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# Scalar derivatives and scalar asymptotic derivatives: properties and some applications

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## 1. Introduction

Scalar derivatives [13,14] were introduced for characterization of monotone operators (in sense of Minty–Browder) which are an important tool for solving operator equations, variational inequalities, complementarity problems and partial differential equations. The asymptotic version of the scalar derivative was defined by Isac in [7] for generalizing a classical fixed point theorem of Krasnoselskii. The scalar asymptotic derivatives generalize the asymptotic derivatives used by Krasnoselskii in his theorem. By introducing the notion of the inversion of a mapping a kind of duality between the scalar derivatives and the scalar asymptotic derivatives will be obtained. This duality will be used for finding scalar asymptotic derivatives of a mapping which in general are not asymptotic derivatives. Replacing assumption 3 of Theorem 3.1 [7] of Isac by these expressions of the scalar asymptotic derivatives various fixed point theorems will be generated. These fixed point theorems will be used for generating surjectivity theorems, solving variational inequalities, complementarity problems and integral equations.

## 2. Preliminaries

Let $E$ be a Banach space and $E^*$ the topological dual of $E$. Let $\langle E, E^*\rangle$ be a duality between $E$ and $E^*$. This duality is with respect to a bilinear functional on $E \times E^*$ denoted

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by $\langle\cdot,\cdot\rangle$ and which satisfies the following separation axioms:

($s_1$) $\langle x_0, y\rangle = 0$ for all $y \in E^*$ implies $x_0 = 0$,
($s_2$) $\langle x, y_0\rangle = 0$ for all $x \in E$ implies $y_0 = 0$.

For the weak topology on $E$ (respectively on $E^*$) we use the Bourbaki's terminology, that is, the weak topology on $E$ is the $\sigma(E, E^*)$-topology and on $E^*$ the $\sigma(E^*, E)$-topology. Denote by $L(E, E^*)$ the set of continuous linear mappings from $E$ into $E^*$. We remark that if $E = H$, where $H$ is a Hilbert space, then $E^*$ can be identified with $H$, the bilinear functional generating the duality between $E$ and $E^*$ with the scalar product of $H$ and $L(E, E^*)$ with the space of continuous linear mappings from $H$ into $H$, which will be denoted by $L(H)$ [11].

Recall the following definitions [8]:

**Definition 2.1.** Let $K \subseteq E$ and $f : K \to E^*$. $f$ is called *completely continuous* if it is continuous and the image of every bounded set is relatively compact.

**Definition 2.2.** We say that a non-empty set $K \subseteq E$ is a convex cone if:

(1) $K + K \subseteq K$,
(2) $\lambda K \subseteq K$ for all $\lambda \in \mathbb{R}_+$.

A convex cone $K$ is called pointed if $K \cap (-K) = \{0\}$ and generating if $K - K = E$.

**Definition 2.3.** Let $K \subseteq E$ be a convex cone. The convex cone

$$K^* = \{y \in E^* \mid \langle x, y\rangle \geqslant 0 \text{ for all } x \in K\}$$

of $E^*$ is called the dual cone of $K$.

For more details about cones the reader is referred to [8].

**Definition 2.4.** Let $\Delta$ be a set, $K \subseteq E$ a pointed convex cone, $x, y \in K$ and $f, g : \Delta \to E$. The relation $x \leqslant_K y$ defined by $y - x \in K$ is an order relation on $E$. Define $f \leqslant_K g$ if $f(z) \leqslant_K g(z)$ for all $z \in \Delta$.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Recall the following definitions:

**Definition 2.5.** A continuous operator $Z : H \to H$ is called skew-adjoint [1] if

$$\langle Z(x), y\rangle = -\langle Z(y), x\rangle, \tag{1}$$

for all $x, y \in H$. In [13] it is proved that relation (1) implies that $Z$ is linear.

**Definition 2.6.** A continuous linear operator $P : H \to H$ is called positive semidefinite [15] if $\langle P(x), x\rangle \geqslant 0$, for all $x \in H$.

## 3. Inversions

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\|\cdot\|$ the norm generated by $\langle\cdot,\cdot\rangle$. The following definition is an extension of Example 5.1 [4, p. 169]:

**Definition 3.1.** The operator

$$i : H \setminus \{0\} \to H \setminus \{0\}, \quad i(x) = \frac{x}{\|x\|^2}$$

is called *inversion* (of pole 0).

It is easy to see that $i$ is one to one and $i^{-1} = i$. Indeed, since $\|i(x)\| = 1/\|x\|$, by the definition of $i$ we have $i(i(x)) = i(x)/\|i(x)\|^2 = \|x\|^2 i(x) = x$. Hence $i$ is a global diffeomorphism of $H \setminus \{0\}$ which can be viewed as a global non-linear coordinate transformation in $H$.

Let $A \subseteq H$ such that $0 \in A$ and $A \setminus \{0\}$ is an invariant set of the inversion $i$, i.e., $i(A \setminus \{0\}) = A \setminus \{0\}$ and $f : A \to H$. Examples of invariant sets of the inversion $i$ are:

(1) $F \setminus \{0\}$ where $F$ is a linear subspace of $H$ (in particular $F$ can be the whole $H$),
(2) $K \setminus \{0\}$ where $K \subseteq H$ is a pointed convex cone.

Now we define the inversion (of pole 0) of the mapping $f$.

**Definition 3.2.** The *inversion* (of pole 0) of the mapping $f$ is the mapping $\mathcal{I}(f) : A \to H$ defined by

$$\mathcal{I}(f)(x) = \begin{cases} \|x\|^2 (f \circ i)(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

**Proposition 3.1.** *The inversion of mappings $\mathcal{I}$ is a one to one operator on the set of mappings $\{f \mid f : A \to H,\ f(0) = 0\}$ and $\mathcal{I}^{-1} = \mathcal{I}$, i.e., $\mathcal{I}(\mathcal{I}(f)) = f$.*

**Proof.** By definition $\mathcal{I}(\mathcal{I}(f))(0) = 0$. Hence, $\mathcal{I}(\mathcal{I}(f))(0) = f(0)$. If $x \neq 0$ then $\mathcal{I}(\mathcal{I}(f))(x) = \|x\|^2 \mathcal{I}(f)(i(x)) = \|x\|^2 \|i(x)\|^2 f(i(i(x))) = f(x)$. Thus, $\mathcal{I}(\mathcal{I}(f))(x) = f(x)$ for all $x \in K$. Therefore $\mathcal{I}(\mathcal{I}(f)) = f$. □

**Proposition 3.2.** *Let $f : A \to A$. Then, $x \neq 0$ is a fixed point of $f$ iff $i(x)$ is a fixed point of $\mathcal{I}(f)$.*

**Proof.** Suppose that $x \neq 0$ is a fixed point of $f$, i.e., $f(x) = x$. Since $i(i(x)) = x$ we have

$$f\bigl(i\bigl(i(x)\bigr)\bigr) = x. \tag{2}$$

Multiplying (2) by $\|i(x)\|^2 = 1/\|x\|^2$ we obtain $\mathcal{I}(f)(i(x)) = i(x)$. Thus, $i(x)$ is a fixed point of $\mathcal{I}(f)$. Similarly can be proved that if $i(x)$ is a fixed point of $\mathcal{I}(f)$, then $x$ is a fixed point of $f$. □

Let $D = \{x \in H \mid \|x\| \leqslant 1\}$ and $C = \{x \in H \mid \|x\| = 1\}$ be the unit ball and the unit sphere of $H$, respectively.

