Convergence of a class of discrete-time semiflows with applications to difference systems✩

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Abstract

In this paper, by employing comparison technique and invariance properties of a positively limited set, we investigate the convergence of precompact orbits of a class of discrete-time semiflows. In particular, we consider the convergence of precompact orbits of discrete-time semiflows generated by some monotone mapping. We then apply these abstract results to a class of difference systems to obtain the large-time behavior of solutions. Our results improve and extend some existing ones.

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1. Introduction

Recently, much progress has been made in applying the theory of monotone dynamical systems to investigate the problem of globally asymptotic behavior of continuous- and discrete-time semiflows. It is commonly hoped that most of the precompact orbits of a strongly monotone semiflow are convergent to a set of equilibria. For strongly monotone continuous-time semiflows, Hirsch [1,2] achieved this goal by

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employing the dichotomy of a positively limited set. For strongly monotone discrete-time semiflows, the limit set dichotomy theorem fails. We refer the reader to [3] for counter-examples about this. However, by making use of some additional hypotheses, convergence to an equilibrium for every precompact orbit of strongly monotone discrete-time semiflows is proved in the papers of [3–7]. Therefore, under certain hypotheses the dynamics of strongly monotone discrete-time semiflows are simple. It is natural to ask whether a similar conclusion holds for not strongly monotone or even non-monotone mappings. In this note we provide a positive answer to this question.

It should be mentioned that Wu [8] gave sufficient conditions for the convergence of the precompact orbits of a class of non-monotone discrete-time semiflows. But unfortunately, there are too many restrictions on the mapping which generates discrete-time semiflows. In fact, most of the restrictive conditions in [8] can be weakened or dropped (see Section 2 below for details). Huang and Yu [9] also investigated the problem of convergence of bounded orbits of a class of difference systems. For related results, we refer to [10–12]. However, it is not difficult to see that the abstract results of the paper [8] cannot be applied to the difference systems considered in [9].

Motivated by the results mentioned above, we study the asymptotic behavior of discrete-time semiflows generated by a class of non-monotone mappings in this paper. By comparing the positively limited set with some quasi-equilibrium (see Section 2 for more details on this definition) and applying invariance properties of a positively limited set, we obtain some results which improve and generalize the corresponding ones due to [8]. Our results show that some of the restrictive conditions in [8] can be weakened or dropped. Moreover, we also study the asymptotic behavior of precompact orbit of a class of monotone discrete-time semiflows. The obtained results improve the corresponding ones in [4].

In addition, we consider the applications of our abstract results to the following difference equation

\[ x_n - x_{n-1} = -F(x_n) + G(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), \quad n \geq 1, \]  

(1.1)

where \( k \) is a positive integer, \( F : R^1 \rightarrow R^1 \) (throughout this paper, \( R^1 \) denotes the set of all real numbers) is continuous and nondecreasing, and \( G : R^k \rightarrow R^1 \) is continuous. By transforming (1.1) equivalently into a class of discrete-time semiflows, we successfully give sufficient conditions for bounded orbits of (1.1) to converge to a constant. We also give the properties of unbounded orbits of (1.1). Obviously, (1.1) contains the following difference equation

\[ x_n - x_{n-1} = -f(x_n) + g(x_{n-k}). \]  

(1.2)

as a special case, where \( k \) is a positive integer, \( f, g : R^1 \rightarrow R^1 \) are continuous, and \( f \) is nondecreasing on \( R^1 \). Asymptotic behavior of solutions of (1.2) with \( f \) strictly increasing on \( R^1 \) has been studied in [9].

In the present paper, as an application of our results, we obtain the same results for (1.2), but we take a rather different point of view in dealing with this problem and the obtained results also improve the corresponding ones in [9].

The paper is organized as follows. In Section 2, we develop some convergence results for a class of abstract discrete-time semiflows. In Section 3, we present some applications of these results obtained in Section 2.

