Radial rapid decay property for cocompact lattices

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Abstract

We study Haagerup inequality for radial functions on uniform lattices in semisimple Lie groups, with respect to Riemannian metrics and, in some case, to word metrics. In particular we extend the Swiatkowski–Valette results to any lattice acting properly and essentially transitively on classical buildings.

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1. Introduction

1.1. Property RD

Consider a locally compact second countable (lcsc) group $G$ equipped with a proper length function $L$, i.e. a proper function $L : G \to \mathbb{R}_+$ with the following property:

\[ L(xy) \leq L(x) + L(y) \quad \forall x, y \in G, \quad (1) \]
\[ L(x^{-1}) = L(x) \quad \forall x \in G, \quad (2) \]
\[ L(1_G) = 0. \quad (3) \]

Consider also a unitary representation $\tau : G \to U(H)$ and a subspace $E$ of the space of continuous functions on $G$ with compact support $C_c(G)$, which is stable under the involution $f^* = \overline{f(x^{-1})}$ (in the following we refer to $E$ as a $*$-subspace). Then we say that the triple

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(τ, L, E) has property RD (rapid decay) if there exist two constants K, C > 0 such that we have the following bound for the operator norm:

\[ \| \tau(F) \|_{op} \leq C \| F(1 + L)^k \|_2 \quad \forall F \in E. \]  \hspace{1cm} (4)

In particular when \( \tau = \lambda \), the left regular representation and \( E = C_c(G) \) we say that \( G \) has property RD with respect to \( L \). In this paper we are also interested in the case when \( \tau = \lambda \) and \( E \) is the space of \( L \)-radial functions; in this case we say that \( G \) has radial rapid decay property (RRD) with respect to \( L \).

**Example 1.1 (E. Breuillard).** Let \( G \) be a finitely generated group, \( L : \Gamma \to \mathbb{N} \) a word length function with respect to a finite symmetric generating set \( S = S^{-1} \) and let \( E \) be the space of \( L \)-radial functions with finite support. Then if \( 1 \) denotes the trivial representation we have that the triple \((1, L, E)\) has property RD if and only if \( G \) is virtually nilpotent. Indeed let \( s_n \) be the characteristic function of \( S_L(n) := \{ x \in G \mid L(x) = n \} \) the sphere of radius \( n \). Then

\[ \| 1(s_n) \|_{op} = \| s_n \|_1 = \# S_L(n) = \| s_n \|_2^2. \]  \hspace{1cm} (5)

On the other hand we see that any \( L \)-radial function \( F \) with finite support can be written in the form \( F(x) = \sum_{k=0}^{n(F)} F_k s_k(x) \) and thus

\[ \| F(1 + L)^k \|_2^2 = \sum_{m=0}^{n(F)} |F_m|^2 (1 + m)^{2k} \| s_m \|_2^2. \]  \hspace{1cm} (6)

Comparing (5) and (6) we obtain that \((1, L, E)\) has property RD if and only if \( G \) has polynomial growth with respect to \( L \) and hence by Gromov’s famous result (see [10]) if and only if the group \( G \) is virtually nilpotent.

Using Corollary 2.8 below we also deduce the well-know fact (see [25]) that for amenable finitely generated groups, RRD property is equivalent to virtual nilpotency (just recall that an lcsc group \( G \) is amenable if and only if the trivial representation \( 1 \) of \( G \) is weakly contained in the left regular representation \( \lambda \) of \( G \)).

**Example 1.2 (U. Haagerup).** Let \( G = \mathbb{F}_n \) be the free group on \( n \) generators and let \( L_S \) denote the word length with respect to a free generating set \( S \). In [11] U. Haagerup shows that

\[ \| \lambda(F) \|_{op} \leq 2 \| F(1 + L)^2 \|_2 \quad \forall F \in C[\mathbb{F}_n]. \]

This inequality is actually known in the literature as the Haagerup inequality.