**Proposition 3.3.** *Let $f, g : A \to H$ such that $f(x) = g(x)$ for all $x \in A \cap C$ and $f(0) = g(0) = 0$. There exists unique extensions $\tilde{f}, \tilde{g} : A \to H$ of $f|_{A\cap D}$ and $g|_{A\cap D}$, respectively, such that $\tilde{g} = \mathcal{I}(\tilde{f})$.*

**Proof.** Let $D^\circ = \{x \in H \mid \|x\| < 1\}$. First we prove the existence of the extensions $\tilde{f}, \tilde{g}$. Define the extensions $\tilde{f}, \tilde{g}$ of $f|_{A\cap D}$ and $g|_{A\cap D}$ by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } \|x\| \leqslant 1, \\ \mathcal{I}(f)(x) & \text{if } \|x\| > 1, \end{cases}$$

and

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } \|x\| \leqslant 1, \\ \mathcal{I}(g)(x) & \text{if } \|x\| > 1, \end{cases}$$

respectively. We have to prove that

$$\tilde{g}(x) = \mathcal{I}(\tilde{f})(x) \tag{3}$$

for all $x \in A$. We consider three cases:

*First case*. $x \in A \cap D^\circ$. In this case $\|x\| < 1$ and hence $\|i(x)\| > 1$. Thus, by definition $\tilde{g}(x) = g(x)$ and $\tilde{f}(i(x)) = \mathcal{I}(g)(i(x))$. By using these relations and the definition of the inversion of a mapping, relation (3) can be proved easily.

*Second case*. $x \in A \setminus D$. In this case $\|x\| > 1$ and hence $\|i(x)\| < 1$. Thus, by definition $\tilde{g}(x) = \mathcal{I}(f)(x)$ and $\tilde{f}(i(x)) = f(i(x))$. Relation (3) can be proved similarly to the previous case.

*Third case*. $x \in A \cap C$. In this case $\|x\| = 1$ and hence $i(x) = x$. Thus, by definition $\tilde{g}(x) = g(x)$ and $\tilde{f}(i(x)) = f(x)$. In this case (3) is equivalent to $f(x) = g(x)$, which by the assumption made on $f$ and $g$ it is true.

Now we prove the uniqueness of the extensions $\tilde{f}, \tilde{g}$. Suppose that $\hat{f}, \hat{g}$ are extensions of $f|_{A\cap D}$ and $g|_{A\cap D}$, respectively, such that $\hat{g} = \mathcal{I}(\hat{f})$. If $\|x\| \leqslant 1$, then $\hat{g}(x) = \tilde{g}(x) = g(x)$ since both $\hat{g}$ and $\tilde{g}$ are extensions of $g|_{A\cap D}$. If $\|x\| > 1$, then $\|i(x)\| < 1$. Since $\hat{f}$ is an extension of $f|_{A\cap D}$, $\hat{f}(i(x)) = f(i(x))$. By using this relation, relation $\hat{g}(x) = \mathcal{I}(\hat{f})(x)$, the definition of the inversion of a mapping and the definition of $\tilde{g}$ we obtain $\hat{g}(x) = \tilde{g}(x)$. Hence, $\hat{g} = \tilde{g}$. Relation $\hat{g} = \mathcal{I}(\hat{f})$ implies $\hat{f} = \mathcal{I}(\hat{g})$. Hence relation $\hat{f} = \tilde{f}$ can be proved by interchanging the roles of $f$ and $g$. □

In the case of $f = g$ Proposition 3.3 has the following corollary:

**Corollary 3.1.** *Let $f : A \to H$, $f(0) = 0$. There exists a unique extension $\tilde{f} : A \to H$ of $f|_{A\cap D}$ such that $\tilde{f}$ is a fixed point of $\mathcal{I}$ (i.e., $\tilde{f} = \mathcal{I}(\tilde{f})$).*

It is easy to see that the inversion of mappings is linear, that if $T \in L(H, H)$ and $j : A \hookrightarrow H$ is the embedding of $A$ into $H$ then $\mathcal{I}(T \circ j) = T \circ j$ and that if $\|x\| \to +\infty$ then $i(x) \to 0$.

## 4. Scalar derivatives

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $G \subseteq H$ a set which contains at least one non-isolated point, $\widetilde{G} \subseteq H$ such that $G \subseteq \widetilde{G}$, $f : \widetilde{G} \to H$ and $x_0$ a non-isolated point of $G$. The following definition is an extension of Definition 2.2 [13]:

**Definition 4.1.** The limit

$$\underline{f}^{\#,G}(x_0) = \liminf_{\substack{x\to x_0\\ x\in G}} \frac{\langle f(x) - f(x_0), x - x_0\rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative* of $f$ at $x_0$ along $G$. Taking $\limsup$ in place of $\liminf$, we can define the upper scalar derivative $\overline{f}^{\#,G}(x_0)$ of $f$ at $x_0$ along $G$ similarly. If $G = \widetilde{G}$, then without confusion, we can shortly say lower scalar derivative and upper scalar derivative instead of lower scalar derivative along $G$ and upper scalar derivative along $G$, respectively. In this case, we omit $G$ from the superscript of the corresponding notations.

We have as follows:

**Lemma 4.1.** *Let $K \subseteq H$ be an unbounded set such that $0 \in K$ and $K \setminus \{0\}$ is an invariant set of the inversion $i$. Let $g : H \to H$. Then we have*

$$\liminf_{\substack{\|x\|\to\infty\\ x\in K}} \frac{\langle g(x), x\rangle}{\|x\|^2} = \underline{\mathcal{I}(g)}^{\#,K}(0).$$

**Proof.** Since $K \subseteq H$ is unbounded and $K \setminus \{0\}$ is an invariant set of $i$, 0 is a non-isolated point of $K$. Hence, $\underline{\mathcal{I}(g)}^{\#,K}(0)$ is well defined. Consider the global non-linear coordinate transformation $y = i(x)$. Then $x = i(y)$ and we have

$$\liminf_{\substack{\|x\|\to\infty\\ x\in K}} \frac{\langle g(x), x\rangle}{\|x\|^2} = \liminf_{\substack{y\to 0\\ y\in K}} \big\langle \mathcal{I}(g)(y), i(y)\big\rangle,$$

from where, by using the definition of the lower scalar derivative along a set, it follows easily the assertion of the lemma. □

## 5. Scalar asymptotic derivatives

Let $(E, \|\cdot\|)$ be a Banach space, $E^*$ the topological dual of $E$, $\langle E, E^*\rangle$ a duality between $E$ and $E^*$ with respect to a bilinear functional on $E \times E^*$ denoted by $\langle\cdot,\cdot\rangle$, $K \subseteq E$ an unbounded set, $\widetilde{K} \subseteq E$ such that $K \subseteq \widetilde{K}$ and $f : \widetilde{K} \to E^*$. The following definition is an extension of the notion of scalar asymptotic derivatives defined in [7]:

**Definition 5.1.** We say that $T \in L(E, E^*)$ is a scalar asymptotic derivative of $f$ along $K$ if

$$\limsup_{\substack{\|x\|\to+\infty\\ x\in K}} \frac{\langle x, f(x) - T(x)\rangle}{\|x\|^2} \leqslant 0.$$

The operator of Definition 5.1 will be denoted by $f'_{s,K}(\infty)$. If $K = \widetilde{K}$, we can shortly say scalar asymptotic derivative instead of scalar asymptotic derivative along $K$. In this case, we omit $K$ from the subscript of the corresponding notation. From now on, in this section we suppose that $E = H$, where $H$ is a Hilbert space, $K = \widetilde{K}$, $0 \in K$ and $K \setminus \{0\}$ is an invariant set of the inversion $i$. $E^*$ can be identified with $H$, the bilinear functional generating the duality between $E$ and $E^*$ with the scalar product of $H$, and $L(E, E^*)$ with $L(H)$. The following proposition follows easily either directly by Definition 5.1 or by Remark 6.1.

