmas 2.3 and 2.6, we obtain that $\mathbf{F}$ is a direct limits preserving functor from the category of sets (with maps) to the category of dimension groups (with positive homomorphisms).

Lemma 3.5. *The canonical map $j_{\mathbf{E}(X)}:\mathbf{E}(X)\to\mathbf{F}(X)$ is an embedding, for every set X.*

Proof. We use the explicit description of $\mathbf{E}(X)$ given in the proof of Lemma 3.3. Denote by B_X the additive group of all bounded maps from $\mathfrak{P}(X)$ to $\mathbb{Z}$. Observe, in particular, that $\bar{e}$ has index 1 in B_X. Hence, $\mathbf{E}(X)\cong F_X$ embeds into the dimension group $(B_X,\bar{e})$ with order-unit of index at most 1. For any interpolator $\imath$ on B_X, the structure $(B_X,\imath)$ is a member of $\mathcal{D}$, in which $\mathbf{E}(X)$ embeds. $\square$

We shall always identify $\mathbf{E}(X)$ *with its image in* $\mathbf{F}(X)$, so that $j_{\mathbf{E}(X)}$ becomes the inclusion map from $\mathbf{E}(X)$ into $\mathbf{F}(X)$. Observe that despite what is suggested by the proof of Lemma 3.5, the element $\boldsymbol{e}$ does not, as a rule, have index 1 in $\mathbf{F}(X)$, but 2. The reason for this discrepancy is that for nonempty X, the canonical map $g:\mathbf{F}(X)\to B_X$ is not one-to-one, even on the positive cone of $\mathbf{F}(X)$. Indeed, picking $\xi\in X$ and putting $\boldsymbol{x}=\bowtie(0,0,\boldsymbol{a}_\xi,\boldsymbol{e}-\boldsymbol{a}_\xi)$, we get $\boldsymbol{x}\in\mathbf{F}(X)^+$. Furthermore, there exists a normalized positive homomorphism $h:(\mathbf{E}(X),\boldsymbol{e})\to(\mathbb{Z},2)$ such that $h(\boldsymbol{a}_\xi)=1$ and there exists an interpolator $\imath$ on $\mathbb{Z}$ such that $\imath(0,0,1,1)=1$, so $h_{[\imath]}(\boldsymbol{x})=\imath(0,0,h(\boldsymbol{a}_\xi),h(\boldsymbol{e}-\boldsymbol{a}_\xi))=\imath(0,0,1,1)=1$, and so $\boldsymbol{x}>0$. However, $2\boldsymbol{x}\leqslant\boldsymbol{a}_\xi+(\boldsymbol{e}-\boldsymbol{a}_\xi)=\boldsymbol{e}$, thus $2g(\boldsymbol{x})\leqslant g(\boldsymbol{e})=\bar{e}$, and so $g(\boldsymbol{x})=0$.

4. Supports and subgroups in $\mathbf{F}(X)$

Throughout this section we shall fix a set X. For all $Y\subseteq X$, we put $f_{Y,X}=\mathbf{I}(e_{Y,X})$, the canonical embedding from $\mathbf{F}(Y)$ into $\mathbf{F}(X)$. A *support* of an element $\boldsymbol{x}\in\mathbf{F}(X)$ is a subset Y of X such that $\boldsymbol{x}\in f_{Y,X}[\mathbf{F}(Y)]$. As the functor $\mathbf{F}$ preserves direct limits, every element of $\mathbf{F}(X)$ has a finite support.

Now put $s_{X,Y}=\mathbf{I}(r_{X,Y})$, $\bar{r}_{X,Y}=e_{Y,X}\circ r_{X,Y}$, and $\bar{s}_{X,Y}=\mathbf{I}(\bar{r}_{X,Y})$. Hence $\bar{r}_{X,Y}$ is an idempotent positive endomorphism of $\mathbf{E}(X)$, and it can be defined as in (3.2). Furthermore, $s_{X,Y}:\mathbf{F}(X)\to\mathbf{F}(Y)$ while $\bar{s}_{X,Y}$ is an idempotent positive endomorphism of $\mathbf{F}(X)$.

Lemma 4.1. *The following equations hold, for all $Y,Z\subseteq X$.*

(i) $f_{Y,X}\circ s_{X,Y}\circ f_{Y,X}=f_{Y,X}$.
(ii) $\bar{s}_{X,Y}\circ\bar{s}_{X,Z}=\bar{s}_{X,Y\cap Z}$.
(iii) $s_{X,Y}\circ f_{Z,X}=f_{Y\cap Z,Y}\circ s_{Z,Y\cap Z}$.

Proof. Apply the functor $\mathbf{I}$ to the following equations, whose verifications are immediate (actually, it is easy to infer the first two equations from the third one):

$$e_{Y,X}\circ r_{X,Y}\circ e_{Y,X}=e_{Y,X},$$
$$\bar{r}_{X,Y}\circ\bar{r}_{X,Z}=\bar{r}_{X,Y\cap Z},$$
$$r_{X,Y}\circ e_{Z,X}=e_{Y\cap Z,Y}\circ r_{Z,Y\cap Z}. \qquad\square$$

Lemma 4.2. *Let $\boldsymbol{x} \in \mathbf{F}(X)$ and let $Y \subseteq X$. Then Y is a support of $\boldsymbol{x}$ iff $\bar{s}_{X,Y}(\boldsymbol{x}) = \boldsymbol{x}$.*