2. Some general convergence results

Let \( X \) be a metrizable topological space endowed with a closed partial order relation \( R \subseteq X \times X \) such that \( \text{Int} R \neq \emptyset \). For any \( x, y, q \in X \) and any subset \( A \subseteq X \), the following notations will be used:

\( x \leq y \) iff \((x, y) \in R\), \( x < y \) if \((x, y) \in R\) and \( x \neq y\), \( x \ll y \) iff \((x, y) \in \text{Int} R\), \( A \leq q \) (\( A < q \)) iff
a ≤ q (a < q) for all a ∈ A, q ≤ A (q < A) iff q ≤ a (q < a) for all a ∈ A, A ≪ q iff a ≪ q for all
a ∈ A, q ≪ A iff q ≪ a for all a ∈ A. We denote \( \overline{Y} \) closure of set \( Y \subseteq X \).

Consider a continuous mapping \( S : X → X \). We define \( E_S = \{ e ∈ X : S(e) = e \} \). If \( x ∈ X \), we
define \( O_S(x) = \{ S^n(x) : n ≥ 0 \} \). If \( \overline{O_S(x)} \) is compact, we define
\[
\omega_S(x) = \bigcap_{j ≥ 1} O_S(S^j(x)).
\]

One can observe that \( \omega_S(x) \) is nonempty, compact, and invariant. In particular, the invariance of \( \omega_S(x) \)
will play a crucial role in the proofs of the main results of this paper. We will always assume that
\( I : R^1 → X \) is continuous, and \( τ_1 < τ_2 \) implies \( I(τ_1) ≪ I(τ_2) \). Throughout this section \( E \)
denotes a closed subset of \( X \) together with \( E_S ≤ E \) and \( I(R^1) ≤ E \). For any given \( e ∈ E \), we define
\( S^e = \{ x ∈ X : x ≥ e \} \) and \( S_e = \{ x ∈ X : x ≤ e \} \).

For convenience, we introduce the following assumptions.

(C1) For any \( e ∈ E \), there exists an integer \( N ≥ 1 \) such that \( e ≪ S^n(x) \) or \( e = S^n(x) \) for any \( x ∈ S^e \)
and \( n > N \).

(C2) For any \( x ∈ X \), there exist \( α, β ∈ R^1 \) such that \( I(α) ≤ x ≤ I(β) \).

If \( E \) satisfies assumption (C1), we call \( E \) the set of quasi-equilibria and call the point in \( E \) a quasi-
equilibrium.

We are now in a position to state one of the main results of this section.

**Theorem 2.1.** Let the mapping \( S : X → X \) be continuous, \( S \) satisfy (C1), and assumption (C2) be
satisfied. If \( x ∈ X \) is given such that \( \overline{O_S(x)} \) is compact, then \( \omega_S(x) = \{ I(α^*) \} \) for some \( α^* ∈ R^1 \).

**Proof.** Since \( \overline{O_S(x)} \) is compact, \( \omega_S(x) \) is nonempty, compact, and invariant. Hence, by (C2), there exist
\( α^*, β ∈ R^1 \) such that \( I(α^*) ≤ \omega_S(x) ≤ I(β) \). Let \( α^* = \sup \{ r ∈ R^1 : I(r) ≤ \omega_S(x) \} \). Then \( α^* ∈ R^1 \).
We want to show that \( I(α^*) ∈ \omega_S(x) \). Suppose not, i.e., \( I(α^*) < \omega_S(x) \). Then by assumption (C1) and
the invariance of \( \omega_S(x) \), we obtain \( I(α^*) ≪ \omega_S(x) \). But this contradicts the definition of \( α^* \). Therefore,
\( I(α^*) ∈ \omega_S(x) \).