**Proposition 5.1.** *If $T$ is a scalar asymptotic derivative of $f$, then for any $c > 0$ the mapping $T + cI$ is also a scalar asymptotic derivative of $f$.*

**Theorem 5.1.** *$T \in L(H)$ is a scalar asymptotic derivative of $f$ iff the upper scalar derivative of $h$ in $0$ is non-positive (i.e., $\overline{h}^{\#}(0) \leqslant 0$) where $h : K \to H$, $h = \mathcal{I}(f - T \circ j) = \mathcal{I}(f) - T \circ j$ and $j : K \hookrightarrow E$ is the embedding of $K$ into $E$.*

**Proof.** We shall suppose that $T \in L(H)$ is a scalar asymptotic derivative of $f$ and prove that $\overline{h}^{\#}(0) \leqslant 0$. The converse implication can be proved similarly. Indeed, since $T \in L(H)$ is a scalar asymptotic derivative of $f$, we have that

$$\limsup_{\substack{\|x\| \to +\infty \\ x \in K}} \langle f(x) - T(x), i(x) \rangle \leqslant 0. \tag{4}$$

Consider the global non-linear coordinate transformation $y = i(x)$ given by the global diffeomorphism $i$. Since $K$ is unbounded and $K \setminus \{0\}$ is invariant under $i$, $0$ is a non-isolated point of $K$. Then, $x = i(y)$ and by (4)

$$\limsup_{\substack{y \to 0 \\ y \in K}} \big\langle (f \circ i)(y) - (T \circ j \circ i)(y), y \big\rangle \leqslant 0.$$

Hence,

$$\limsup_{\substack{y \to 0 \\ y \in K}} \big\langle \mathcal{I}(f)(y) - \mathcal{I}(T \circ j)(y), i(y) \big\rangle \leqslant 0.$$

Thus, by the definition of the upper scalar derivative we have $\overline{h}^{\#}(0) \leqslant 0$. □

**Corollary 5.1.** *$0$ is a scalar asymptotic derivative of $f$ iff $\overline{\mathcal{I}(f)}^{\#}(0) \leqslant 0$.*

The following theorem shows the surprising fact that every $f$ with finite upper scalar derivative in 0 is asymptotically scalarly differentiable.

**Theorem 5.2.** *If $\overline{\mathcal{I}(f)}^{\#}(0) < +\infty$, then $f$ is asymptotically scalarly differentiable and*

$$T = \overline{\mathcal{I}(f)}^{\#}(0) I$$

*is a scalar asymptotic derivative of $f$, where $I : H \to H$ is the identity operator.*

**Proof.** Indeed, $\bar{h}^{\#}(0) = 0$, where $h = \mathcal{I}(f) - T \circ j = \mathcal{I}(f) - \overline{\mathcal{I}(f)}^{\#}(0)(I \circ j)$. Hence, the result follows by using Theorem 5.1. □

The following remark follows easily by using Proposition 5.1.

**Remark 5.1.** Every operator $cI$ is a scalar asymptotic derivative of $f$ where $c \geqslant \overline{\mathcal{I}(f)}^{\#}(0)$ is a constant.

## 6. Properties

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\|\cdot\|$ the norm generated by $\langle\cdot,\cdot\rangle$ and $f : H \to H$. Recall the following notion [10]:

**Definition 6.1.** $f$ is called *$\psi$-additive* if there exist $\theta \geqslant 0$ and a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty}(\psi(t)/t) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\big(\psi(\|x\|) + \psi(\|y\|)\big),$$

for all $x, y \in H$.

**Theorem 6.1.** *Suppose that $f(tx)$ is continuous in $t$ for each fixed $x$. If $f$ is $\psi$-additive and $\psi$ satisfies*

(1) $\psi(ts) \leqslant \psi(t)\psi(s)$, *for all* $t, s \in \mathbb{R}_+$,
(2) $\psi(t) < t$, *for all* $t > 1$,

*then there exist a linear mapping $T : H \to H$ such that*

$$\big|\langle f(x) - T(x), x\rangle\big| \leqslant \frac{2\theta\psi(\|x\|)\|x\|}{2 - \psi(2)}, \tag{5}$$

*for all $x \in H$. $S$ is another linear mapping satisfying* (5) *iff $T - S$ is skew-adjoint.*

**Proof.** By Theorem 1 [10] there exists a unique linear mapping $T$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\theta\psi(\|x\|)}{2 - \psi(2)}, \tag{6}$$

for all $x \in H$. Moreover, by [10] $T(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$, for all $x \in H$. Hence, by using the Cauchy inequality in (6), we obtain (5). Suppose that $S$ is another linear mapping satisfying (5). Hence,

$$\left|\langle T(x)-S(x),x\rangle\right| \leqslant \left|\langle T(x)-f(x),x\rangle\right| + \left|\langle f(x)-S(x),x\rangle\right| \leqslant \frac{4\theta\psi(\|x\|)\|x\|}{2-\psi(2)}.$$

Then,

$$\left|\langle T(x)-S(x),x\rangle\right| = \left|\left\langle \frac{1}{n}T(nx)-\frac{1}{n}S(nx),x\right\rangle\right| \leqslant \frac{\psi(n)}{n}\frac{4\theta\psi(\|x\|)\|x\|}{2-\psi(2)}.$$

Since $\lim_{n\to\infty}(\psi(n)/n)=0$, we obtain that $\langle T(x)-S(x),x\rangle=0$. Thus, $T-S$ is skew-adjoint. Conversely, if $T-S$ is skew-adjoint, then $\langle T(x)-S(x),x\rangle=0$. Hence,

$$\begin{aligned}\left|\langle f(x)-S(x),x\rangle\right| &\leqslant \left|\langle f(x)-T(x),x\rangle\right| + \left|\langle T(x)-S(x),x\rangle\right| \\ &= \left|\langle f(x)-T(x),x\rangle\right| \leqslant \frac{2\theta\psi(\|x\|)\|x\|}{2-\psi(2)}. \qquad \square\end{aligned}$$

**Remark 6.1.** By the definition of the scalar asymptotic derivative, it follows easily that, if $U$ is a scalar asymptotic derivative of $f$ and $g: H\to H$ satisfies the relation

$$\langle g(x),x\rangle \leqslant 0, \tag{7}$$

for all $x\in H$, then $U$ is also a scalar asymptotic derivative of $f+g$. Particularly, for any skew-adjoint mapping $Z$, the mapping $U$ is a scalar asymptotic derivative of $f+Z$, or equivalently $U+Z$ is a scalar asymptotic derivative of $f$. Moreover, for any $P$ continuous linear positive semidefinite operator, $U+P$ is also a scalar asymptotic derivative of $f$. An example for a non-linear mapping $g$ satisfying (7) is $g: R^3\to R^3$:

$$g(u,v,w)=(-u+vw,-v+uw,-w-2uv).$$

It would be interesting to study the properties of mappings satisfying the condition (7). Of course, 0 is a scalar asymptotic derivative of these mappings.

**Remark 6.2.** By the Cauchy inequality it follows easily that every asymptotic derivative of $f$ is a scalar asymptotic derivative of $f$. However, the converse is not true. Indeed, it can be easily checked that if $f: R^3\to R^3$:

$$f(u,v,w)=(vw,uw,-2uv),$$

then 0 is a scalar asymptotic derivative of $f$ but it is not an asymptotic derivative of $f$.