Proof. Suppose first that $\bar{s}_{X,Y}(\boldsymbol{x}) = \boldsymbol{x}$, and put $\boldsymbol{y} = s_{X,Y}(\boldsymbol{x})$. Then $\boldsymbol{x} = \bar{s}_{X,Y}(\boldsymbol{x}) = f_{Y,X}(\boldsymbol{y})$ belongs to $f_{Y,X}[\mathbf{F}(Y)]$. Conversely, suppose that $\boldsymbol{x} = f_{Y,X}(\boldsymbol{y})$, for some $\boldsymbol{y} \in \mathbf{F}(Y)$. Then, using Lemma 4.1(i), we obtain

$$\bar{s}_{X,Y}(\boldsymbol{x}) = f_{Y,X} \circ s_{X,Y} \circ f_{Y,X}(\boldsymbol{y}) = f_{Y,X}(\boldsymbol{y}) = \boldsymbol{x}. \qquad \square$$

Corollary 4.3. *Every element of $\mathbf{F}(X)$ has a least support, which is a finite subset of X.*

Proof. Let Y and Z be supports of $\boldsymbol{x} \in \mathbf{F}(X)$. It follows from Lemmas 4.2 and 4.1(ii) that $\boldsymbol{x} = \bar{s}_{X,Y}(\boldsymbol{x}) = \bar{s}_{X,Z}(\boldsymbol{x})$, thus $\boldsymbol{x} = \bar{s}_{X,Y} \circ \bar{s}_{X,Z}(\boldsymbol{x}) = \bar{s}_{X,Y\cap Z}(\boldsymbol{x})$, and so, again by Lemma 4.2, $Y \cap Z$ is a support of $\boldsymbol{x}$. As $\boldsymbol{x}$ has a finite support, the conclusion follows. $\square$

We shall denote by $\operatorname{supp}(\boldsymbol{x})$ the least support of an element $\boldsymbol{x}$ of $\mathbf{F}(X)$.

Lemma 4.4. *Let $\boldsymbol{x} \in \mathbf{F}(X)$ and let $Y \subseteq X$. Then* $\operatorname{supp}(s_{X,Y}(\boldsymbol{x})) \subseteq \operatorname{supp}(\boldsymbol{x}) \cap Y$.

Proof. Put $Z = \operatorname{supp}(\boldsymbol{x})$. There is $z \in \mathbf{F}(Z)$ such that $\boldsymbol{x} = f_{Z,X}(z)$, thus, by Lemma 4.1(iii), $s_{X,Y}(\boldsymbol{x}) = s_{X,Y} \circ f_{Z,X}(z) = f_{Y\cap Z,Y} \circ s_{Z,Y\cap Z}(z)$, and so $s_{X,Y}(\boldsymbol{x})$ belongs to the image of $f_{Y\cap Z,Y}$. $\square$

Now we shall define certain additive subgroups G_Z^X of $\mathbf{F}(X)$, for $Z \in [X]^{\leqslant 2}$. First, we put $G_\emptyset^X = \mathbb{Z}\boldsymbol{e}$. Next, for any $\xi \in X$, we denote by $G_{\{\xi\}}^X$ the subgroup of $\mathbf{F}(X)$ generated by $\{\boldsymbol{a}_\xi, \boldsymbol{b}_\xi\}$. Finally, for all distinct $\xi, \eta \in X$, we put

$$\boldsymbol{c}_{\xi,\eta} = \bowtie(0, \boldsymbol{a}_\xi + \boldsymbol{a}_\eta - \boldsymbol{e}, \boldsymbol{a}_\xi, \boldsymbol{a}_\eta),$$

and we denote by $G_{\{\xi,\eta\}}^X$ the subgroup of $\mathbf{F}(X)$ generated by $\{\boldsymbol{a}_\xi, \boldsymbol{a}_\eta, \boldsymbol{b}_\xi, \boldsymbol{b}_\eta, \boldsymbol{c}_{\xi,\eta}\}$. As, by axiom (SYMM) (see Section 2), $\boldsymbol{c}_{\xi,\eta} = \boldsymbol{c}_{\eta,\xi}$, this definition is correct. For $\xi \in X$, we define a positive homomorphism $\varphi_\xi : \mathbb{Z}^2 \to G_{\{\xi\}}^X$, and for $\xi \neq \eta$ in X, we define a positive homomorphism $\psi_{\xi,\eta} : \mathbb{Z}^4 \to G_{\{\xi,\eta\}}^X$, by the rules

$$\varphi_\xi(x_0, x_1) = x_0\boldsymbol{a}_\xi + x_1\boldsymbol{b}_\xi, \tag{4.1}$$

$$\begin{aligned}\psi_{\xi,\eta}(x_0, x_1, x_2, x_3) = {} & x_0\boldsymbol{c}_{\xi,\eta} + x_1(\boldsymbol{a}_\xi - \boldsymbol{c}_{\xi,\eta}) + x_2(\boldsymbol{a}_\eta - \boldsymbol{c}_{\xi,\eta}) \\ & + x_3(\boldsymbol{c}_{\xi,\eta} + \boldsymbol{e} - \boldsymbol{a}_\xi - \boldsymbol{a}_\eta),\end{aligned} \tag{4.2}$$

for all $x_0, x_1, x_2, x_3 \in \mathbb{Z}$.

Lemma 4.5.

(i) *All maps φ_ξ, for $\xi \in X$, and $\psi_{\xi,\eta}$, for $\xi \neq \eta$ in X, are isomorphisms.*
(ii) *$G_Y^X \cap G_Z^X = G_{Y\cap Z}^X$, for all $Y, Z \in [X]^{\leqslant 2}$.*

Proof. By the definition of $\mathbf{E}(X)$, there exists a unique positive homomorphism $\tau_\xi : \mathbf{E}(X) \to \mathbb{Z}^2$ that sends $\boldsymbol{e}$ to $(1,1)$, $\boldsymbol{a}_\xi$ to $(1,0)$, and $\boldsymbol{a}_\zeta$ to $(0,0)$ for all $\zeta \in X \setminus \{\xi\}$. Fix any interpolator ι on $\mathbb{Z}^2$ and set $\pi_\xi = (\tau_\xi)_{[\iota]}$. Then $\pi_\xi \circ \varphi_\xi$ fixes both vectors $(1,0)$ and $(1,1)$, thus it is the identity. Therefore, φ_ξ is an embedding, and thus an isomorphism.

