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# On the $L_2$-Stability of a Class of Nonlinear Systems

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Sufficient conditions are given for the $L_2$-stability of a class of feedback systems consisting of a linear operator **G** and a nonlinear gain function, either odd monotone or restricted by a power-law, in cascade, in a negative feedback loop. The criterion takes the form of a frequency-domain inequality,

$$\text{Re}[1 + Z(j\omega)]\, G(j\omega) \geqslant \delta > 0 \;\forall \omega \in (-\infty, +\infty),$$

where $Z(j\omega)$ is given by, $Z(j\omega) = \beta[Y_1(j\omega) + Y_2(j\omega)] + (1 - \beta)[Y_3(j\omega) - Y_3(-j\omega)]$, with $0 \leqslant \beta \leqslant 1$ and the functions $y_1(\cdot)$, $y_2(\cdot)$ and $y_3(\cdot)$ satisfying the time-domain inequalities,

$$\int_{-\infty}^{+\infty} | y_1(t) + y_2(t)| \, dt \leqslant 1 - \epsilon, \qquad y_1(\cdot) = 0, \quad t < 0,$$
$$y_2(\cdot) = 0, \quad t > 0 \quad \text{and} \quad \epsilon > 0,$$

and

$$\int_0^{\infty} | y_3(t)| \, dt < \frac{1}{2c_2},$$

$c_2$ being a constant depending on the order of the power-law restricting the nonlinear function. The criterion is derived using Zames' passive operator theory and is shown to be more general than the existing criteria.

## 1. INTRODUCTION

Following the advent of Popov's [1] frequency-domain stability criterion for the feedback system which can be posed in the Lur'e form, many recent publications have aimed at broadening the class of multipliers by imposing more restrictions on the nonlinear function. O'Shea [2] introduced a class of multipliers with certain time-domain conditions, for systems with monotone and odd monotone nonlinearities and proved stability in these cases by using bounds on the input–output cross correlation function of the nonlinearity. Closely related results, with considerably more generalization, were obtained in the framework of operator theory by Zames and Falb [3]. In a recent publication, Thathachar [4] gave a stability criterion for systems with power-law restricted nonlinear functions which indeed form a subclass of odd monotone functions.

However, the proof in that paper requires the a priori assumption that the solutions are bounded and appears to take into account a limited class of nonlinearities. In the present paper, a similar system is considered and a criterion for $L_2$-stability is derived by using what is now well known as Zames' positive operator theory.

The main contribution here is considered to be the following:

(1) The condition on the multiplier $\mathbf{M} = \mathbf{E} + \mathbf{Z}$ in [3] that the norm of the operator $\mathbf{Z}$ be less than unity is removed.

(2) The stability criterion derived here for systems with odd monotone nonlinearities is more general than that in [3] and further takes into account the case of power-law nonlinearities.

(3) The criterion is also general than that in [4] for the case of power-law nonlinearities and at the same time overcomes the boundedness assumption made in that paper.

## 2. Problem Formulation

### 2a. *Notations and Definitions*

It is assumed that the reader is familiar with the notions of normed spaces, linear spaces, inner products, $L_2$-spaces, extended $L_2$-spaces ($L_{2e}$) and extended norms on $L_{2e}$. (These concepts are defined in Zames and Falb [3].)

An "operator $\mathbf{M}$" on $L_2$ is a single-valued mapping of $L_2$ into itself.

The "gain of an operator $\mathbf{M}$", denoted by $\nu(\mathbf{M})$, is given by,

$$\nu(\mathbf{M}) = \sup_{\substack{x \in L_2 \\ x \neq 0}} \frac{\| \mathbf{M}x(\cdot)\|}{\| x(\cdot)\|} .$$

An operator $\mathbf{M}$ is said to be "positive" if $\langle x(\cdot), \mathbf{M}x(\cdot)\rangle \geqslant 0 \ \forall x(\cdot) \in L_2$. If a stronger inequality of the form $\langle x(\cdot), \mathbf{M}x(\cdot)\rangle \geqslant \delta \langle x(\cdot), x(\cdot)\rangle$ holds $\forall x(\cdot) \in L_2$, $\delta > 0$, then the operator $\mathbf{M}$ is termed "strongly positive".