**Remark 6.3.** Every continuous operator $S$ satisfying (5) is a scalar asymptotic derivative of $f$. Indeed, we have

$$\limsup_{\|x\|\to+\infty}\frac{\langle f(x)-T(x),x\rangle}{\|x\|^2} \leqslant \frac{2\theta}{2-\psi(2)}\lim_{\|x\|\to+\infty}\frac{\psi(\|x\|)}{\|x\|}=0.$$

## 7. Applications

### *7.1. Fixed point theorems*

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $K\subseteq H$ a generating closed pointed convex cone and $f: K\to K$. If in Theorem 3.1 proved in [7] we replace assumptions 1 and 2 by "1. $f$ is completely continuous" we obtain as follows:

**Theorem 7.1.** *If the following assumptions are satisfied*:

(1) $f$ *is completely continuous,*
(2) *there exists a scalarly differentiable mappings* $f_0 : K \to H$ *such that* $f_0 : K \to H$, $f \leqslant_{K^*} f_0$ *and* $\|f_s'(\infty)\| < 1$,

*then* $f$ *has a fixed point.*

By Theorem 7.1 and Theorem 5.2 we have the following fixed point theorem:

**Theorem 7.2.** *If the following assumptions are satisfied*:

(1) $f$ *is completely continuous,*
(2) *there exists a mapping* $f_0 : K \to H$ *such that* $f \leqslant_{K^*} f_0$ *and* $\overline{\mathcal{I}(f_0)}^{\#}(0) < 1$,

*then* $f$ *has a fixed point.*

**Proof.** By Theorem 5.2 the linear operator $T = \overline{\mathcal{I}(f_0)}^{\#}(0)I$ is a scalar asymptotic derivative of $f_0$. We have $\|T\| = |\overline{\mathcal{I}(f_0)}^{\#}(0)|$. We consider two cases:

(1) $\overline{\mathcal{I}(f_0)}^{\#}(0) \leqslant 0$. In this case choose a $c \in ]-1, 0] \cap [\overline{\mathcal{I}(f_0)}^{\#}(0), +\infty[$. By Remark 5.1, $T = cI$ is a scalar asymptotic derivative of $f_0$ with $\|T\| = -c < 1$.
(2) $0 < \overline{\mathcal{I}(f_0)}^{\#}(0) < 1$. In this case $\|T\| = \overline{\mathcal{I}(f_0)}^{\#}(0) < 1$.

It follows that $\|T\| < 1$. By using Theorem 7.1, $f$ has a fixed point. □

**Corollary 7.1.** *If the following assumptions are satisfied*:

(1) $f$ *is completely continuous,*
(2) $\overline{\mathcal{I}(f)}^{\#}(0) < 1$,

*then* $f$ *has a fixed point.*

Corollary 7.1 has the following interesting consequence:

**Proposition 7.1.** *Let* $q : K \to K$ *be a completely continuous mapping such that* $I \leqslant_K q$ *and* $f : K \to K$, $f = q - I$. *Then,* $\overline{\mathcal{I}(f)}^{\#}(0) \geqslant 0$.

**Proof.** Suppose that $\overline{\mathcal{I}(f)}^{\#}(0) < 0$. Since $K$ is generating $K \neq \{0\}$. Let $a \in K \setminus \{0\}$. Since $K + K \subseteq K$, $x + f(x) + a \in K$ for all $x \in K$. Define $q_a : K \to K$ by $q_a(x) = x + f(x) + a$. Since $q_a = q + a$, $q_a$ is completely continuous. We also have $\overline{\mathcal{I}(q_a)}^{\#}(0) = 1 + \overline{\mathcal{I}(f)}^{\#}(0) < 1$. Hence, by Corollary 7.1, $q_a$ has a fixed point, that is the equation $f(x) = -a$ has a solution. It follows that $a \in -K$. Since $K \cap (-K) = \{0\}$, it follows that $a = 0$. But this is in contradiction with $a \in K \setminus \{0\}$. Hence, $\overline{\mathcal{I}(f)}^{\#}(0) \geqslant 0$. □

*7.2. Surjectivity theorems*

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subseteq H$ a generating closed pointed convex cone and $f : K \to K$.

**Theorem 7.3.** *If the following assumptions are satisfied*:

(1) $f = I - q$, *where* $q : K \to K$ *is completely continuous and* $q \leqslant_K I$,
(2) *There exists a mapping* $f_0 : K \to H$ *such that* $f_0 \leqslant_{K^*} f$ *and* $\underline{\mathcal{I}(f_0)}^{\#}(0) > 0$,

*then* $f$ *is surjective.*

**Proof.** Let $y \in K$ arbitrary but fixed. Define the mapping $q_{y,0} : K \to H$ by $q_{y,0} = x - f_0(x) + y$. Since $K + K \subseteq K$, $x - f(x) + y = q(x) + y \in K$ for all $x \in K$. Define the mapping $q_y : K \to K$ by $q_y(x) = x - f(x) + y$. It is easy to see that $q_y$ is completely continuous, $q_y \leqslant_{K^*} q_{y,0}$ and

$$\overline{\mathcal{I}(q_{y,0})}^{\#}(0) = 1 - \underline{\mathcal{I}(f_0)}^{\#}(0) < 1.$$

Hence, by Theorem 7.2, $q_y$ has a fixed point, that is the equation $f(x) = y$ has a solution. Since $y$ was arbitrarily chosen, $f$ is surjective. □

**Corollary 7.2.** *If the following assumptions are satisfied*:

(1) $f = I - q$, *where* $q : K \to K$ *is completely continuous and* $q \leqslant_K I$,
(2) $\underline{\mathcal{I}(f)}^{\#}(0) > 0$,

*then* $f$ *is surjective.*

**Theorem 7.4.** *If the following assumptions are satisfied*:

(1) $f = bI - q$, *where* $b > 0$, $q : K \to K$ *is completely continuous and* $q \leqslant_K bI$,
(2) *there exists a mapping* $f_0 : K \to H$ *such that* $f_0 \leqslant_{K^*} f$ *and* $\underline{\mathcal{I}(f_0)}^{\#}(0) > 0$,

*then* $f$ *is surjective.*

**Proof.** By using Theorem 7.3 with $(1/b)f_0$, $(1/b)f$ and $(1/b)q$ replacing $f_0$, $f$ and $q$, respectively, we obtain that $(1/b)f$ is surjective. Hence, $f$ is also surjective. □

**Corollary 7.3.** *If the following assumptions are satisfied*:

(1) $f = bI - q$, *where* $b > 0$, $q : K \to K$ *is completely continuous and* $q \leqslant_K bI$,
(2) $\underline{\mathcal{I}(f)}^{\#}(0) > 0$,

*then* $f$ *is surjective.*

**Lemma 7.1.** *Let $A \subseteq H$ such that $A \setminus \{0\}$ is an invariant set of the inversion $i$ and $\Upsilon = \{\tau \mid \tau : A \to H\}$. The inversion of mappings $\mathcal{I}$ is $K^*$-monotone on $\Upsilon$, i.e., $\mathcal{I}(\tau_1) \leqslant_{K^*} \mathcal{I}(\tau_2)$, for all $\tau_1, \tau_2 : A \to H$ with $\tau_1 \leqslant_{K^*} \tau_2$.*

**Proof.** Let $\tau_1, \tau_2 : A \to H$ such that $\tau_1 \leqslant_{K^*} \tau_2$. We have to prove that

$$\big\langle \mathcal{I}(\tau_1)(x) - \mathcal{I}(\tau_2)(x), y \big\rangle \geqslant 0, \tag{8}$$

for all $x \in A$ and $y \in K$. For $x = 0$ the inequality is trivial. Suppose that $x \neq 0$. Since $A \setminus \{0\}$ is an invariant set of $i$, $i(x) \in A$. By the inequality $\tau_1 \leqslant_{K^*} \tau_2$, we have

$$\big\langle \tau_1\big(i(x)\big) - \tau_2\big(i(x)\big), y \big\rangle \geqslant 0. \tag{9}$$

Multiplying inequality (9) by $\|x\|^2$, we obtain the required inequality (8). □

We remark that it is easy to see that $\mathcal{I}$ is also $K$-monotone on $\Upsilon$.