In [15] and [14] P. Jolissaint generalizes Haagerup result, defining property RD and showing that a group \( G \) has RD with respect to \( L \) if and only if the Sobolev space associated to \( L \),

\[ H^\infty_L(G) = \{ F \in L^2(G) \mid F(1 + L)^k \in L^2(G) \ \forall k \geq 0 \}, \]
embeds continuously in the reduced $C^*$-algebra of the group $G$ (see also [16]). In this case $\mathcal{H}_L^\infty(G)$ is a dense subalgebra of $C^*_{r}(G)$, closed under holomorphic calculus (the locally compact case is due to Schweitzer and Ji, see [13]).

In [7] property RD is established for word-hyperbolic groups; A. Connes and H. Moscovici use these results to prove Novikov conjecture for word-hyperbolic groups in [4].

In [15] again it is shown that if $\Gamma < G$ is a uniform lattice in $G$ and $\Gamma$ has RD with respect to the restriction $L|_{\Gamma}$ of a length function $L$ on $G$, then $G$ has RD with respect to $L$. The converse has been conjectured by A. Valette in [25] in the case of uniform lattices in semisimple Lie groups. This conjecture is solved only in the rank 1 case (already in [15]) and in the case $G = SL_3(F)$ where $F = \mathbb{Q}_p$, $F = \mathbb{R}$ and $F = \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ in [22,18,1] respectively (here $\mathbb{O}$ denotes the division algebra of octonions). This question takes particular relevance after V. Lafforgue’s work, that shows in [19] that uniform lattices in semisimple Lie groups with property RD satisfy the Baum–Connes conjecture (see [26]). V. Lafforgue obtained in this way the first examples of discrete groups with Kazhdan property $T$ that satisfy the Baum–Connes conjecture (namely uniform lattices in $SL_3(\mathbb{R})$).

In the recent paper [3] all connected Lie groups with property RD are classified. In particular it is shown that semisimple Lie groups have property RD. This fact is due to C. Herz (see [5] and [6]), as explained in [3].

Let $G$ be a semisimple Lie group, let $\Gamma$ be a uniform lattice inside $G$ and consider the Riemannian metric $d$ on $G$. A direct corollary of the Valette conjecture is that the operator norm of the $L^2$-normalized characteristic function of the set $\Gamma_T := \{ \gamma \in \Gamma : d(1_G, \gamma) \leq T \}$ is polynomial in $T$. In this paper we confirm this fact.

The following theorem, that we prove in Section 3, is the central statement of the paper.

**Theorem 1.3.** Let $G$ be an lcsc group, compactly generated, $\Gamma < G$ a cocompact lattice. Let $1 \in \Omega = \Omega^{-1} \subset G$ a relatively compact symmetric Borel set such that $G = \bigcup_{n \geq 0} \Omega^n$. Define the length function $L(x) := \min\{n \in \mathbb{N} \mid x \in \Omega^n\}$, and consider $E_L$ the space of $L$-radial functions. Let $\tau : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ be a positive unitary representation (see Definition 2.4 below) of $\Gamma$. Suppose that $(\text{ind}_{G}^{\Gamma} \tau, L, E_L)$ has RD. Then $(\tau, L|_{\Gamma}, E_{L|_{\Gamma}})$ has RD. Moreover $G$ has RRD property with respect to $L$ if and only if $\Gamma$ has RRD property with respect to $L|_{\Gamma}$.

Theorem 0.1 of [3] shows that connected semisimple Lie groups have RD with respect to the length function associated to any relatively compact symmetric generating Borel set, giving the following

**Corollary 1.4.** Let $G$ be a connected semisimple Lie group. Then any cocompact lattice $\Gamma < G$ has the radial RD property with respect to any $G$-coherent (see Definition 3.9 below) word length function.

Our work was essentially motivated by the work of Swiatkowski and Valette (see [24] and [25]), which shows the connection between radial property RD and the return probability of a simple random walk on the group (see also [2]). In particular Proposition 4 of [25] establishes the equivalence between the radial RD property and the strict $N$-loop inequality. The following theorem (see Section 3 for the proof) generalizes Theorem 0.6(a) of [24] and Theorem 1 of [25].
Theorem 1.5. Let $\Gamma$ be a uniform lattice in a semisimple algebraic group $G$, defined over a non-archimedean local field $F$. Consider the Bruhat–Tits building $\Delta$ associated to $G$. Fix a base vertex $v_0 \in \Delta^{(0)}$ and define the length function $L_\Delta(\gamma) := d_\Delta(v_0, \gamma v_0)$, where $d_\Delta$ is the combinatorial distance on the 1-skeleton of $\Delta$. Then $\Gamma$ has the radial RD property with respect to $L_\Delta$.