The "adjoint" of an operator $\mathbf{M}$, denoted $\mathbf{M}^*$ is a mapping of $L_2$ into itself such that,

$$\langle x_1(\cdot), \mathbf{M}x_2(\cdot)\rangle = \langle \mathbf{M}^* x_1(\cdot), x_2(\cdot)\rangle \ \forall x_1, x_2 \in L_2 .$$

### 2b. *The Main Problem*

Consider the feedback system illustrated in Fig. 1. Assuming that $\mathbf{G}$ and $\mathbf{N}$ are operators in $L_2$, the feedback equations may be formulated as,

$$e_1 = x_1 - w_2 , \tag{2.1}$$

$$e_2 = x_2 + w_1 , \tag{2.2}$$

$$w_1(t) = \mathbf{G}e_1(t) = \int_0^\infty g(\tau)\, e_1(t-\tau)\, d\tau, \tag{2.3}$$

$$w_2(t) = \mathbf{N}e_2(t) = N(e_2(t)), \tag{2.4}$$

satisfying the following assumptions.

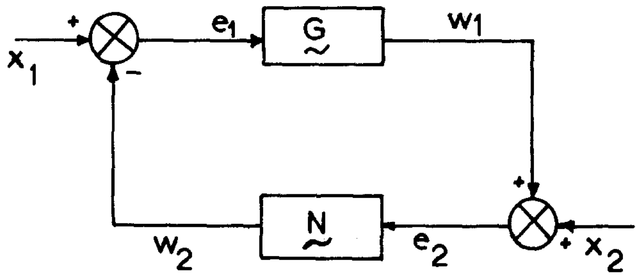


FIG. 1. The feedback system under consideration $\underset{\sim}{G}$ and $\underset{\sim}{N}$ are written as **G** and **N**, respectively, in the text.

(A) $x_1(\cdot)$ and $x_2(\cdot) \in L_2(-\infty, +\infty)$,

(B) $g(\cdot) \in L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$.

It must be noted that the initial conditions are assumed to be zero in (2.3) and this does not result in any loss of generality (see [5]).

Let $G(j\omega)$ denote the Fourier transform of $g(\cdot)$.

(C) In (2.4) $N(\cdot)$ is a real-valued function on $(-\infty, +\infty)$ such that $N(0) = 0$ and $N(\cdot)$ is restricted by a power-law, i.e. it satisfies the following conditions:

$$\text{(i)} \quad \sigma N(\sigma) \geqslant 0 \forall \sigma, \tag{2.5}$$

$$\text{(ii)} \quad N(\sigma) = -N(-\sigma) \text{ and } dN(\sigma)/d\sigma \geqslant 0 \forall \sigma, \tag{2.6}$$

$$\text{(iii)} \quad 1/m \leqslant [d \log N(\sigma)]/(d \log \sigma) \leqslant m \forall \sigma > 0, \tag{2.7}$$

where "$m$" is the order of the power-law.

$$\text{(iv)} \quad |N(\sigma)| \leqslant K|\sigma| \text{ for some } K > 0 \text{ and all } \sigma. \tag{2.8}$$

When the nonlinear function satisfies the above conditions, it will be said to belong to the class $P^m$. The particular case of $m = 1$ corresponds to linear feedback and as $m \to \infty$, the nonlinear function approaches the class of odd monotone functions. An important property of this kind of nonlinearity is that it satisfies an inequality of the form

$$\sigma_1 N(\sigma_2) - \sigma_2 N(\sigma_1) \leqslant c_2[\sigma_1 N(\sigma_1) + \sigma_2 N(\sigma_2)] \, \forall \sigma_1, \sigma_2 . \tag{2.9}$$

where $c_2$ is a constant,

$$c_2 = \max_{0<y<\infty} \left| \frac{y^m - y}{1 + y^{m+1}} \right|. \tag{2.10}$$

For a derivation of this, see [4].