**Proposition 7.2.** *If there exist $a, b \in \mathbb{R}$ with $0 < a \leqslant b$ and $q : K \to K$ completely continuous with $q \leqslant_K bI$, such that $f = bI - q$ and*

$$aI \leqslant_{K^*} f, \tag{10}$$

*for all $x \in K$, then $f$ is surjective.*

**Proof.** We shall use Corollary 7.3. The first assumption of Corollary 7.3 is obviously satisfied. It remains to prove that $\underline{\mathcal{I}(f)}^{\#}(0) > 0$. By inequality (10) and Lemma 7.1 with $A = K$, we have

$$ax \leqslant_{K^*} \mathcal{I}(f)(x), \tag{11}$$

for all $x \in K \setminus \{0\}$. Since $K \setminus \{0\}$ is invariant under $i$, we also have $i(x) \in K$. Hence, multiplying scalarly inequality (11) by $i(x)$, we obtain

$$\big\langle \mathcal{I}(f)(x), i(x) \big\rangle \geqslant a. \tag{12}$$

Tending with $x$ to 0 in (12) it yields

$$\underline{\mathcal{I}(f)}^{\#}(0) \geqslant a > 0. \qquad \square$$

**Corollary 7.4.** *Consider the case when $H = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, where*

$$\mathbb{R}^n_+ = \big\{x = (x_1, \dots, x_n) \mid x_i \geqslant 0 \text{ for all } i = 1, \dots, n\big\}$$

*is the non-negative orthant of $\mathbb{R}^n$. If $f$ is continuous and there exist $a, b \in \mathbb{R}$, such that $0 < a \leqslant b$ and*

$$aI \leqslant_K f \leqslant_K bI, \tag{13}$$

*then $f$ is surjective.*

**Proof.** It is easy to see that $K = K^*$. Hence, Corollary 7.4 is a straightforward consequence of Proposition 7.2. □

We remark that Corollary 7.4 remains true for the subcones[2] of the orthants and their images through orthogonal transformations.[3] For these cones we have $K \subseteq K^*$ and therefore we can apply Proposition 7.2.

**Example.** Let $H = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $a, b \in \mathbb{R}$, $0 < a \leqslant b$ and $\alpha, \beta : \mathbb{R}^2_+ \to [a, b]$ two arbitrary continuous mappings. Define $f : K \to K$ by the relation

$$f(x_1, x_2) = \big(\alpha(x_1, x_2)x_1, \beta(x_1, x_2)x_2\big),$$

for every $x = (x_1, x_2) \in \mathbb{R}^2_+$. It is easy to see that the conditions of Corollary 7.4 are satisfied. Hence, $f$ is surjective.

### 7.3. Integral equations

Let $\Omega \subseteq \mathbb{R}$ be a bounded open set, $L^2(\Omega)$ the set of functions on $\Omega$ whose square is integrable on $\Omega$ and

$$L^2_+(\Omega) = \big\{u \in L^2(\Omega) \mid u(t) \geqslant 0 \text{ for almost all } t \in \Omega\big\}.$$

$L^2(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle = \int_\Omega u(s)v(s)\,ds$$

and $L^2_+(\Omega)$ is a generating closed convex pointed cone of $L^2(\Omega)$. Let $\mathcal{L} : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $\mathcal{K} : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ and $\mathcal{F} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. Denote by $\mathcal{I}_3$ and $\mathcal{I}_2$ the inversions with respect to the third and second variable, respectively. We recall the following definition and result [17]:

**Definition 7.1.** We say that $\mathcal{L}$ is a *Charatheodory* function if $\mathcal{L}(s, t, u)$ is continuous with respect to $u$ for almost all $(s, t) \in \overline{\Omega} \times \overline{\Omega}$ and is measurable in $(s, t)$ for each $u \in \mathbb{R}$.

**Theorem 7.5.** *If the following conditions are satisfied*:

(1) *$\mathcal{L}$ is a Charatheodory function,*
(2) $|\mathcal{L}(s, t, u)| \leqslant \mathcal{R}(s, t)(a + b|u|)$ *for almost all* $s, t \in \Omega$, $\forall u \in \mathbb{R}$, *where* $a, b > 0$ *and* $\mathcal{R} \in L^2(\Omega \times \Omega)$,
(3) *for any* $\alpha > 0$ *the function* $\mathcal{R}_\alpha(s, t) = \max_{|u| \leqslant \alpha} |\mathcal{L}(s, t, u)|$ *is sumable with respect to* $t$ *for almost all* $s \in \Omega$,
(4) *for any* $\alpha > 0$,

$$\lim_{\operatorname{mes}(D) \to 0} \sup_{|u| \leqslant \alpha} \left\| \mathcal{P}_D \int_\Omega \mathcal{L}\big(s, t, u(t)\big)\,dt \right\|_{L^2(\Omega)} = 0,$$

*where* $\operatorname{mes}(D)$ *is the Lebesgue measure of* $D$ *and* $\mathcal{P}_D$ *is the operator of multiplication by the characteristic function of the set* $D \subseteq \Omega$,

[2] A *subcone* of a cone $K$ is a subset of $K$ which is a cone.
[3] A linear transformation of $\mathbb{R}^n$ is called *orthogonal* if it is non-singular and the transpose of its matrix is equal to the inverse of its matrix.

(5) *for any* $\beta > 0$,

$$\lim_{\mathrm{mes}(D)\to 0} \sup_{\|u\|_{L^2(\Omega)}\leqslant\beta} \left\| \int_\Omega \mathcal{L}\big(s,t,u(t)\big)\,dt \right\|_{L^2(\Omega)} = 0,$$

*then the operator*

$$\mathcal{A}(u)(s) = \int_\Omega \mathcal{L}\big(s,t,u(t)\big)\,dt$$

*is a completely continuous operator from* $L^2(\Omega)$ *into* $L^2(\Omega)$.

Since the integral of an almost everywhere non-negative function is non-negative, by Theorem 7.5 we have as follows:

**Corollary 7.5.** *If conditions* (1)–(5) *of Theorem* 7.5 *and condition*

(6) $\mathcal{L}(s,t,u) \geqslant 0$ *for all* $u \in \mathbb{R} \cap [0,+\infty[$, *for all* $s \in \Omega$ *and for almost all* $t \in \Omega$

*are satisfied, then the operator*

$$\mathcal{A}(u)(s) = \int_\Omega \mathcal{L}\big(s,t,u(t)\big)\,dt$$

*is a completely continuous operator from* $L^2_+(\Omega)$ *into* $L^2_+(\Omega)$.

By using Corollary 7.1, Corollary 7.5, Theorem 7.5 and the definition of the upper scalar derivative it can be shown as follows:

**Theorem 7.6.** *If conditions* (1)–(6) *of Corollary* 7.5 *and condition*

(7) $\exists \varepsilon, \delta > 0$ *such that*

$$\frac{\mathcal{I}_3(\mathcal{L})(s,t,u) - \mathcal{I}_3(\mathcal{L})(s,t,0)}{u} \leqslant 1 - \delta,$$

*for almost all* $s, t \in \Omega$ *and for all* $u \in [-\varepsilon, \varepsilon] \cap \mathbb{R}$

*are satisfied, then the integral equation*

$$u(s) = \int_\Omega \mathcal{L}\big(s,t,u(t)\big)\,dt$$

*has a solution* $u \in L^2_+(\Omega)$.