The proof presented here is quite different from the proof of Swiatkowski and Valette, that uses elaborated arguments involving the geometry of the building $\Delta$ together with combinatorial techniques.

2. Operator norm and induced representation

2.1. $G$-invariant positive cone

Positivity plays a crucial role in our approach. We introduce the notion of a $G$-invariant positive cone, we give various examples, and finally we show that the operator norm preserves positivity (Lemma 2.7) in the case of positive unitary representations.

Let $\mathcal{H}$ be a Hilbert space. A positive cone $\mathcal{H}_+$ is a closed convex subset of $\mathcal{H}$, stable by the $\mathbb{R}_+$-scalar multiplication and such that $\langle \eta, \xi \rangle \geq 0 \forall \eta, \xi \in \mathcal{H}_+$. Observe that in this case there exists an $\mathbb{R}_+$-linear map:

$$p : \mathcal{H}_+^4 \rightarrow \mathcal{H}$$

$$p(\xi_1, \ldots, \xi_4) = \sum_{j=1}^{4} (i)_j \xi_j$$

such that the image $p(\mathcal{H}_+^4)$ is a vector subspace of $\mathcal{H}$. We say that the positive cone $\mathcal{H}_+$ is generating if $p(\mathcal{H}_+^4) = \mathcal{H}$.

Definition 2.1. Let $G$ be an lcsc group, let $\tau : G \rightarrow U(\mathcal{H})$ be a unitary representation. A $G$-pc is a pair $(\mathcal{H}_+, s)$ where $\mathcal{H}_+$ is a $G$-stable generating positive cone of $\mathcal{H}$ (i.e. $G \cdot \mathcal{H}_+ \subset \mathcal{H}_+$) and $s$ is a $G$-equivariant Lipschitz section:

$$s : \mathcal{H}_+^4 \rightarrow \mathcal{H}$$

$$\|s(\xi)\| \leq C_s \|\xi\| \quad \forall \xi \in \mathcal{H}$$

such that $p \circ s = Id_{\mathcal{H}}$ and $x_4 \circ s|_{\mathcal{H}_+} = Id_{\mathcal{H}_+}$, where $x_k : \mathcal{H}_+^4 \rightarrow \mathcal{H}_+$ is the projection on the $k$th coordinate of $\mathcal{H}_+^4$.

For any $j \in \{1, \ldots, 4\}$ we will write in what follows for simplicity $s_j$ instead of $x_j \circ s$. This definition is modeled on the following standard example.

Example 2.2. Let $(X, \nu)$ be a standard Borel $G$-space with $\nu$ a quasi-invariant measure (i.e. $[g_* \nu] = [\nu] \forall g \in G$). Then the set $L^2_+(X, \nu) := \{ F \in L^2(X, \nu) \mid F \geq 0 \}$ is a $G$-stable generating...
positive cone for the representation:

\[ \tau : G \longrightarrow \mathcal{U}(L^2(X, \nu)) \]

\[ \tau(g)F(x) = \sqrt{\frac{dg^{-1} \nu}{d\nu}}(x)F(g^{-1}x). \]

Moreover the map:

\[ s : L^2(X, \nu) \longrightarrow L^2_+(X, \nu)^4 \]

\[ F(x) \mapsto \begin{pmatrix} \max\{\Im(F(x)), 0\} \\ -\min\{\Im(F(x)), 0\} \\ -\min\{\Re(F(x)), 0\} \\ \max\{\Re(F(x)), 0\} \end{pmatrix} \]

is a \( G \)-equivariant section and satisfies \( \|s(F)\|_2 = \|F\|_2 \), as a simple verification shows.