The problem of interest may now be stated as follows. Given the system described by (2.1)–(2.4) with the assumptions A, B, and C, find sufficient conditions on $\mathbf{G}$ for $e_1(\cdot)$ and $e_2(\cdot)$ to be in $L_2(-\infty, +\infty)$ and

$$\lim_{t\to\infty} w_1(t) = 0.$$

## 3. The Stability Criterion

### 3a. *Main Theorem*

If there exist elements $y_1(\cdot)$, $y_2(\cdot)$, and $y_3(\cdot)$ in $L_1(-\infty, +\infty)$ such that,

$$y_1(t) = 0, \quad t < 0; \qquad y_2(t) = 0, \quad t > 0; \qquad y_3(t) = 0, \quad t < 0 \tag{3.1}$$

and

$$\int_{-\infty}^{+\infty} | y_1(t) + y_2(t)| \, dt \leqslant 1 - \epsilon, \qquad \epsilon > 0 \tag{3.2}$$

$$\int_0^{\infty} | y_3(t)| \, dt < \frac{1}{2c_2}, \tag{3.3}$$

$c_2$ being given by (2.10), and the inequality,

$$\mathrm{Re}[1 + Z(j\omega)] \, G(j\omega) \geqslant \delta > 0 \forall \omega \in (-\infty, +\infty), \tag{3.4}$$

is satisfied for $Z(j\omega)$ given by,

$$Z(j\omega) = \beta[Y_1(j\omega) + Y_2(j\omega)] + (1 - \beta) \, [Y_3(j\omega) - Y_3(-j\omega)], \tag{3.5}$$

where $\beta$ is a constant and $0 \leqslant \beta \leqslant 1$, then $e_1(\cdot)$ and $e_2(\cdot)$ are in $L_2(-\infty, +\infty)$, and also $\lim_{t\to\infty} w_1(t) = 0$.

### 3b. *A few Remarks*

*Remark* 1. A more general function $Z_g(j\omega) = Z(j\omega) + \alpha j\omega$, $\alpha > 0$ can be used in (3.4) if an additional condition $\lim_{\omega\to\infty} | \omega G(j\omega)| = 0$ is satisfied (see [8]).

*Remark* 2. It must be noted that the present criterion is more general than the existing criteria. For the case of $\beta = 1$, the multiplier $[1 + Z(j\omega)]$ reduces to the multiplier in [2] (in view of Remark 1) and [3][1] and for the case of $\beta = 0$, the multiplier used in [4] is obtained.

*Remark* 3. For the case of the nonlinearity being odd monotone, $m = \infty$ and $c_2$ from (2.10) is equal to unity. Hence the inequality (3.4) holds with (3.1), (3.2), (3.5), and

$$\int_0^\infty |y_3(t)|\, dt < \tfrac{1}{2}.$$

*Remark* 4. Since [4] is a special case[2] of the present criterion, an additional feature of the result given here is that for linear feedback, it reduces to the well known necessary and sufficient conditions corresponding to the Nyquist criterion.

*Remark* 5. The multiplier $[1 + Z(j\omega)]$ used here is very general in the sense that any rational function $P(s)$ which satisfies $\text{Re}\, P(j\omega) > 0 \forall \omega$, can be put in this form by spectral factorization (see [6]).

## 4. Proof of the Stability Criterion

Before going to the actual proof of the stability criterion, a few preliminary results will be established.

### 4a. *A Stability Theorem for Feedback Systems*

Here, a theorem given by Zames and Falb [3] for the stability of a general feedback system with the configuration as in Fig. 1 is invoked.