**Proof.** Consider the integral operator $\mathcal{A}$ defined by the relation

$$\mathcal{A}(u)(s) = \int_\Omega \mathcal{L}\big(s,t,u(t)\big)\,dt.$$

By Corollary 7.5, $\mathcal{A}$ is a completely continuous operator from $L^2_+(\Omega)$ into $L^2_+(\Omega)$. It is easy to see that

$$\mathcal{I}(\mathcal{A})(u)(s) = \int_\Omega \mathcal{I}_3(\mathcal{L})\big(s,t,u(t)\big)\,dt. \tag{14}$$

By (14)

$$\frac{\langle \mathcal{I}(\mathcal{A})(u) - \mathcal{I}(\mathcal{A})(0), u\rangle}{\|u\|^2} = \frac{\int_\Omega\int_\Omega (\mathcal{I}_3(\mathcal{L})(s,t,u(t)) - \mathcal{I}_3(\mathcal{L})(s,t,0))u(s)\,ds\,dt}{\int_\Omega u^2(s)\,ds}$$
$$= \frac{\int_\Omega\int_\Omega \frac{(\mathcal{I}_3(\mathcal{L})(s,t,u(t)) - \mathcal{I}_3(\mathcal{L})(s,t,0))}{u(t)} u(s)u(t)\,ds\,dt}{\int_\Omega u^2(s)\,ds}.$$

By the Cauchy inequality

$$\int_\Omega\int_\Omega u(s)u(t)\,ds\,dt = \left(\int_\Omega u(s)\,ds\right)^2 \leqslant \int_\Omega u^2(s)\,ds. \tag{15}$$

By using (15) and the definition of the upper scalar derivative, we have $\overline{\mathcal{I}(\mathcal{A})}^{\#}(0) < 1$, if (6) holds. Hence, Theorem 7.6 is a consequence of Corollary 7.1 and Theorem 7.5. □

**Corollary 7.6.** *If conditions* (1)–(6) *of Corollary* 7.5 *with* $\mathcal{K}(s,t)\mathcal{F}(t,u)$ *in place of* $\mathcal{L}(s,t,u)$ *and condition*

(7) $\exists \varepsilon, \delta > 0$ *such that*

$$\mathcal{K}(s,t)\frac{\mathcal{I}_2(\mathcal{F})(t,u) - \mathcal{I}_2(\mathcal{F})(t,0)}{u} \leqslant 1-\delta,$$

*for almost all* $s,t \in \Omega$ *and all* $u \in [-\varepsilon,\varepsilon] \cap \mathbb{R}$

*are satisfied, then the integral equation*

$$u(s) = \int_\Omega \mathcal{K}(s,t)\mathcal{F}\big(t,u(t)\big)\,dt$$

*has a solution* $u \in L^2_+(\Omega)$.

By using Corollary 7.2 it can be proved similarly to Theorem 7.6 and Corollary 7.6 as follows:

**Theorem 7.7.** *If conditions* (1)–(6) *of Corollary* 7.5 *with*

$$\frac{1}{\operatorname{mes}(\Omega)}u - \mathcal{L}(s,t,u)$$

*in place of* $\mathcal{L}(s,t,u)$ *and condition*

(7) $\exists \varepsilon, \delta > 0$ *such that*

$$\frac{\mathcal{I}_3(\mathcal{L})(s,t,u) - \mathcal{I}_3(\mathcal{L})(s,t,0)}{u} \geqslant \delta,$$

*for almost all* $s, t \in \Omega$ *and all* $u \in [-\varepsilon, \varepsilon] \cap \mathbb{R}$

*are satisfied, then the integral equation*

$$v(s) = \int_\Omega \mathcal{L}\big(s, t, u(t)\big)\, dt$$

*has a solution* $u \in L^2_+(\Omega)$ *for every* $v \in L^2_+(\Omega)$.

**Corollary 7.7.** *If conditions* (1)–(6) *of Corollary* 7.5 *with*

$$\frac{1}{\operatorname{mes}(\Omega)} u - \mathcal{K}(s,t)\mathcal{F}(t,u)$$

*in place of* $\mathcal{L}(s,t,u)$ *and condition*

(7) $\exists \varepsilon, \delta > 0$ *such that*

$$\mathcal{K}(s,t)\frac{\mathcal{I}_2(\mathcal{F})(t,u) - \mathcal{I}_2(\mathcal{F})(t,0)}{u} \geqslant \delta,$$

*for almost all* $s, t \in \Omega$ *and all* $u \in [-\varepsilon, \varepsilon] \cap \mathbb{R}$

*are satisfied, then the integral equation*

$$v(s) = \int_\Omega \mathcal{K}(s,t)\mathcal{F}\big(t, u(t)\big)\, dt$$

*has a solution* $u \in L^2_+(\Omega)$ *for every* $v \in L^2_+(\Omega)$.

### *7.4. Variational inequalities and complementarity problems*

Let $(E, \|\cdot\|)$ be a Banach space, $E^*$ the topological dual of $E$, $\langle E, E^*\rangle$ a duality between $E$ and $E^*$ and $\langle \cdot, \cdot \rangle$ the bilinear mapping which defines the duality $\langle E, E^*\rangle$.

**Lemma 7.2.** *If* $\{x_n\}_{n\in\mathbb{N}} \subseteq E$, $\{y_n\}_{n\in\mathbb{N}} \subseteq E^*$ *are sequences such that* $\{x_n\}_{n\in\mathbb{N}}$ *is weakly convergent to* $x_* \in E$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *is strongly convergent to* $y_* \in E^*$, *then* $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x_*, y_*\rangle$.

**Proof.** The lemma is a consequence of the following formula:

$$\langle x_n, y_n\rangle - \langle x_*, y_*\rangle = \langle x_n - x_*, y_n - y_*\rangle + \langle x_*, y_n\rangle + \langle x_n, y_*\rangle - 2\langle x_*, y_*\rangle. \qquad \square$$

We recall the following classical results:

**Theorem 7.8** (Eberlein–Šmulian). *A set $M \subseteq E$ is relatively weakly compact iff every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $M$ has a weakly convergent subsequence.*

**Proof.** For a proof of this theorem the reader is referred to [16]. □

**Proposition 7.3.** *Any closed ball in $E^*$ is $\sigma(E^*, E)$-compact.*

**Proof.** This proposition is Proposition 1 in [3, Chapter IV, p. 112]. □

Recall the following definition [9]:

**Definition 7.2.** We say that a mapping $T_1 : E \to E^*$ satisfies condition $(S)^1_+$ if any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ with the following properties:

(1) $\{x_n\}_{n\in\mathbb{N}}$ is $\sigma(E, E^*)$-convergent to $x_* \in E$,
(2) $\{T_1(x_n)\}_{n\in\mathbb{N}}$ is $\sigma(E^*, E)$-convergent to $u_* \in E^*$,
(3) $\limsup_{n\to\infty}\langle x_n, T_1(x_n)\rangle \leqslant \langle x_*, u_*\rangle$

has a subsequence convergent to $x_*$.

**Remark 7.1.** Examples of mappings satisfying condition $(S)^1_+$ are given in [9].

**Definition 7.3.** We say that a mapping $T_2 : E \to E^*$ is demicompletely continuous if the following conditions are satisfied:

(1) $T_2$ is continuous,
(2) for every weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$, a strongly convergent subsequence exists in $\{T_2(x_n)\}_{n\in\mathbb{N}}$.

**Remark 7.2.** If $E$ is a reflexive Banach space, then demicomplete continuity and complete continuity are equivalent. However, if $E$ is a non-reflexive Banach space, then this fact is not true.

In this section we shall give some application to variational inequalities and in particular to complementarity problems.

Given a mapping $f : E \to E^*$ and a closed convex set $D \subseteq E$ the *variational inequality* defined by $f$ and $D$ is the following problem:

$$VI(f, D): \quad \text{find } x_* \in D \text{ such that } \langle f(x_*), x - x_*\rangle \geqslant 0, \text{ for all } x \in D.$$

If in particular the set $D = K$ where $K$ is a closed convex cone in $E$, and the dual cone of $K$ is $K^*$, then in this case it is known [6,8] that the problem $VI(f, K)$ is equivalent to the following *non-linear complementarity problem*

$$NCP(f, K): \quad \text{find } x_* \in K \text{ such that } f(x_*) \in K^* \text{ and } \langle x_*, f(x_*)\rangle = 0.$$

The theory of variational inequalities is one of the most popular domains of applied mathematics [2,12].

The complementarity theory is a relatively new domain of applied mathematics with many application in economics, optimization, game theory, mechanics, engineering, etc. [5,6,8,9].