The following lemma is useful to produce new example of \( G \)-pc.

**Lemma 2.3.** Let \( G \) and \( G' \) be two lcsc groups, let \( (X, \nu) \) be a standard Borel \( G \)-space, and let \( \tau : G' \rightarrow \mathcal{U}(\mathcal{H}) \) be a unitary representation of \( G' \). If \((\mathcal{H}_+, s)\) is a \( G' \)-pc for \( \tau \) then for any Borel cocycle \( \alpha : G \times X \rightarrow G' \) the set

\[ L^2_+(X, \mathcal{H}, \nu) := \{ F \in L^2(X, \mathcal{H}, \nu) \mid F(x) \in \mathcal{H}_+, \ \nu\text{-a.e.} \} \]

together with the map \( \sigma : L^2(X, \mathcal{H}, \nu) \rightarrow L^2_+(X, \mathcal{H}, \nu)^4 \) defined by \( \sigma(F)(x) = s(F(x)) \ \forall x \in X \), is a \( G \)-pc for the representation \( \tau_\alpha \) defined by:

\[ \tau_\alpha : G \longrightarrow \mathcal{U}(L^2(X, \mathcal{H}, \nu)) \]

\[ \tau_\alpha(g)F(x) = \sqrt{\frac{dg^{-1} \nu}{d\nu}}(x)\tau(\alpha(g, x))F(g^{-1}x). \]

**Proof.** The fact that \( \mathcal{H}_+ \) is \( G' \)-invariant together with the formula (7) imply that \( L^2_+(X, \mathcal{H}, \nu) \) is \( G \)-invariant. Moreover we have that for any \( F, H \in L^2_+(X, \mathcal{H}, \nu) \) we get \( \langle F, H \rangle_{L^2(X, \mathcal{H}, \nu)} \geq 0 \). Indeed, recall that this scalar product is defined by the formula

\[ \langle F, H \rangle_{L^2(X, \mathcal{H}, \nu)} = \int_X \langle F(x), G(x) \rangle_{\mathcal{H}} d\nu(x) \]

and that by construction \( \langle F(x), G(x) \rangle_{\mathcal{H}} \geq 0 \ \nu\text{-a.e.} \). On the other hand it is clear that \( \sigma \) is well defined as a section and \( G \)-equivariant. Indeed for any \( j \in \{1, \ldots, 4\} \):
\[
\tau_\alpha(g)(\sigma_j(F))(x) = \sqrt{\frac{dg^{-1}v}{dv}}(x)\tau(\alpha(g,x))s_j(F(g^{-1}x)) \\
= s_j\left(\sqrt{\frac{dg^{-1}v}{dv}}(x)\tau(\alpha(g,x))F(g^{-1}x)\right) \\
= \sigma_j(\tau_\alpha(g)F)(x).
\]

Finally we have that:

\[
\|\sigma(F)\|_2^2 = 4\sum_{j=1}^4 \|\sigma_j(F)\|_2^2 = 4\sum_{j=1}^4 \int_X \|\sigma_j(F)(x)\|^2 d\nu(x) \leq \int_X Cs \|F(x)\|^2 d\nu(x) = Cs \|F\|_2^2. \quad \Box
\]

**Definition 2.4.** Let \( G \) be an lcsc group and let \( \tau : G \to \mathcal{U}(H) \) be a unitary representation. We say that \( \tau \) is a positive unitary representation if it admits a \( G \)-pc \((\mathcal{H}+, s)\).

**Lemma 2.5.** Let \( G \) be an lcsc group and let \( H \) be a closed subgroup. If \( \tau : H \to \mathcal{U}(H) \) is a positive unitary representation then the induced representation \( \text{ind}_G^H \tau : G \to \mathcal{U}(L^2(G/H, \mathcal{H})) \) is a positive unitary representation of \( G \).

**Proof.** Given a Borel section \( \sigma : G/H \to G \), we apply Lemma 2.3 for the cocycle \( \alpha : G \times G/H \to H, \alpha(g,x) = s(gx)^{-1}gs(x) \). Then \( \tau_\alpha \cong \text{ind}_G^H \tau \). \( \Box \)

If \( E \) is a subspace of \( C_c(G) \), denote by \( E_+ \) the set \( \{|F| : F \in E\} \) where \( |F|(x) := |F(x)| \).