Now let $\xi \neq \eta$ in X. There exists a unique positive homomorphism $\sigma_{\xi,\eta} : \mathbf{E}(X) \to \mathbb{Z}^4$ such that

$$\sigma_{\xi,\eta}(\boldsymbol{a}_\xi) = (1,1,0,0), \qquad \sigma_{\xi,\eta}(\boldsymbol{a}_\eta) = (1,0,1,0),$$
$$\sigma_{\xi,\eta}(\boldsymbol{e}) = (1,1,1,1), \qquad \sigma_{\xi,\eta}(\boldsymbol{a}_\zeta) = (0,0,0,0), \quad \text{for all } \zeta \in X \setminus \{\xi,\eta\}.$$

Let ι be any interpolator on $\mathbb{Z}^4$ and set $\rho_{\xi,\eta} = (\sigma_{\xi,\eta})_{[\iota]}$. As

$$(0,0,0,0), (1,0,0,-1) \leqslant \rho_{\xi,\eta}(\boldsymbol{c}_{\xi,\eta}) \leqslant (1,1,0,0), (1,0,1,0),$$

the only possibility is $\rho_{\xi,\eta}(\boldsymbol{c}_{\xi,\eta}) = (1,0,0,0)$. It follows that $\rho_{\xi,\eta} \circ \psi_{\xi,\eta}$ fixes each of the vectors $(1,0,0,0)$, $(1,1,0,0)$, $(1,0,1,0)$, and $(1,1,1,1)$, whence it is the identity. In particular, $\psi_{\xi,\eta}$ is an embedding, but it is obviously surjective, thus it is an isomorphism.

Now let $\xi \neq \eta$ in X, and let $\boldsymbol{z} \in G^X_{\{\xi\}} \cap G^X_{\{\eta\}}$. There are $x, y, x', y' \in \mathbb{Z}$ such that

$$\boldsymbol{z} = x\boldsymbol{a}_\xi + y\boldsymbol{b}_\xi = x'\boldsymbol{a}_\eta + y'\boldsymbol{b}_\eta.$$

Applying $\rho_{\xi,\eta}$ yields $(x,x,y,y) = (x',y',x',y')$, whence $x = x' = y' = y$, and so $\boldsymbol{z} = x\boldsymbol{e} \in G^X_\emptyset$. Therefore, $G^X_{\{\xi\}} \cap G^X_{\{\eta\}} = G^X_\emptyset$.

Finally, let ξ, η, ζ be distinct elements of X, and let $\boldsymbol{z} \in G^X_{\{\xi,\eta\}} \cap G^X_{\{\xi,\zeta\}}$. There are $x_i, y_i \in \mathbb{Z}$, for $i < 4$, such that

$$\begin{aligned} \boldsymbol{z} &= x_0\boldsymbol{c}_{\xi,\eta} + x_1(\boldsymbol{a}_\xi - \boldsymbol{c}_{\xi,\eta}) + x_2(\boldsymbol{a}_\eta - \boldsymbol{c}_{\xi,\eta}) + x_3(\boldsymbol{c}_{\xi,\eta} + \boldsymbol{e} - \boldsymbol{a}_\xi - \boldsymbol{a}_\eta) \\ &= y_0\boldsymbol{c}_{\xi,\zeta} + y_1(\boldsymbol{a}_\xi - \boldsymbol{c}_{\xi,\zeta}) + y_2(\boldsymbol{a}_\zeta - \boldsymbol{c}_{\xi,\zeta}) + y_3(\boldsymbol{c}_{\xi,\zeta} + \boldsymbol{e} - \boldsymbol{a}_\xi - \boldsymbol{a}_\zeta). \end{aligned} \tag{4.3}$$

From $\rho_{\xi,\eta}(\boldsymbol{a}_\zeta) = (0,0,0,0)$ it follows that $\rho_{\xi,\eta}(\boldsymbol{c}_{\xi,\zeta}) = (0,0,0,0)$. Hence, applying $\rho_{\xi,\eta}$ to (4.3) yields that $(x_0,x_1,x_2,x_3) = (y_1,y_1,y_3,y_3)$, and thus $x_0 = x_1$, $x_2 = x_3$, whence $\boldsymbol{z} = x_0\boldsymbol{a}_\xi + x_2\boldsymbol{b}_\xi \in G^X_{\{\xi\}}$. All other instances of (ii) can be easily deduced from the two above. □

5. Smoothening interpolators on $\mathbf{F}(X)$

In the present section we shall also fix a set X.

Definition 5.1. An interpolator ι on $\mathbf{F}(X)$ is *smoothening of level* 2, if all subgroups G^X_Z (see Section 4), for $Z \in [X]^{\leqslant 2}$, are closed under ι.

Lemma 5.2. *There exists a smoothening interpolator of level* 2 *on* $\mathbf{F}(X)$.