THEOREM 1. *If there is a mapping* $\mathbf{M}$ *of* $L_2$ *into* $L_2$ *such that:*

(i) *there are linear maps* $\mathbf{M}_+$ *and* $\mathbf{M}_-$ *of* $L_2$ *into* $L_2$ *such that* $\mathbf{M} = \mathbf{M}_-\mathbf{M}_+$ *where* $\mathbf{M}_+$ *and* $\mathbf{M}_-$ *are invertible and* $\mathbf{M}_+$, $\mathbf{M}_+^{-1}$, $\mathbf{M}_-^*$, $\mathbf{M}_-^{*-1}$ *are nonanticipative and have finite gains,*

(ii) $\mathbf{MG}$ *and* $\mathbf{M}^*\mathbf{N}$ *are positive,*

(iii) $\mathbf{MG}$ *is strongly positive and* $\mathbf{G}$ *has finite gain, then* $e_1(\cdot)$ *and* $e_2(\cdot)$ *are in* $L_2$.

This theorem is fundamental and the rest of the proof involves recasting the information available so as to satisfy the conditions of this theorem.

[1] It must be noted that the condition (3.2) is not different from the condition in [2] or [3] since $\epsilon$ is arbitrary.

[2] The condition in [4] is slightly different from (3.4) in that $\delta$ is allowed to be zero but involves somewhat more restrictions on the nonlinearity.

### 4b. *Operator Factorization*

The importance of the factorization of an operator $\mathbf{M}$ into a product of two operators $\mathbf{M}_+$ and $\mathbf{M}_-$ is well broughtforth in the above theorem. The conditions for such a factorization will now be established.

LEMMA 1. *Let $\mathscr{B}$ be a commutative Banach algebra with an identity* $\mathbf{E}$. *If* $\mathbf{Z}(\cdot)$ *is a nonzero element of* $\mathscr{B}$ *such that the frequency function* $Z(j\omega)$ *is of the form* (3.5), *then* $(\mathbf{E} + \mathbf{Z})$ *is a strongly positive operator.*

*Proof.*

$$\begin{aligned}\langle x(\cdot), \mathbf{Z}x(\cdot)\rangle &= \int_{-\infty}^{+\infty} z(\tau) \int_{-\infty}^{+\infty} x(t)\, x(t-\tau)\, dt\, d\tau \\ &= \beta \int_{-\infty}^{+\infty} [y_1(\tau) + y_2(\tau)] \int_{-\infty}^{+\infty} x(t)\, x(t-\tau)\, dt\, d\tau \\ &\quad + (1-\beta) \int_0^{\infty} y_3(\tau) \left[\int_{-\infty}^{+\infty} x(t)\, x(t-\tau)\, dt \right. \\ &\quad \left. - \int_{-\infty}^{+\infty} x(t)\, x(t+\tau)\, dt\right] d\tau\end{aligned}$$

The integrals within the square brackets in the last term of the RHS cancel, and hence,

$$\begin{aligned}|\langle x(\cdot), \mathbf{Z}x(\cdot)\rangle| &\leqslant \beta \int_{-\infty}^{+\infty} |y_1(\tau) + y_2(\tau)| \left| \int_{-\infty}^{+\infty} x(t)\, x(t-\tau)\, dt \right| d\tau \\ &\leqslant \beta(1-\epsilon) \left| \int_{-\infty}^{+\infty} x(t)\, x(t-\tau)\, dt \right|, \qquad \text{from (3.2)} \\ &\leqslant \beta(1-\epsilon) \int_{-\infty}^{+\infty} x^2(t)\, dt,\end{aligned}$$

since

$$|R(\tau)| \leqslant R(0),$$

where $R(\tau)$ is the autocorrelation function $\int_0^T x(t)\, x(t-\tau)\, dt$.

This implies,

$$\langle x(\cdot), \mathbf{Z}x(\cdot)\rangle \geqslant -\beta(1-\epsilon)\langle x(\cdot), x(\cdot)\rangle.$$

Hence,

$$\begin{aligned}\langle x(\cdot), (\mathbf{E}+\mathbf{Z})\, x(\cdot)\rangle &= \langle x(\cdot), x(\cdot)\rangle + \langle x(\cdot), \mathbf{Z}x(\cdot)\rangle \\ &\geqslant [1-\beta(1-\epsilon)]\langle x(\cdot), x(\cdot)\rangle \\ &\geqslant \epsilon\langle x(\cdot), x(\cdot)\rangle,\end{aligned}$$

since $0 \leqslant \beta \leqslant 1$. Q.E.D.