**Theorem 7.9.** *Let $T_1, T_2 : E \to E^*$ be two mappings. If the following assumptions are satisfied*:

(1) *$T_1$ is continuous, bounded* (*i.e., for any bounded set $B \subseteq E$, $T(B)$ is bounded*) *and satisfies condition $(S)^1_+$,*
(2) *$T_2$ is demicompletely continuous,*

*then, for every weakly compact non-empty convex set $D \subseteq E$, the variational inequality $VI(T_1 - T_2, D)$ has a solution.*

**Proof.** Let $\Lambda$ be the family of all finite dimensional subspaces $F$ of $E$ such that $F \cap D$ is non-empty. Consider the family $\Lambda$ ordered by inclusion. Denote by $f(x) = T_1(x) - T_2(x)$ for all $x \in D$ and by $D(F) = F \cap D$, for each $F \in \Lambda$. For each $F \in \Lambda$ we define

$$A_F := \{ y \in D \mid \langle x - y, f(y) \rangle \geqslant 0 \quad \text{for all } x \in D(F) \}.$$

For each $F \in \Lambda$ the set $A_F$ is non-empty. Indeed, to show this it is sufficient to show that the problem $VI(f, D(F))$ has a solution (since the solution set of the problem $VI(f, D(F))$ is a subset of $A_F$). We show now that the solution set of the problem $VI(f, D(F))$ is non-empty. Indeed, let $j : F \to E$ denote the inclusion and $j^* : E^* \to F^*$ the adjoint (transpose) of $j$. By our assumption we have that the mapping

$$j^* \circ f \circ j : D(F) \to F^*$$

is continuous and

$$\langle x - y, (j^* \circ f \circ j)(y) \rangle = \langle j(x - y), (f \circ j)(y) \rangle = \langle x - y, f(y) \rangle,$$

for all $x, y \in D(F)$. Applying the classical Hartman–Stampacchia theorem [6] to the mapping $j^* \circ f \circ j$ and the set $D(F)$ we obtain that the problem $VI(f, D(F))$ has a solution. So, for any $F \in \Lambda$, the set $A_F$ is non-empty. Denote by $\bar{A}^\sigma_F$ the weak closure of $A_F$. We have that $\bigcap_{F \in \Lambda} \bar{A}^\sigma_F \neq 0$. Indeed, let $\bar{A}^\sigma_{F_1}, \bar{A}^\sigma_{F_2}, \dots, \bar{A}^\sigma_{F_n}$ be a finite subfamily of the family $\{\bar{A}^\sigma_F\}_{F \in \Lambda}$. Let $F_0$ be the finite dimensional subspace in $E$ generated by $F_1, F_2, \dots, F_n$. Because $F_k \subseteq F_0$ for all $k = 1, 2, \dots, n$, we have that $D(F_k) \subseteq D(F_0)$ for all $k = 1, 2, \dots, n$. We have $A_{F_0} \subseteq A_{F_k}$, which implies $\bar{A}^\sigma_{F_0} \subseteq \bar{A}^\sigma_{F_k}$ for all $k = 1, 2, \dots, n$, and finally we have that $\bigcap_{k=1}^n \bar{A}^\sigma_{F_k} \neq 0$. Since $D$ is weakly compact we conclude that $\bigcap_{F \in \Lambda} \bar{A}^\sigma_F \neq 0$. Let $y_* \in \bigcap_{F \in \Lambda} \bar{A}^\sigma_F$, i.e., for every $F \in \Lambda$, $y_* \in \bar{A}^\sigma_F$. Let $x \in D$ be an arbitrary element. There exists some $F \in \Lambda$ such that $x, y_* \in F$. Since $y_* \in \bar{A}^\sigma_F$, there exists a net $\{y_j\} \subseteq A_F$ such that $\{y_j\}$ is weakly convergent to $y_*$. By Theorem 7.8, we can suppose that the net $\{y_j\}$ is a sequence $\{y_n\}_{n \in \mathbb{N}}$ weakly convergent to $y_*$. We have

$$\langle y_* - y_n, f(y_n) \rangle \geqslant 0 \quad \text{and} \quad \langle x - y_n, f(y_n) \rangle \geqslant 0,$$

or

$$\langle y_n - y_*, T_1(y_n) \rangle \leqslant \langle y_n - y_*, T_2(y_n) \rangle \tag{16}$$

and

$$\langle x - y_n, T_1(y_n)\rangle \geqslant \langle x - y_n, T_2(y_n)\rangle. \tag{17}$$

By assumption (2) there exists a subsequence of $\{T_2(y_n)\}_{n\in\mathbb{N}}$, denoted again by $\{T_2(y_n)\}_{n\in\mathbb{N}}$, strongly convergent to an element $u_0 \in E^*$. From formula (16) and considering Lemma 7.2 we have

$$\limsup_{n\to\infty}\langle y_n - y_*, T_1(y_n)\rangle \leqslant 0. \tag{18}$$

Because $T_1$ is bounded and considering Proposition 7.3, we can suppose (taking eventually a subsequence of $\{y_n\}_{n\in\mathbb{N}}$) that $\{T_1(y_n)\}_{n\in\mathbb{N}}$ is weakly convergent to an element $v_0 \in E^*$. Because

$$\langle y_n, T_1(y_n)\rangle = \langle y_n - y_*, T_1(y_n)\rangle + \langle y_*, T_1(y_n)\rangle,$$

and considering formula (18), we obtain

$$\limsup_{n\to\infty}\langle y_n, T_1(y_n)\rangle \leqslant \langle y_*, v_0\rangle.$$

Hence by condition $(S)^1_+$ we obtain that the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a subsequence, denoted again by $\{y_n\}_{n\in\mathbb{N}}$, strongly convergent to $y_*$. By assumption (2) we must have $T_2(y_*) = u_0$. From inequality (17) we obtain $\langle x - y_*, T_1(y_*) - T_2(y_*)\rangle \geqslant 0$ for all $x \in D$, and the proof is complete. □

For every $n \in \mathbb{N}$, we denote by

$$B(0,n) = \{x \in E \mid \|x\| \leqslant n\}.$$

**Definition 7.4.** We say that a non-empty subset $K$ of $E$ is a weakly Lindelöf set if the following properties are satisfied:

(1) $K$ is a closed convex unbounded set,
(2) for any $n \in \mathbb{N}$ such that $D_n = B(0,n) \cap K$ is non-empty, we have that $D_n$ is a weakly compact set.

*Examples for Lindelöf sets*:

(1) Any closed convex unbounded set in a reflexive Banach space.
(2) Any closed pointed convex cone with a weakly compact base in an arbitrary Banach space.
(3) Any closed convex unbounded subset of a closed pointed convex cone $K$ generated by a weakly compact convex set $D$ with $0 \notin D$.

**Theorem 7.10.** *Let $K \subseteq E$ be a weakly Lindelöf subset and $T_1, T_2 : E \to E^*$ two mappings. If the following assumptions are satisfied*:

(1) *$T_1$ is continuous bounded and satisfies condition $(S)^1_+$*,
(2) *$T_2$ is demicompletely continuous,*

(3) *there exists a real number* $c > 0$ *such that*

$$c \leqslant \liminf_{\substack{\|x\| \to \infty \\ x \in K}} \frac{\langle x, T_1(x) \rangle}{\|x\|^2},$$

(4) $T_2$ *has a scalar asymptotic derivative* $T'_{2,s,K}(\infty)$ *along* $K$ *such that* $\|T'_{2,s,K}(\infty)\| < c$,

*then the problem* $VI(T_1 - T_2, K)$ *has a solution.*

**Proof.** We may suppose that for any $n \in \mathbb{N}$, $D_n = B(0, n) \cap K$ is non-empty. We have $K = \bigcup_{n=1}^{\infty} D_n$. For each $n \in \mathbb{N}$, $D_n$ is weakly compact and convex. By Theorem 7.9 the problem $VI(T_1 - T_2, D_n)$ has a solution $y_n \in D_n$ for every $n \in \mathbb{N}$. Therefore we have

$$\langle x - y_n, (T_1 - T_2)(y_n) \rangle \geqslant 0 \quad \text{for all } x \in D_n. \tag{19}$$