Proof. For all $p = (x, x', y, y') \in \mathbf{F}(X)^4$, we put $\operatorname{rng} p = \{x, x', y, y'\}$. It follows from Lemma 4.5(ii) that the set

$$I(p) = \left\{Z \in [X]^{\leqslant 2} \mid \operatorname{rng} p \subseteq G_Z^X\right\}$$

is closed under intersection, hence it has a greatest lower bound Z_p in $(\mathfrak{P}(X), \subseteq)$, which belongs to $I(p)$ in case $I(p)$ is nonempty (otherwise $Z_p = X$). Put

$$H_p = G_{Z_p}^X,$$

where we define $G_X^X = \mathbf{F}(X)$. So H_p contains $\operatorname{rng} p$, and it follows from Lemma 4.5(i) that H_p is a dimension group. Now we consider the equivalence relation $\sim$ on $\mathbf{F}(X)^4$ generated by all pairs $(x, x', y, y') \sim (x', x, y, y')$ and $(x, x', y, y') \sim (x, x', y', y)$, for $x, x', y, y' \in \mathbf{F}(X)$, and we pick a subset C of $\mathbf{F}(X)^4$ such that for each $p \in \mathbf{F}(X)^4$ there exists a unique $\overline{p} \in C$ such that $p \sim \overline{p}$. For each $p \in \mathbf{F}(X)^4$, we put

$$\bar{\imath}(p) = \begin{cases} \text{any } z \in H_p \text{ such that } x, x' \leqslant z \leqslant y, y', & \text{if } x, x' \leqslant y, y', \\ 0, & \text{otherwise,} \end{cases}$$

and then we define $\imath(p) = \bar{\imath}(\overline{p})$, for all $p \in \mathbf{F}(X)^4$. Observe that if $\operatorname{rng} p \subseteq G_Z^X$, with $Z \in [X]^{\leqslant 2}$, then $Z_p \subseteq Z$, thus

$$H_p = G_{Z_p}^X \subseteq G_Z^X,$$

and thus $\imath(p) \in H_p \subseteq G_Z^X$. Hence, all G_Z^X, for $Z \in [X]^{\leqslant 2}$, are closed under $\imath$. Therefore, $\imath$ is a smoothening interpolator of level 2 on $\mathbf{F}(X)$. $\square$

Lemma 5.3. *Let $\imath$ be a smoothening interpolator of level* 2 *on* $\mathbf{F}(X)$. *Then for all $Z \in [X]^{\leqslant 2}$ and all $\boldsymbol{x} \in \mathbf{F}(X)$ with support Z, the element $(j_{\mathbf{E}(X)})_{[\imath]}(\boldsymbol{x})$ belongs to G_Z^X.*

Proof. Put $g = j_{\mathbf{E}(X)}$. By the definition of a support, there exists $\boldsymbol{z} \in \mathbf{F}(Z)$ such that $\boldsymbol{x} = f_{Z,X}(\boldsymbol{z})$. Therefore, by using Lemma 2.4,

$$g_{[\imath]}(\boldsymbol{x}) = g_{[\imath]} \circ f_{Z,X}(\boldsymbol{z}) = g_{[\imath]} \circ \mathbf{I}(e_{Z,X})(\boldsymbol{z}) = (g \circ e_{Z,X})_{[\imath]}(\boldsymbol{z}).$$

However, $\operatorname{im}(g \circ e_{Z,X}) = \mathbf{E}(Z) \subseteq G_Z^X$ and G_Z^X is closed under $\imath$, hence, by Lemma 2.5, the image of $(g \circ e_{Z,X})_{[\imath]}$ is contained in G_Z^X. In particular, using Lemma 2.4, we obtain that $g_{[\imath]}(\boldsymbol{x}) = (g \circ e_{Z,X})_{[\imath]}(\boldsymbol{z})$ belongs to G_Z^X. $\square$

6. Proof of the main theorem

Let $P(X, L, \mu)$ denote the following statement:

X is a set, L is a lattice, and $\mu : \operatorname{Dim} L \twoheadrightarrow \mathbf{F}(X)^+$ is a surjective V-homomorphism.

We say that μ is *zero-separating*, if $\mu^{-1}\{0\} = \{0\}$.

Lemma 6.1. *If $P(X, L, \mu)$ holds, then $P(X, L', \mu')$ holds for some modular lattice L' and some zero-separating μ'.*

Proof. It follows from Lemma 1.1 that $I = \{\boldsymbol{x} \in \operatorname{Dim} L \mid \mu(\boldsymbol{x}) = 0\}$ is an o-ideal of $\operatorname{Dim} L$ and the map $\overline{\mu} : (\operatorname{Dim} L)/I \twoheadrightarrow \mathbf{F}(X)^+$, $[\boldsymbol{x}]_I \mapsto \mu(\boldsymbol{x})$ is a V-homomorphism. However, it follows from Propositions 2.1 and 2.4 in [16] that $(\operatorname{Dim} L)/I \cong \operatorname{Dim}(L/\theta)$, where θ is the congruence of L defined by $x \equiv_\theta y$ iff $\Delta(x, y) \in I$, for all $x, y \in L$. Hence, replacing L by $L' = L/\theta$, it suffices to prove that if μ separates zero, then L is modular. If $\{o, a, b, c, i\}$ is a (possibly degenerate) pentagon of L, that is, $o \leqslant c \leqslant a \leqslant i$, $a \wedge b = o$, and $b \vee c = i$, then

$$\mu\Delta(o, c) = \mu\Delta(b, i) = \mu\Delta(o, a) = \mu\Delta(o, c) + \mu\Delta(c, a),$$

thus, since $\mathbf{F}(X)^+$ is cancellative, $\mu\Delta(c, a) = 0$. Therefore, since μ separates zero, $\Delta(c, a) = 0$, and hence $a = c$. This proves the modularity of L. □

Our main theorem is a consequence of the following more precise result.

Theorem 6.2. *Let X be a set, let L a lattice, and let $\mu : \operatorname{Dim} L \twoheadrightarrow \mathbf{F}(X)^+$ be a V-homomorphism with image containing $\boldsymbol{e}$. Then $|X| \leqslant \aleph_1$.*

Proof. Suppose, to the contrary, that $|X| \geqslant \aleph_2$. It follows from Lemma 6.1 that we may assume that L is modular and μ is zero-separating.