**Proposition 2.6.** Let \( \tau : G \to \mathcal{U}(H) \) be a positive unitary representation and let \( \mathcal{H}_+ \) be a \( G \)-pc. Then the triple \((\tau, L, E)\) has property RD if and only if there exist \( K, C \geq 0 \) such that:

\[
\langle \tau(F)\xi, \eta \rangle \leq C\left\| F(1+L)^K \right\|_2 \|\xi\| \|\eta\| \quad \forall \eta, \xi \in \mathcal{H}_+ \forall F \in E_+.
\]

Let us recall that if \( A \in \mathcal{B}(\mathcal{H}) \) is a bounded operator on a Hilbert space then

\[
\|A\|_{op} = \sup_{\eta, \xi \in \mathcal{H}\setminus\{0\}} \frac{|\langle A\eta, \xi \rangle|}{\|\eta\| \|\xi\|}. \quad (9)
\]

So the proposition is a straightforward consequence of the following lemma:
**Lemma 2.7.** Let $G$ be an lcsc group and let $\tau : G \to \mathcal{U}(\mathcal{H})$ be a positive unitary representation. Then there exists $C > 0$, which depends on $\tau$ such that:

$$\|\tau(F)\|_{op} \leq C \|\tau(|F|)\|_{op} \quad \forall F \in C_c(G).$$

Moreover if $0 \leq F \leq F' \in L^1(G)$ then

$$\|\tau(F)\|_{op} \leq \|\tau(F')\|_{op}. \quad (10)$$

**Proof.** Fix a $G$-pc $(\mathcal{H}_+, s)$ for the representation $\tau$. Consider the continuous $G$-equivariant map:

$$q : \mathcal{H}_+^4 \longrightarrow \mathcal{H}$$

$$q(\xi_1, \ldots, \xi_4) = \sum_{j=1}^4 \xi_j$$

and put $C = \sup_{\|\xi\| \leq 1} \|q \circ s(\xi)\|^2 \leq \infty$.

Let $F \in C_c(G)$ be a continuous function, then for any $\xi, \eta \in \mathcal{H}$:

$$\langle \tau(F)\xi, \eta \rangle = \left| \int_G F(g) \left( \sum_{j=1}^4 (i)^j s_j(\xi), \sum_{k=1}^4 (i)^k s_k(\eta) \right) dg \right|$$

$$= \left| \int_G F(g) \sum_{j,k=1}^4 (i)^{j-k} \langle \tau(g)s_j(\xi), s_k(\eta) \rangle dg \right|$$

$$\leq \int_G |F(g)| \sum_{j,k=1}^4 (i)^{j-k} \langle \tau(g)s_j(\xi), s_k(\eta) \rangle dg$$

$$\leq \int_G |F(g)| \sum_{j,k=1}^4 \langle \tau(g)s_j(\xi), s_k(\eta) \rangle dg$$

$$= \langle \tau(|F|)q \circ s(\xi), q \circ s(\eta) \rangle.$$
A direct consequence of this lemma is the following corollary, that summarizes some well-known facts (compare to [23]). Recall that given two unitary representations $\tau, \sigma$ of $G$ we say that $\tau$ is weakly contained in $\sigma$ -- in symbol $\tau \prec \sigma$ -- if any matrix coefficients of $\tau$ can be approximated by a sequence of matrix coefficients of $\sigma$ uniformly on compact subset of $G$.

**Corollary 2.8.** Let $\pi : G \to U(\mathcal{H})$ be a positive unitary representation such that $\pi \prec \lambda$, where $\lambda$ denotes the left regular representation. Then for every $\ast$-subspace $E \subseteq C_0(G)$ and for any length function $L : G \to \mathbb{R}_+$ the triple $(\lambda, L, E)$ has RD if and only if the triple $(\pi, L, E)$ has RD.