Next we will show that \( \omega_S(x)^2 = 1 \), where \( \omega_S(x)^2 \) denotes the cardinal numbers of \( \omega_S(x) \). Otherwise,
\( \omega_S(x)^2 > 1 \). Then by assumption (C1) and the invariance of \( \omega_S(x) \), there exist \( q ∈ \omega_S(x) \) such that
\( q ≷ I(α^*) \). By the definition of \( \omega_S(x) \), there exists \( n_1 > 1 \) such that \( S^{n_1}(x) ≷ I(α^*) \). Because of
the continuity of \( I \), we can find a real number \( β^* > α^* \) such that \( S^{n_1}(x) ≷ I(β^*) ≷ I(α^*) \). By assumption (C1) and the fact that \( I(β^*) ∈ E \), there exists \( n_2 > n_1 \) such that \( S^n(x) ≥ I(β^*) ≷ I(α^*) \)
for all \( n ≥ n_2 \). This implies that \( ω(x) ≥ I(β^*) ≷ I(α^*) \), a contradiction to \( I(α^*) ∈ \omega_S(x) \). Therefore,
\( \omega_S(x) = \{ I(α^*) \} \). This completes the proof. \( \square \)

**Remark 2.1.** It is clear that Theorem 2.1 extends and improves Theorem 2.1 in [8] in many aspects such as:
(a) We do not require that \( I(R^1) ≤ E_S \) holds; (b) assumption (ii) of Theorem 2.1 in [8] has been
weakened by assumption (C1) drastically in our paper; (c) assumption (i) of Theorem 2.1 in [8] has been
dropped in our Theorem 2.1.

**Remark 2.2.** It should be noted that the mapping \( I \) in assumption (C2) is not necessarily defined in \( R^1 \)
itself. In fact, it may be defined in an arbitrary interval of \( R^1 \) such as \( [0, 1] \) and \( (0, 1) \).

In many applications, it is necessary to consider the symmetric form of Theorem 2.1. To do this, we
introduce the following assumption.
For any $e \in E$, there exists an integer $N \geq 1$ such that $e \gg S^n(x)$ or $e = S^n(x)$ for any $x \in S_e$ and $n > N$.

**Theorem 2.2.** Let the mapping $S : X \rightarrow X$ be continuous, assumption (C2) be satisfied and $S$ satisfy (C3). Then the conclusion of Theorem 2.1 holds.

**Proof.** Let $\tilde{R} = \{(x, y) \in X \times X : (y, x) \in R\}$. For any $\alpha \in R^1$, let $\tilde{I}(\alpha) = I(-\alpha)$. Then replacing $R$ and $I$ in Theorem 2.1 by $\tilde{R}$ and $\tilde{I}$, respectively, we can conclude that $S$ satisfies the conditions of Theorem 2.1, and so it follows from Theorem 2.1 that the conclusion of Theorem 2.2 holds. The proof is now complete. □

In [4], the convergence of discrete-time semiflows generated by strongly monotone mappings has been discussed (see Theorem 1.3 in [4] for more details). Before proceeding, we need the following assumption.

(C4) Let the mapping $T : X \rightarrow X$ be continuous. There exists an integer $N \geq 1$ such that for any $x, y \in X$ with $x \geq y$, $T^n(x) \gg T^n(y)$ or $T^n(x) = T^n(y)$ for $n \geq N$.

We say that $T$ is semi-strongly monotone if $T$ satisfies assumption (C4).

As a direct consequence of Theorems 2.1 and 2.2, we obtain the following convergence of discrete-time semiflows generated by semi-strongly monotone mappings, which improves Theorem 1.3 in [4].

**Corollary 2.1.** Let the mapping $T : X \rightarrow X$ be continuous and semi-strongly monotone, also let assumption (C2) be satisfied and $I(R^1) \subseteq E_T$. If $x \in X$ is given such that $\overline{O_T}(x)$ is compact, then $\omega_T(x) = \{I(\alpha)\}$ for some $\alpha \in R^1$.

**Proof.** Let $E = E_T$. It is not difficult to check that $T$ satisfies the conditions of Theorem 2.1 or 2.2. Therefore, we can apply Theorem 2.1 or 2.2 to obtain the conclusion of Corollary 2.1. This completes the proof. □

3. Applications to some difference equations

In this section, we apply the abstract results in Section 2 to consider the large-time behavior of solutions for the difference equations (1.1) and (1.2).

To simplify the following argument, we introduce the following auxiliary mappings and establish several important lemmas that will play a major role in our analysis.