If in (19) we put $x = 0$, we obtain

$$\langle y_n, T_1(y_n) \rangle \leqslant \langle y_n, T_2(y_n) \rangle.$$

The sequence $\{y_n\}_{n \in \mathbb{N}}$ is bounded. Indeed, if we suppose that $\|y_n\| \to \infty$ as $n \to \infty$, then by assumptions (3) and (4) we have (supposing that $\|y_n\| \neq 0$ for all $n \in \mathbb{N}$)

$$\begin{aligned}
c &\leqslant \liminf_{\|y_n\| \to \infty} \frac{\langle y_n, T_1(y_n) \rangle}{\|y_n\|^2} \leqslant \liminf_{\|y_n\| \to \infty} \frac{\langle y_n, T_2(y_n) \rangle}{\|y_n\|^2} \\
&\leqslant \limsup_{\|y_n\| \to \infty} \frac{\langle y_n, T_2(y_n) - T_{2,s}(\infty)(y_n) \rangle}{\|y_n\|^2} + \limsup_{\|y_n\| \to \infty} \frac{\langle y_n, T_{2,s}(\infty)(y_n) \rangle}{\|y_n\|^2} \\
&\leqslant \|T_{2,s}(\infty)\|^2 < c,
\end{aligned}$$

which is a contradiction. Therefore we conclude that $\{y_n\}_{n \in \mathbb{N}}$ is a bounded sequence. Hence, there exists $m \in \mathbb{N}$ such that $\{y_n\} \subseteq D_m$. Because $D_m$ is weakly compact, by Theorem 7.8, we have that $\{y_n\}_{n \in \mathbb{N}}$ has a subsequence, denoted again by $\{y_n\}_{n \in \mathbb{N}}$, weakly convergent to an element $y_* \in K$. Since $T_1$ is bounded, by Proposition 7.3, and considering eventually again a subsequence, we can suppose that $\{T_1(y_n)\}_{n \in \mathbb{N}}$ is weakly convergent in $E^*$ (i.e., $\sigma(E^*, E)$-convergent) to an element $u \in E^*$. Let $x \in K$ be an arbitrary element. There exists $n_0 \in \mathbb{N}$ such that $n_0 > m$ and $\{y_*, x\} \subseteq D_{n_0}$, and obviously $\{y_*, x\} \subseteq D_n$ for all $n \geqslant n_0$. From formula (19) we deduce

$$\langle y_* - y_n, (T_1 - T_2)(y_n) \rangle \geqslant 0 \tag{20}$$

and

$$\langle x - y_n, (T_1 - T_2)(y_n) \rangle \geqslant 0. \tag{21}$$

Because there exists a subsequence $\{T_2(y_{n_k})\}_{k \in \mathbb{N}}$ in $\{T_2(y_n)\}_{n \in \mathbb{N}}$ strongly convergent to an element $w \in E^*$ and since

$$\langle y_* - y_{n_k}, T_2(y_{n_k}) \rangle = \langle y_* - y_{n_k}, T_2(y_{n_k}) - w \rangle + \langle y_* - y_{n_k}, w \rangle,$$

by using Lemma 7.2 we obtain that

$$\langle y_* - y_{n_k}, T_2(y_{n_k}) \rangle \to 0 \quad \text{as } k \to \infty.$$

Therefore, by using (20) we have

$$\limsup_{k\to\infty}\langle y_{n_k} - y_*, T_1(y_{n_k})\rangle \leqslant \limsup_{k\to\infty}\langle y_{n_k} - y_*, T_2(y_{n_k})\rangle = 0.$$

From the last inequality and the equality

$$\langle y_{n_k}, T_1(y_{n_k})\rangle = \langle y_{n_k} - y_*, T_1(y_{n_k})\rangle + \langle y_*, T_1(y_{n_k})\rangle,$$

we deduce the inequality

$$\limsup_{k\to\infty}\langle y_{n_k}, T_1(y_{n_k})\rangle \leqslant \langle y_*, u\rangle.$$

Because $T_1$ satisfies condition $(S)^1_+$, we obtain that $\{y_{n_k}\}_{k\in\mathbb{N}}$ contains a subsequence, denoted again by $\{y_{n_k}\}_{k\in\mathbb{N}}$, strongly convergent to an element, which obviously must be $y_*$. Now computing the limit in (21), considering the properties of $T_1$ and $T_2$ and applying again Lemma 7.2, we obtain that

$$\langle x - y_*, (T_1 - T_2)(y_*)\rangle \geqslant 0 \quad \text{for all } x \in K,$$

i.e., the problem $VI(T_1 - T_2, K)$ has a solution. □

**Corollary 7.8.** *If either $E$ is a reflexive Banach space and $K \subseteq E$ is an arbitrary closed convex pointed cone, or $E$ is an arbitrary Banach space and $K \subseteq E$ is a closed convex pointed cone with a weakly compact base, and the assumptions* (1)–(4) *of Theorem* 7.10 *are satisfied, then the problem $NCP(T_1 - T_2, K)$ has a solution.*

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

**Theorem 7.11.** *Let $K \in H$ be a closed convex unbounded set such that $K \setminus \{0\}$ is an invariant set of the inversion $i$ and $T_1, T_2 : H \to H$ two mappings. If the assumptions*

(1) *$T_1$ is continuous bounded and satisfies condition $(S)^1_+$,*
(2) *$T_2$ is completely continuous,*
(3) *there exists a real number $c > 0$ such that $c \leqslant \underline{\mathcal{I}(T_1)}^{\#,K}(0)$,*
(4) *$\overline{\mathcal{I}(T_2)}^{\#,K}(0) < c$*

*are satisfied, then the problem $VI(T_1 - T_2, K)$ has a solution.*

**Proof.** Since $K \in H$ is unbounded, closed and $K \setminus \{0\}$ is an invariant set of $i$, $0 \in K$ and 0 is a non-isolated point of $K$. Hence, $\underline{\mathcal{I}(T_1)}^{\#,K}(0)$ and $\overline{\mathcal{I}(T_2)}^{\#,K}(0)$ are well defined. The proof of Theorem 7.11 follows by Theorem 7.10, by using Lemma 4.1 and a similar argument to the proof of Theorem 7.2. □

By Corollary 7.8 and Theorem 7.11 we have as follows:

**Corollary 7.9.** *If $K \subseteq H$ is a closed pointed convex cone and the assumptions* (1)–(4) *of Theorem* 7.11 *are satisfied, then the problem $NCP(T_1 - T_2, K)$ has a solution.*

## 8. Comments

(1) In [14] formulae for computing the scalar derivatives of mappings in interior points of the domain of definition were given (formulae which can also be used to calculate the scalar derivatives along a set, in interior points of this set). Throughout the paper we gave some theorems containing assumptions concerning the scalar derivatives of mappings in 0, where 0 was not an interior point of the domain of definition (or of the set along which the scalar derivatives were taken). It would be interesting to give computational formulae for the scalar derivatives in non-interior points of the domain of definition (or of the set along which the scalar derivatives are taken). This could lead to a series of new results.
(2) By Proposition 3.1 in the fixed point theorems and surjectivity theorems, containing assumptions concerning the scalar derivatives of $\mathcal{I}(f)$, we can firstly start with a mapping $g$ and after that set $f = \mathcal{I}(g)$. Then, the assumptions concerning the scalar derivatives of $\mathcal{I}(f)$ can be rewritten as assumptions imposed to the scalar derivatives of $g$.

## 9. Conclusions

By using a kind of duality between the scalar derivatives and scalar asymptotic derivatives, a novel method for calculating the scalar asymptotic derivatives was found and used for proving various fixed point theorems. These fixed point theorems were generated by a fixed point theorem of Isac, which extends a classical fixed point theorem of Krasnoselskii. Applications for surjectivity theorems, integral equations, variational inequalities and complementarity problems were given.

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