As μ is a monoid homomorphism and $\boldsymbol{e} \in \operatorname{im} \mu$, there are a natural number n and elements $u_i < v_i$ in L, for $i < n$, such that $\boldsymbol{e} = \sum(\mu\Delta(u_i, v_i) \mid i < n)$. For all $\xi \in X$, we obtain, by applying refinement in $\mathbf{F}(X)^+$ to the equation

$$\boldsymbol{a}_\xi + \boldsymbol{b}_\xi = \sum \bigl(\mu\Delta(u_i, v_i) \mid i < n\bigr),$$

decompositions of the form

$$\boldsymbol{a}_\xi = \sum(\boldsymbol{a}_{\xi,i} \mid i < n), \qquad \boldsymbol{b}_\xi = \sum(\boldsymbol{b}_{\xi,i} \mid i < n) \tag{6.1}$$

in $\mathbf{F}(X)^+$ such that

$$\boldsymbol{a}_{\xi,i} + \boldsymbol{b}_{\xi,i} = \mu\Delta(u_i, v_i), \quad \text{for all } i < n. \tag{6.2}$$

Since L is modular and μ is a V-homomorphism, we are entitled to apply Lemma 1.3 to the latter equation, and hence we obtain a positive integer $\ell_{\xi,i}$ and a finite chain in L of the form

$$u_i = x^0_{\xi,i} \leqslant x^1_{\xi,i} \leqslant \cdots \leqslant x^{2\ell_{\xi,i}}_{\xi,i} = v_i$$

such that

$$\boldsymbol{a}_{\xi,i} = \sum \bigl(\mu\Delta\bigl(x_{\xi,i}^{2j}, x_{\xi,i}^{2j+1}\bigr) \mid j < \ell_{\xi,i}\bigr), \tag{6.3}$$

$$\boldsymbol{b}_{\xi,i} = \sum \bigl(\mu\Delta\bigl(x_{\xi,i}^{2j+1}, x_{\xi,i}^{2j+2}\bigr) \mid j < \ell_{\xi,i}\bigr). \tag{6.4}$$

Now we define

$$\Phi(\xi) = \bigcup \bigl(\operatorname{supp} \mu\Delta\bigl(x_{\xi,i}^{j}, x_{\xi,i}^{j+1}\bigr) \mid i < n,\ j < 2\ell_{\xi,i}\bigr), \quad \text{for all } \xi \in X.$$

By applying Lázár's theorem (see Lemma 1.5), we obtain a subset X_1 of X of cardinality $\aleph_2$ such that

$$\eta \notin \Phi(\xi), \quad \text{for all distinct } \xi, \eta \in X_1. \tag{6.5}$$

By Lemma 5.2, there exists a smoothening interpolator ι of level 2 on $\mathbf{F}(X_1)$. Now we put

$$\pi = (j_{\mathbf{E}(X_1)})_{[\iota]} \circ s_{X,X_1}, \qquad \mu' = \pi \circ \mu, \tag{6.6}$$

$$\boldsymbol{a}'_{\xi,i} = \pi(\boldsymbol{a}_{\xi,i}), \quad \boldsymbol{b}'_{\xi,i} = \pi(\boldsymbol{b}_{\xi,i}), \quad \text{for all } \xi \in X_1 \text{ and all } i < n. \tag{6.7}$$

For all $\xi \in X$, $i < n$, and $j < 2\ell_{\xi,i}$, it follows from Lemma 4.4 that $\Phi(\xi) \cap X_1$ is a support of the element $s_{X,X_1}\mu\Delta(x_{\xi,i}^{j}, x_{\xi,i}^{j+1})$, hence, if $\xi \in X_1$ and by using (6.5), we obtain that $\{\xi\}$ is a support of $s_{X,X_1}\mu\Delta(x_{\xi,i}^{j}, x_{\xi,i}^{j+1})$. Therefore, by applying $(j_{\mathbf{E}(X_1)})_{[\iota]}$ and using Lemma 5.3, we obtain

$$\mu'\Delta\bigl(x_{\xi,i}^{j}, x_{\xi,i}^{j+1}\bigr) \in G_{\{\xi\}}^{X_1}, \quad \text{for all } \xi \in X_1,\ i < n, \text{ and } j < 2\ell_{\xi,i}. \tag{6.8}$$

By applying π to Eqs. (6.1)–(6.4) and observing that all elements of $\mathbf{E}(X_1)$ are fixed under π, we obtain the equations

$$\boldsymbol{a}_{\xi} = \sum \bigl(\boldsymbol{a}'_{\xi,i} \mid i < n\bigr) \quad \text{and} \quad \boldsymbol{b}_{\xi} = \sum \bigl(\boldsymbol{b}'_{\xi,i} \mid i < n\bigr), \quad \text{for all } \xi \in X_1, \tag{6.9}$$

$$\boldsymbol{a}'_{\xi,i} = \sum \bigl(\mu'\Delta\bigl(x_{\xi,i}^{2j}, x_{\xi,i}^{2j+1}\bigr) \mid j < \ell_{\xi,i}\bigr), \quad \text{for all } \xi \in X_1 \text{ and all } i < n, \tag{6.10}$$

$$\boldsymbol{b}'_{\xi,i} = \sum \bigl(\mu'\Delta\bigl(x_{\xi,i}^{2j+1}, x_{\xi,i}^{2j+2}\bigr) \mid j < \ell_{\xi,i}\bigr), \quad \text{for all } \xi \in X_1 \text{ and all } i < n. \tag{6.11}$$