Let $F$ and $G$ be defined as in (1.1). We define the mappings

$$\varphi : R^1 \rightarrow R^1 \quad \text{by} \quad \varphi(x) = x + F(x)$$

and

$$H : R^k \rightarrow R^k \quad \text{by} \quad H(z_1, z_2, \ldots, z_k) = (z_2, \ldots, z_k, \varphi^{-1}(z_k + G(z_1, z_2, \ldots, z_k))).$$

(3.1)

It follows that $\varphi(x)$ and $\varphi^{-1}(x)$ are continuous and strictly increasing on $R^1$, and hence $H(z)$ is continuous on $R^k$.

**Lemma 3.1.** Let the mapping $\varphi$ be defined as above, $\alpha$ be a given constant and define the mapping $\gamma : R^1 \rightarrow R^1$ by $\gamma(x) = \varphi^{-1}(x + F(\alpha))$. Then for any given $M > 0$, there exists $N > M$ such that $\gamma^i(N) > M$ for all $i \in \{1, 2, \ldots, k\}$.
We next distinguish two cases to finish the proof of (3.3).

Therefore, the conclusion (i) is a consequence of Theorem 2.1.

Proof. Let \( p(x) \equiv \min_{1 \leq i \leq k} \gamma_i(x) \), where \( x \in \mathbb{R}^1 \). Since \( \lim_{x \to +\infty} \gamma(x) = +\infty \), it follows that \( \lim_{x \to +\infty} p(x) = +\infty \). Therefore, the conclusion of the lemma is true. \( \square \)

In what follows, we will use \( \mathbb{R}^k_+ \) to denote the set of all nonnegative vectors in \( \mathbb{R}^k \). It then follows that \( \mathbb{R}^k_+ \) is an order cone in \( \mathbb{R}^k \). For any \( u, v \in \mathbb{R}^k \), the following notations will be used: \( u \leq v \) iff \( v - u \in \mathbb{R}^k_+ \), \( u < v \) iff \( u \leq v \) and \( u \neq v \), \( u \ll v \) iff \( v - u \in \text{Int} \mathbb{R}^k_+ \).

**Lemma 3.2.** Let the mappings \( F, G \) and \( H \) be as above. If \( G(z) \geq F(\alpha) \) for all \( \alpha \in \mathbb{R}^1 \) and \( z \in \mathbb{R}^k \) such that \( z \geq (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^k \), then we have

(i) if \( z \in \mathbb{R}^k \) is given such that \( O_H(z) \) is compact, then there exists \( c \in \mathbb{R}^1 \) such that

\[
\lim_{n \to \infty} H^n(z) = (c, c, \ldots, c) \in \mathbb{R}^k,
\]

where \( H^n(z) = (H^n(z)_1, (H^n(z)_2)_1, \ldots, (H^n(z)_k)_1) = H(H^{n-1}(z)) \) for \( n = 1, 2, \ldots \), and \( H^0 \equiv \text{Id}_{\mathbb{R}^k} \), in which \( \text{Id}_{\mathbb{R}^k} \) denotes the identical mapping from \( \mathbb{R}^k \) to \( \mathbb{R}^k \);

(ii) if \( z \in \mathbb{R}^k \) is given such that \( O_H(z) \) is unbounded, then

\[
\lim_{n \to \infty} (H^n(z))_i = +\infty \quad \text{for all } i \in \{1, \ldots, k\}.
\]

Proof. Let \( E = \{(\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^k : \alpha \in \mathbb{R}^1\} \) and define the mapping \( I : \mathbb{R}^1 \to \mathbb{R}^k \) by \( I(\alpha) = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^k \). By the definition of \( E \) and \( I \), assumption (C2) holds. Assume that \( e = (\alpha, \alpha, \ldots, \alpha) \in E \), \( z \in \mathbb{R}^k \) and \( z \geq e \). We want to show that

\[
z_k > \alpha \quad \text{implies} \quad (H(z))_k > \alpha. \tag{3.2}
\]

Indeed, by the definition of \( H \), we get

\[
(H(z))_k = \varphi^{-1}(z_k + G(z))
\geq \varphi^{-1}(z_k + F(\alpha))
> \varphi^{-1}(\alpha + F(\alpha))
= \alpha.
\]

Hence, by (3.2) and the continuity of \( H \), we have \( H(z) \geq e \). We will prove that

\[
H^n(z) = e \quad \text{or} \quad H^n(z) \gg e, \quad \text{for all } n \geq 2k + 2. \tag{3.3}
\]

We next distinguish two cases to finish the proof of (3.3).