**Proof.** It is well known that if $\pi \prec \lambda$ then

$$\|\pi(F)\|_{op} \leq \|\lambda(F)\|_{op} \quad \forall F \in C_c(G). \tag{11}$$

On the other hand if $\pi$ is a positive unitary representation, then Lemma 2.3 of [23] establishes that:

$$\|\pi(F)\|_{op} \geq \|\lambda(F)\|_{op} \quad \forall F \geq 0 \in C_c(G). \tag{12}$$

Lemma 2.7 together with (12) proves the corollary. $\Box$

We recall that a locally compact group $G$ has polynomial growth with respect to a proper length function $L$ if there exist two constants $C, K > 0$ such that $\mu(B_L(R)) \leq C(1 + R)^K \forall R > 0$, where $\mu$ is any Haar measure on $G$ and $B_L(R) = \{g \in G \mid L(g) \leq R\}$.

**Corollary 2.9.** Let $G$ be an lcsc amenable group, let $L : G \to \mathbb{R}_+$ be a proper length function and let $E \subseteq C_0(G)$ be a $\ast$-subspace containing $L$-radial functions. Then the following are equivalent:

1. There exists a positive unitary representation $\pi : G \to U(\mathcal{H})$ such that the triple $(\pi, L, E)$ has RD.
2. For any positive unitary representation $\tau : G \to U(\mathcal{H})$ the triple $(\tau, L, E)$ has RD.
3. The group $G$ has polynomial growth with respect to $L$.

**Proof.** The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious. So to prove the statement it is enough to show the implication (1) $\Rightarrow$ (3). Fix $R > 0$ and consider the characteristic function $\mathbb{1}_{B_L(R)}$ of the ball of radius $R$. By amenability of the group $G$ and positivity of $\pi$, using inequalities (11) and (12), we have that

$$\|\pi(\mathbb{1}_{B_L(R)})\|_{op} \geq \|\mathbb{1}(\mathbb{1}_{B_L(R)})\|_{op} = \mu(B_L(R)). \tag{13}$$

On the other hand, as the function $\mathbb{1}_{B_L(R)}$ is in $E$ and the triple $(\pi, L, E)$ has RD, there exist constants $C, K > 0$, not depending on $R$, such that

$$\|\pi(\mathbb{1}_{B_L(R)})\|_{op} \leq C(1 + R)^K \mu(B_L(R))^{1/2}. \tag{14}$$

Inequalities (13) and (14) imply (3). $\Box$
2.2. Cocompact lattices and induced representations

Let $G$ be an lcsc group and let $\Gamma < G$ be a lattice; let $D$ be a fundamental domain for the right action of $\Gamma$ on $G$, i.e. the image of a Borel section: $\sigma : G/\Gamma \to G$. Let $\mu$ be the Haar measure on $G$ such that $\mu(D) = 1$. In the following, in the case that $\Gamma < G$ is cocompact (i.e. $G/\Gamma$ is a compact topological space) we always choose the fundamental domain $D$ to be relatively compact.

**Proposition 2.10.** Let $G$ be an lcsc group, let $\Gamma < G$ be a cocompact lattice. We define the following linear map

$$\Psi : C_c(G) \longrightarrow \mathbb{C}[\Gamma]$$

$$\Psi(F)(\gamma) = \int_G F(g)\varphi(\gamma, g) \, d\mu(g),$$

where

$$\varphi : \Gamma \times G \longrightarrow \mathbb{R}_+$$

$$\varphi(\gamma, g) = \mu(D\gamma \cap gD).$$

Then

1. $\Psi$ extends to a continuous map: $L^1(G) \to l^1(\Gamma)$,
2. $\Psi$ extends to a continuous map: $L^2(G) \to l^2(\Gamma)$,
3. $\Psi(F^*) = \Psi(F)^*$ for all $F$ in $C_c(G)$.

**Proof.** Remark that if $F \in C_c(G)$, i.e. the support of $F$ is contained in a compact set $\text{supp}(F) \subset K \subset G$, then $DK \subset \bigcup_{\gamma \in \Lambda} \gamma D$ with $\Lambda \subset \Gamma$ finite and $\text{supp}(\Psi(F)) \subset \Lambda$; this shows that the map $\Psi$ is well defined.