This lemma is the crucial step in establishing the factorizability of the operator $(\mathbf{E} + \mathbf{Z})$. Since $(\mathbf{E} + \mathbf{Z})$ is strongly positive, its spectrum is contained entirely in the open RHP of the complex plane. We can take a simply connected domain $D$ in the complex plane including all the spectrum points of $(\mathbf{E} + \mathbf{Z})$ but excluding the origin. Let $\Gamma$ be a simple closed curve in $D$ enclosing all the spectrum points. Since $f(\xi) = \log \xi$ is a holomorphic function on $D$, the logarithm of $(\mathbf{E} + \mathbf{Z})$ exists[3] and is given by the Dunford–Taylor integral (see [7])

$$f(\mathbf{M}) = \frac{1}{2\pi i} \int_{\Gamma} f(\xi)\,(\xi\mathbf{E} - \mathbf{M})^{-1}\,d\xi.$$

Hence, $(\mathbf{E} + \mathbf{Z})$ can be factorized into $(\mathbf{E} + \mathbf{Z}) = \mathbf{Z}_{-}\mathbf{Z}_{+}$ where

$$\mathbf{Z}_{+} = \exp[P_{+} \log(\mathbf{E} + \mathbf{Z})] \qquad \text{and} \qquad \mathbf{Z}_{-} = \exp[P_{-} \log(\mathbf{E} + \mathbf{Z})],$$

$P_{+}$ and $P_{-}$ being projections on $\mathscr{B}$ such that $P_{-} + P_{+} = E_{\mathscr{L}}$, the identity of the space $\mathscr{L}(\mathscr{B}, \mathscr{B})$ which is the space of all continuous linear maps of $\mathscr{B}$ into itself. It can be seen that $\mathbf{Z}_{+}$ and $\mathbf{Z}_{-}$ are invertible.

### 4c. *Positivity Conditions*

LEMMA 2. *If $N(\cdot)$ is an odd monotone function and $x(\cdot) \in L_2(-\infty, +\infty)$, then*

$$\left| \int_{-\infty}^{+\infty} x(t)\, N(x(t-\tau))\, dt \right| \leqslant \int_{-\infty}^{+\infty} x(t)\, N(x(t))\, dt. \tag{5.1}$$

The proof of this lemma is given in [3].

LEMMA 3. *If $N(\cdot) \in P^m$ and $x(\cdot) \in L_2(-\infty, +\infty)$, then*

$$\left| \int_{-\infty}^{+\infty} x(t)\, N(x(t-\tau))\, dt - \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt \right| \leqslant 2c_2 \int_{-\infty}^{+\infty} x(t)\, N(x(t))\, dt, \tag{5.2}$$

*where $c_2$ is given by* (2.10).

The proof of this lemma follows from inequality (2.9) and is given in [4].

[3] It must be noted here that Zames and Falb [3] have shown the existence of the logarithm of $(\mathbf{E} + \mathbf{Z})$ with the assumption, norm of $\mathbf{Z}$ less than 1. Such an assumption cannot be made here since an element $\mathbf{Z}(.)$ having $Z(j\omega)$ of the form (3.5) can have a norm greater than 1 and hence the existence of the logarithm is to be shown from different considerations.

LEMMA 4. *If* $\mathbf{Z}$ *is an element of a commutative Banach algebra* $\mathscr{B}$ *such that* $Z(j\omega)$ *has the form* (3.5) *and* $N(\cdot) \in P^m$, *then* $(\mathbf{E} + \mathbf{Z})^* \mathbf{N}$ *is positive where* $\mathbf{E}$ *is the identity element of* $\mathscr{B}$.