Case 1. \( H^{k+1}(z) = e \).

Let \( n_0 = \inf\{n \geq k + 1 : H^n(z) > e\} \). If \( n_0 = +\infty \), then \( H^n(z) = e \) for all \( n \geq k + 1 \), and hence, the proof is complete. If \( n_0 < +\infty \), then we can conclude \( n_0 = k + 2 \). Otherwise, \( H^{k+2}(z) = e \) and \( n_0 > k + 2 \). Thus, \( H^{k+2}(z) = H(H^{k+1}(z)) = H(e) = e \), and so \( H^{n_0}(e) = e \), a contradiction. By the definition of \( n_0 \), we have \( H^{k+2}(z) = H(H^{k+1}(z)) = H(e) > e \), and hence, \( (H^{k+2}(z))_k > \alpha \). Therefore, from (3.2), we get \( H^n(z) \gg e \), for all \( n \geq 2k + 2 \).

Case 2. \( H^{k+1}(z) > e \).

It follows that there exists \( i \in \{1, 2, \ldots, k\} \) such that \( (H^{k+1}(z))_i > \alpha \). By the definition of \( H \), we obtain \( (H^{i+1}(z))_i > \alpha \). Hence, from (3.2), we get \( H^n(z) \gg e \) for all \( n \geq k + i \).

From the above discussion, we can conclude that (3.3) holds and hence, \( H \) satisfies assumption (C1). Therefore, the conclusion (i) is a consequence of Theorem 2.1.
We next show that the conclusion (ii) holds. Indeed, let \( z \in R^k \) be given such that \( O_H(z) \) is unbounded. Then choose \( \beta \in R^1 \) such that \( z \geq (\beta, \beta, \ldots, \beta) \in R^k \), from which it follows that \( H^\alpha(z) \geq (\beta, \beta, \ldots, \beta) \in R^k \). By Lemma 3.1 and the fact that \( O_H(z) \) is unbounded, for any \( M > 0 \), there exists \( n_1 > 1 \) such that
\[
H^n(z) \geq (M, M, \ldots, M) \in R^k.
\]
Hence,
\[
H^n(z) \geq (M, M, \ldots, M) \in R^k \quad \text{for all } n \geq n_1.
\]
Therefore, \( \lim_{n \to \infty} H^n(z)_i = +\infty \) for all \( i \in \{1, \ldots, k\} \). This proves the lemma. \( \square \)

**Remark 3.1.** It should be noted that Theorem 2.1 in \([8]\) cannot be applied to the mapping \( H \) in Lemma 3.2.

**Theorem 3.1.** Let \( \{x_n\}_{n=-k}^\infty \) be a solution of (1.1). If \( G(z) \geq F(\alpha) \) for all \( z \in R^k \) and \( \alpha \in R^1 \) such that \( z \geq (\alpha, \alpha, \ldots, \alpha) \in R^k \), then either \( \lim_{n \to \infty} x_n = +\infty \) or \( \lim_{n \to \infty} x_n = c \) for some \( c \in R^1 \).

**Proof.** Note that system (1.1) is equivalent to the system
\[
z^{(n)} = H^n(z),
\]
where \( H \) is defined as (3.1). The desired conclusion follows immediately from Lemma 3.2 and thus, the proof is complete. \( \square \)

We are now ready to state a symmetric form of Theorem 3.1, the proof of which is similar to that of Theorem 3.1 and therefore, it is omitted.

**Theorem 3.2.** Let \( \{x_n\}_{n=-k}^\infty \) be a solution of (1.1). If \( G(z) \leq F(\alpha) \) for all \( z \in R^k \) and \( \alpha \in R^1 \) such that \( z \leq (\alpha, \alpha, \ldots, \alpha) \in R^k \), then either \( \lim_{n \to \infty} x_n = -\infty \) or \( \lim_{n \to \infty} x_n = c \) for some \( c \in R^1 \).