Fix $\xi \in X_1$ and $i < n$. It follows from (6.8), (6.10), and (6.11) that both $\boldsymbol{a}'_{\xi,i}$ and $\boldsymbol{b}'_{\xi,i}$ belong to $G_{\{\xi\}}^{X_1}$. However, it follows from (6.9) that $0 \leqslant \boldsymbol{a}'_{\xi,i} \leqslant \boldsymbol{a}_{\xi}$. Since the isomorphism $\varphi_{\xi}^{-1} : G_{\{\xi\}}^{X_1} \to \mathbb{Z}^2$ (see (4.1)) carries $\boldsymbol{a}_{\xi}$ to $(1,0)$, it follows that

$$\boldsymbol{a}'_{\xi,i} \in \{0, \boldsymbol{a}_{\xi}\}. \tag{6.12}$$

It follows from (6.12), (6.8), and (6.10) that there exists $j < \ell_{\xi,i}$ such that

$$\mu'\Delta\bigl(x_{\xi,i}^{2j'}, x_{\xi,i}^{2j'+1}\bigr) = 0, \quad \text{for all } j' < \ell_{\xi,i} \text{ with } j' \neq j. \tag{6.13}$$

Similarly, $\boldsymbol{b}'_{\xi,i} \in \{0, \boldsymbol{b}_\xi\}$ and there exists $k < \ell_{\xi,i}$ such that

$$\mu' \Delta\bigl(x_{\xi,i}^{2k'+1}, x_{\xi,i}^{2k'+2}\bigr) = 0, \quad \text{for all } k' < \ell_{\xi,i} \text{ with } k' \neq k. \tag{6.14}$$

We define an element $z_{\xi,i} \in L$ as follows:

$$z_{\xi,i} = \begin{cases} x_{\xi,i}^{2j+1}, & \text{if } j \leqslant k, \\ x_{\xi,i}^{2k+2}, & \text{if } j > k. \end{cases}$$

It follows easily from (6.10), (6.11), (6.13), and (6.14) that the following statements hold:

$$\begin{aligned} \boldsymbol{a}'_{\xi,i} &= \mu' \Delta(u_i, z_{\xi,i}) \quad \text{and} \quad \boldsymbol{b}'_{\xi,i} = \mu' \Delta(z_{\xi,i}, v_i), \quad \text{if } j \leqslant k, \\ \boldsymbol{b}'_{\xi,i} &= \mu' \Delta(u_i, z_{\xi,i}) \quad \text{and} \quad \boldsymbol{a}'_{\xi,i} = \mu' \Delta(z_{\xi,i}, v_i), \quad \text{if } j > k. \end{aligned}$$

Let $A(\xi, i)$ hold, if $\boldsymbol{a}'_{\xi,i} = \mu' \Delta(u_i, z_{\xi,i})$ and $\boldsymbol{b}'_{\xi,i} = \mu' \Delta(z_{\xi,i}, v_i)$, and let $B(\xi, i)$ hold, if $\boldsymbol{b}'_{\xi,i} = \mu' \Delta(u_i, z_{\xi,i})$ and $\boldsymbol{a}'_{\xi,i} = \mu' \Delta(z_{\xi,i}, v_i)$. What will matter for us is that the following property is satisfied:

$$\text{Either } A(\xi, i) \text{ or } B(\xi, i) \text{ holds}, \quad \text{for all } \xi \in X_1 \text{ and all } i < n. \tag{6.15}$$

Now we denote by U the powerset of $n = \{0, 1, \ldots, n-1\}$, and we put

$$Y_u = \bigl\{\xi \in X_1 \mid (\forall i \in u) A(\xi, i) \text{ and } (\forall i \in n \setminus u) B(\xi, i)\bigr\}, \quad \text{for all } u \in U.$$

Claim 1. $X_1 = \bigcup(Y_u \mid u \in U)$.

Proof. Let $\xi \in X_1$, and put $u = \{i < n \mid A(\xi, i)\}$. It follows from (6.15) that $B(\xi, i)$ holds, for all $i \in n \setminus u$. Therefore, $\xi \in Y_u$. □

Now we put $\boldsymbol{d}_{\xi,\eta} = \sum(\mu' \Delta(z_{\xi,i}, z_{\eta,i}) \mid i < n)$, for all $\xi, \eta \in X_1$.

Claim 2. *The following inequalities hold*:

(i) $\boldsymbol{d}_{\xi,\zeta} \leqslant \boldsymbol{d}_{\xi,\eta} + \boldsymbol{d}_{\eta,\zeta}$, *for all* $\xi, \eta, \zeta \in X_1$;
(ii) $\boldsymbol{d}_{\xi,\eta} \leqslant \boldsymbol{a}_\xi + \boldsymbol{a}_\eta, \boldsymbol{b}_\xi + \boldsymbol{b}_\eta$, *for all* $u \in U$ *and all* $\xi, \eta \in Y_u$;
(iii) $\boldsymbol{e} \leqslant \boldsymbol{a}_\eta + \boldsymbol{b}_\xi + \boldsymbol{d}_{\xi,\eta}, \boldsymbol{a}_\xi + \boldsymbol{b}_\eta + \boldsymbol{d}_{\xi,\eta}$, *for all* $u \in U$ *and all* $\xi, \eta \in Y_u$.

Proof. Item (i) follows immediately from Lemma 1.2(iii).