*Proof.* If $x(\cdot) \in L_2(-\infty, +\infty)$,

$$\begin{aligned}
\langle x(\cdot), \mathbf{Z}^*\mathbf{N}x(\cdot)\rangle &= \int_{-\infty}^{+\infty} z(\tau) \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt\, d\tau \\
&= \int_{-\infty}^{+\infty} [\beta\{y_1(\tau) + y_2(\tau)\} + (1-\beta)\{y_3(\tau) - y_3(-\tau)\}] \\
&\qquad \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt\, d\tau \\
&= \beta \int_{-\infty}^{+\infty} [y_1(\tau) + y_2(\tau)] \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt\, d\tau \\
&\quad + (1-\beta) \int_0^{\infty} y_3(\tau) \Big[\int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt \\
&\quad - \int_{-\infty}^{+\infty} x(t)\, N(x(t-\tau))\, dt\Big]\, d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle x(\cdot), \mathbf{Z}^*\mathbf{N}x(\cdot)\rangle| &\leqslant \beta \int_{-\infty}^{+\infty} |y_1(\tau) + y_2(\tau)| \left| \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt \right| d\tau \\
&\quad + (1-\beta) \int_0^{\infty} |y_3(\tau)| \left| \int_{-\infty}^{+\infty} x(t)\, N(x(t+\tau))\, dt \right. \\
&\quad \left. - \int_{-\infty}^{+\infty} x(t)\, N(x(t-\tau))\, dt \right| d\tau \\
&\leqslant \beta \int_{-\infty}^{+\infty} |y_1(\tau) + y_2(\tau)| \int_{-\infty}^{+\infty} x(t)\, N(x(t))\, dt\, d\tau \\
&\quad + (1-\beta) \int_0^{\infty} |y_3(\tau)| \left\{ 2c_2 \int_{-\infty}^{+\infty} x(t)\, N(x(t))\, dt \right\} d\tau,
\end{aligned}$$

from (5.1) and (5.2)

$$< \langle x(\cdot), \mathbf{N}x(\cdot)\rangle, \qquad \text{from (3.2) and (3.3).}$$

Hence,

$$\begin{aligned}
\langle x(\cdot), (\mathbf{E} + \mathbf{Z})^* \mathbf{N}x(\cdot)\rangle &= \langle (\mathbf{E} + \mathbf{Z})\, x(\cdot), \mathbf{N}x(\cdot)\rangle \\
&= \langle x(\cdot), \mathbf{N}x(\cdot)\rangle + \langle x(\cdot), \mathbf{Z}^*\mathbf{N}x(\cdot)\rangle \\
&\geqslant 0.
\end{aligned}$$

Q.E.D.

LEMMA 5. *If* $\mathbf{Z}$ *and* $\mathbf{G}$ *are elements of* $\mathscr{B}$ *with* $Z(j\omega)$ *of the form* (3.5) *and if,*

$$\operatorname{Re}[1 + Z(j\omega)]\, G(j\omega) \geqslant \delta > 0 \forall \omega \in (-\infty, +\infty),$$

*then* $(\mathbf{E} + \mathbf{Z})\,\mathbf{G}$ *is strongly positive.*

The proof is quite straightforward and follows from an application of Parseval's theorem,

$$\langle x(\cdot), y(\cdot) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, Y(-j\omega)\, d\omega.$$

### 4d. *Proof of the Main Result*

It can now be seen that the aim of the previous divisions (viz. 4b and 4c) was to systematically bring forth the satisfaction of the conditions of the fundamental theorem in Section 4a. Condition (i) is satisfied by Lemma 1 and conditions (ii) and (iii) by Lemmas 4 and 5. Hence it follows that $e_1(\cdot)$ and $e_2(\cdot)$ are in $L_2$. The proof of the assertion $\lim_{t\to\infty} w_1(t) = 0$ when $g(\cdot) \in L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ is given in Zames and Falb [3]. This completes the proof of the stability theorem.

## 5. CONCLUSIONS

A frequency-time domain stability criterion was established for the $L_2$-stability of a feedback system comprising of a linear transfer function and a nonlinearity of a certain class. A new form of multiplier was used to prove stability. The proof involved the factorization of operators in a Banach algebra of operators and this was related to the positivity properties of the operator. Finally, it was shown that the present criterion is very general and the earlier criteria are merely special cases of this.

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