As an application of Theorems 3.1 and 3.2, for the special case of (1.1), we can get the following results.

**Corollary 3.1.** Let \( f \) and \( g \) be defined as in (1.2) and let \( \{x_n\}_{n=-k}^\infty \) be a solution of (1.2). If \( g(x) \geq f(x) \) for all \( x \in R^1 \), then either \( \lim_{n \to \infty} x_n = +\infty \) or \( \lim_{n \to \infty} x_n = c \) for some \( c \in R^1 \).

**Proof.** Let \( F(x) = f(x) \) for any \( x \in R^1 \) and \( G(z_1, z_2, \ldots, z_k) = g(z_1) \) for any \( (z_1, z_2, \ldots, z_k) \in R^k \). Then, by exploiting Theorem 3.1, the conclusion of Corollary 3.1 is immediate. \( \square \)

**Corollary 3.2.** Let \( \{x_n\}_{n=-k}^\infty \) be a solution of (1.2). If \( f(x) \geq g(x) \) for all \( x \in R^1 \), then either \( \lim_{n \to \infty} x_n = -\infty \) or \( \lim_{n \to \infty} x_n = c \) for some \( c \in R^1 \).

**Proof.** The proof is similar to that of Corollary 3.1, and so it is omitted. \( \square \)

**Corollary 3.3.** Let \( \{x_n\}_{n=-k}^\infty \) be a solution of (1.2). If \( f \equiv g \), then \( \lim_{n \to \infty} x_n = c \) for some \( c \in R^1 \).

**Proof.** Applying Corollaries 3.1 and 3.2, we can conclude that Corollary 3.3 holds. \( \square \)

**Remark 3.2.** Corollary 2.1 can also be applied to conclude that Corollary 3.3 holds. Indeed, define the following auxiliary mapping \( \psi : R^1 \to R^1 \) by \( \psi(x) = x + f(x) \). It follows that \( \psi \) and \( \psi^{-1} \) are continuous and strictly increasing on \( R^1 \). Also, define the mapping \( h : R^k \to R^k \) by \( h(z_1, z_2, \ldots, z_k) = (z_2, z_3, \ldots, z_k, \psi^{-1}(z_k + f(z_1))) \).
We claim that $h$ satisfies assumption (C4). Indeed, assume that $z, z' \in R^k$ satisfy $z \geq z'$. Since $\psi$ is strictly increasing and $f$ is nondecreasing, it follows from the definition of $h$ that $h(z) \geq h(z')$. We next distinguish two cases to finish the proof of the above claim.

Case 1. $h^{k+1}(z) = h^{k+1}(z')$.
It follows that $h^n(z) = h^n(z')$ for all $n \geq k + 1$, and hence the above claim is established.

Case 2. $h^{k+1}(z) > h^{k+1}(z')$.
It follows that there exists $i \in \{1, 2, \ldots, k\}$ such that $(h^{i+1}(z))_i > (h^{i+1}(z'))_i$. Hence, by the definition of $h$, we have $(h^{i+1}(z))_k > (h^{i+1}(z'))_k$. Again by the definition of $h$, we get

\[
(h^{i+2}(z))_k = \psi^{-1}((h^{i+1}(z))_k + f((h^{i+1}(z))_1)) \\
> \psi^{-1}((h^{i+1}(z'))_k + f((h^{i+1}(z'))_1)) \\
= (h^{i+2}(z'))_k.
\]
Thus, by induction, we have $h^n(z) > h^n(z')$ for all $n \geq k + i$.

Therefore, from the above discussion, we can conclude that $h$ satisfies assumption (C4). It then follows from Corollary 2.1 that Corollary 3.3 is established.

**Remark 3.3.** Corollaries 3.1–3.3 improve the results obtained in [9] since $f$ is required to be strictly increasing in [9]. In particular, our proofs are quite different from those of [9]. We refer to [9] for a detailed description of the applications of Corollaries 3.1–3.3.

**References**