Now let $u \in U$ and let $\xi, \eta \in Y_u$. Let $i < n$. If $i \in u$, then, by using again Lemma 1.2,

$$\begin{aligned} \mu' \Delta(z_{\xi,i}, z_{\eta,i}) &\leqslant \mu' \Delta(z_{\xi,i}, u_i) + \mu' \Delta(u_i, z_{\eta,i}) = \boldsymbol{a}'_{\xi,i} + \boldsymbol{a}'_{\eta,i}, \\ \mu' \Delta(u_i, v_i) &\leqslant \mu' \Delta(u_i, z_{\eta,i}) + \mu' \Delta(z_{\eta,i}, z_{\xi,i}) + \mu' \Delta(z_{\xi,i}, v_i) \\ &= \boldsymbol{a}'_{\eta,i} + \boldsymbol{b}'_{\xi,i} + \mu' \Delta(z_{\xi,i}, z_{\eta,i}), \end{aligned}$$

while if $i \in n \setminus u$,

$$\begin{aligned}\mu'\Delta(z_{\xi,i}, z_{\eta,i}) &\leqslant \mu'\Delta(z_{\xi,i}, v_i) + \mu'\Delta(v_i, z_{\eta,i}) = \boldsymbol{a}'_{\xi,i} + \boldsymbol{a}'_{\eta,i},\\ \mu'\Delta(u_i, v_i) &\leqslant \mu'\Delta(u_i, z_{\xi,i}) + \mu'\Delta(z_{\xi,i}, z_{\eta,i}) + \mu'\Delta(z_{\eta,i}, v_i)\\ &= \boldsymbol{b}'_{\xi,i} + \boldsymbol{a}'_{\eta,i} + \mu'\Delta(z_{\xi,i}, z_{\eta,i}),\end{aligned}$$

so that in any case,

$$\mu'\Delta(z_{\xi,i}, z_{\eta,i}) \leqslant \boldsymbol{a}'_{\xi,i} + \boldsymbol{a}'_{\eta,i}, \tag{6.16}$$

$$\mu'\Delta(u_i, v_i) \leqslant \boldsymbol{a}'_{\eta,i} + \boldsymbol{b}'_{\xi,i} + \mu'\Delta(z_{\xi,i}, z_{\eta,i}). \tag{6.17}$$

Symmetrically, we can obtain

$$\mu'\Delta(z_{\xi,i}, z_{\eta,i}) \leqslant \boldsymbol{b}'_{\xi,i} + \boldsymbol{b}'_{\eta,i}, \tag{6.18}$$

$$\mu'\Delta(u_i, v_i) \leqslant \boldsymbol{a}'_{\xi,i} + \boldsymbol{b}'_{\eta,i} + \mu'\Delta(z_{\xi,i}, z_{\eta,i}), \tag{6.19}$$

Adding together all inequalities (6.16)–(6.19), for $i < n$, establishes both (ii) and (iii). □

By Claim 1, there exists $u \in U$ such that $|Y_u| = \aleph_2$. For the rest of the proof we fix such a subset u. We define $\Psi(\{\xi, \eta\}) = \operatorname{supp} \boldsymbol{d}_{\xi,\eta}$, for all distinct $\xi, \eta \in Y_u$. Applying Kuratowski's theorem (see Lemma 1.6) to the map Ψ, we obtain distinct elements $\alpha, \beta, \gamma \in Y_u$ such that $\alpha \notin \Psi(\{\beta, \gamma\})$, $\beta \notin \Psi(\{\alpha, \gamma\})$, and $\gamma \notin \Psi(\{\alpha, \beta\})$.

Put $X_2 = \{\alpha, \beta, \gamma\}$. It follows from Lemma 5.2 that there exists a smoothening interpolator $\jmath$ of level 2 on $\mathbf{F}(X_2)$. Put $\pi' = (\jmath_{\mathbf{E}(X_2)})_{[\jmath]} \circ s_{X_1,X_2}$, a positive homomorphism from $\mathbf{F}(X_1)$ to $\mathbf{F}(X_2)$. For all distinct $\xi, \eta \in Y_u$, it follows from Lemma 4.4 that $\Psi(\{\xi, \eta\}) \cap X_2$ is a support of the element $s_{X_1,X_2}(\boldsymbol{d}_{\xi,\eta})$. Hence, we obtain that the pair $\{\xi, \eta\}$ is a support of $s_{X_1,X_2}(\boldsymbol{d}_{\xi,\eta})$, for all distinct $\xi, \eta \in X_2$. Therefore, putting $\boldsymbol{d}'_{\xi,\eta} = \pi'(\boldsymbol{d}_{\xi,\eta})$, applying $(\jmath_{\mathbf{E}(X_1)})_{[\jmath]}$, and using Lemma 5.3, we obtain that

$$\boldsymbol{d}'_{\xi,\eta} \in G^{X_2}_{\{\xi,\eta\}}, \quad \text{for all distinct } \xi, \eta \in X_2. \tag{6.20}$$

Applying the positive homomorphism π' to the inequalities in Claim 2, we obtain the following new inequalities, for all distinct $\xi, \eta, \zeta \in X_2$:

(i) $\boldsymbol{d}'_{\xi,\zeta} \leqslant \boldsymbol{d}'_{\xi,\eta} + \boldsymbol{d}'_{\eta,\zeta}$;
(ii) $\boldsymbol{d}'_{\xi,\eta} \leqslant \boldsymbol{a}_\xi + \boldsymbol{a}_\eta, \boldsymbol{b}_\xi + \boldsymbol{b}_\eta$;
(iii) $\boldsymbol{e} \leqslant \boldsymbol{a}_\eta + \boldsymbol{b}_\xi + \boldsymbol{d}'_{\xi,\eta}, \boldsymbol{a}_\xi + \boldsymbol{b}_\eta + \boldsymbol{d}'_{\xi,\eta}$.

By applying the isomorphism $\psi^{-1}_{\xi,\eta}$ (see (4.2)) to the inequalities (ii) and (iii) above, we obtain the inequalities

$$\begin{aligned}
&\psi_{\xi,\eta}^{-1}\big(\boldsymbol{d}'_{\xi,\eta}\big) \leqslant (1,1,0,0)+(1,0,1,0),\\
&\psi_{\xi,\eta}^{-1}\big(\boldsymbol{d}'_{\xi,\eta}\big) \leqslant (0,0,1,1)+(0,1,0,1),\\
&(1,1,1,1) \leqslant \psi_{\xi,\eta}^{-1}\big(\boldsymbol{d}'_{\xi,\eta}\big)+(1,0,1,0)+(0,0,1,1),\\
&(1,1,1,1) \leqslant \psi_{\xi,\eta}^{-1}\big(\boldsymbol{d}'_{\xi,\eta}\big)+(1,1,0,0)+(0,1,0,1),
\end{aligned}$$

which leaves the only possibility

$$\psi_{\xi,\eta}^{-1}\big(\boldsymbol{d}'_{\xi,\eta}\big)=(0,1,1,0),$$

that is,

$$\boldsymbol{d}'_{\xi,\eta}=\boldsymbol{a}_\xi+\boldsymbol{a}_\eta-2\boldsymbol{c}_{\xi,\eta}.$$

Therefore, applying the inequality (i) above with $(\xi,\eta,\zeta)=(\alpha,\beta,\gamma)$, we obtain

$$\boldsymbol{c}_{\alpha,\beta}+\boldsymbol{c}_{\beta,\gamma} \leqslant \boldsymbol{a}_\beta+\boldsymbol{c}_{\alpha,\gamma} \tag{6.21}$$

in $\mathbf{F}(X_2)$. However, we shall now prove that (6.21) does not hold. Indeed, the structure $(\mathbb{Z}^2,(2,1))$ is a dimension group with order-unit of index 2, thus it expands to some member $(\mathbb{Z}^2,(2,1),\iota)$ of $\mathcal{D}$ (where ι is an interpolator on $\mathbb{Z}^2$) such that

$$\iota\big((0,0),(0,-1),(1,0),(1,0)\big)=(0,0) \quad \text{and} \quad \iota\big((0,0),(0,0),(1,0),(1,1)\big)=(1,0).$$

Now there exists a unique normalized positive homomorphism $h:(\mathbf{E}(X_2),\boldsymbol{e})\to(\mathbb{Z}^2,(2,1))$ such that

$$h(\boldsymbol{a}_\alpha)=h(\boldsymbol{a}_\gamma)=(1,0), \quad \text{and} \quad h(\boldsymbol{a}_\beta)=(1,1).$$

By definition, $h(\boldsymbol{e})=(2,1)$, so we can compute

$$\begin{aligned}
h_{[\iota]}(\boldsymbol{c}_{\alpha,\gamma}) &= \iota\big((0,0),h(\boldsymbol{a}_\alpha+\boldsymbol{a}_\gamma-\boldsymbol{e}),h(\boldsymbol{a}_\alpha),h(\boldsymbol{a}_\gamma)\big)\\
&= \iota\big((0,0),(0,-1),(1,0),(1,0)\big)\\
&= (0,0),\\
h_{[\iota]}(\boldsymbol{c}_{\alpha,\beta}) &= \iota\big((0,0),h(\boldsymbol{a}_\alpha+\boldsymbol{a}_\beta-\boldsymbol{e}),h(\boldsymbol{a}_\alpha),h(\boldsymbol{a}_\beta)\big)\\
&= \iota\big((0,0),(0,0),(1,0),(1,1)\big)\\
&= (1,0),\\
h_{[\iota]}(\boldsymbol{c}_{\beta,\gamma}) &= \iota\big((0,0),h(\boldsymbol{a}_\beta+\boldsymbol{a}_\gamma-\boldsymbol{e}),h(\boldsymbol{a}_\beta),h(\boldsymbol{a}_\gamma)\big)\\
&= \iota\big((0,0),(0,0),(1,1),(1,0)\big)\\
&= (1,0).
\end{aligned}$$

Therefore, applying $h_{[\iota]}$ to the inequality (6.21) yields the inequality $(2,0) \leqslant (1,1)$ (in $\mathbb{Z}^2$!), a contradiction. □

Corollary 6.3. *For any set X, the following conditions are equivalent.*

(i) *There exists a lattice L such that* $\operatorname{Dim} L \cong \mathbf{F}(X)^+$.
(ii) *There exists a complemented modular lattice L such that* $\operatorname{Dim} L \cong \mathbf{F}(X)^+$.
(iii) *There exists a von Neumann regular ring R such that $V(R) \cong \mathbf{F}(X)^+$.*
(iv) *There exists a locally matricial ring R such that $K_0(R) \cong \mathbf{F}(X)$.*
(v) $|X| \leqslant \aleph_1$.

Proof. (i) ⇒ (v) follows immediately from Theorem 6.2.

Now suppose that $|X| \leqslant \aleph_1$. Then $\mathbf{F}(X)$ is a dimension group of cardinality at most $\aleph_1$; moreover, it has an order-unit (namely, $\boldsymbol{e}$). By [6, Theorem 1.5], for any field $\mathbb{F}$, there exists a locally matricial algebra R over $\mathbb{F}$ such that $K_0(R) \cong \mathbf{F}(X)$. Hence (v) implies (iv).

(iv) ⇒ (iii) is trivial, as $V(R) \cong K_0(R)^+$ for any locally matricial ring (and, more generally, for any unit-regular ring) R.

Now assume (iii). Since R is von Neumann regular, it follows from Lemma 1.4 that $V(R) \cong \operatorname{Dim} L$, where L is the (complemented modular) lattice of all principal right ideals of $M_2(R)$. Hence $\operatorname{Dim} L \cong \mathbf{F}(X)^+$, and so (ii) holds.

Finally, (ii) ⇒ (i) is a tautology. □

We conclude the paper with a problem.

Problem. Is every conical refinement monoid of cardinality at most $\aleph_1$ isomorphic to $\operatorname{Dim} L$, for some modular lattice L?

Even for countable monoids the question above is open. It is formally similar to the fundamental open problem raised by K.R. Goodearl in his survey paper [5], that asks which refinement monoids are isomorphic to $V(R)$ for some von Neumann regular ring R. A positive answer to Goodearl's question would yield a positive answer to the problem above, with L sectionally complemented modular.

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