1. By definition $G = \bigcup_{\gamma \in \Gamma} \gamma D$. Then since the union is disjoint:

$$\sum_{\gamma \in \Gamma} \varphi(\gamma, g) = \mu\left( \bigcup_{\gamma \in \Gamma} D\gamma \cap gD \right)$$

$$= \mu(gD)$$

$$= 1.$$

Hence

$$\|\Psi(F)\|_1 = \sum_{\gamma \in \Gamma} \left| \int_G F(g)\varphi(\gamma, g) \, d\mu(g) \right|$$

$$\leq \sum_{\gamma \in \Gamma} \int_G \left| F(g)\varphi(\gamma, g) \right| \, d\mu(g)$$
\[ \leq \int_{G} |F(g)| \sum_{\gamma \in \Gamma} \varphi(\gamma, g) \, d\mu(g) \]
\[ \leq \|F\|_{1}. \]

(2) First we observe that \( G \) must be unimodular because it contains a lattice, so \( \mu \) is also right invariant. Hence according to Fubini

\[ \int_{G} \varphi(\gamma, g) \, d\mu(g) = \int_{G} \int_{D} \chi_{D\gamma}(xg) \, d\mu(x) \, d\mu(g) \]
\[ = \int_{D} \int_{G} \chi_{D\gamma}(g) \, d\mu(g) \, d\mu(x) \]
\[ = \mu(D) \mu(D\gamma) \]
\[ = 1. \]

And hence, by Cauchy–Schwarz,

\[ \left\| \Psi(F) \right\|_{2}^{2} = \sum_{\gamma \in \Gamma} \left\| \int_{G} F(g) \varphi(\gamma, g) \, d\mu(g) \right\|^{2} \]
\[ \leq \sum_{\gamma \in \Gamma} \left( \int_{G} |F(h)|^{2} \varphi(\gamma, h) \, d\mu(h) \right) \left( \int_{G} \varphi(\gamma, g) \, d\mu(g) \right) \]
\[ \leq \int_{G} |F(h)|^{2} \sum_{\gamma \in \Gamma} \varphi(\gamma, h) \, d\mu(h) \]
\[ \leq \|F\|_{2}^{2}. \]

(3) Using once more the right invariance of \( \mu \):

\[ \mu(D\gamma \cap gD) = \mu(D \cap gD\gamma^{-1}) \]
\[ = \mu(g^{-1} D \cap D\gamma^{-1}). \]

Hence

\[ \Psi(F)^{*}(\gamma) = \overline{\Psi(F)(\gamma^{-1})} \]
\[ = \int_{G} F(g) \varphi(\gamma^{-1}, g) \, d\mu(g) \]
\[ = \int_{G} \bar{F}(g^{-1}) \varphi(\gamma, g) \, d\mu(g) \]
\[ = \Psi(F^{*})(\gamma). \]
Remark 1. We observe that in the proof of Proposition 2.10 we show some useful property of the function $\varphi : \Gamma \times G \to \mathbb{R}_+$, namely:

$$
\forall g \in G \quad \| \gamma \mapsto \varphi(\gamma, g) \|_{L^1(\Gamma)} = 1,
$$

$$
\forall \gamma \in \Gamma \quad \| g \mapsto \varphi(\gamma, g) \|_{L^1(G)} = 1,
$$

$$
\forall g \in G \forall \gamma \in \Gamma \quad \varphi(\gamma^{-1}, g) = \varphi(\gamma, g^{-1}).
$$

The linear map $\Psi$ plays a crucial role in this paper, as illustrated in the following theorem (compare with Proposition 3.9 of [20]).

**Theorem 2.11.** Let $G, \Gamma$ be as before and let $\tau : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Let $\text{ind}_G^\Gamma \tau$ denote the induced representation from $\Gamma$ to $G$. Then:

$$
\| \tau(\Psi(F)) \|_{op} \leq \| \text{ind}_G^\Gamma \tau(F) \|_{op} \quad \forall F \in C_c(G).
$$

**Proof.** Given a Borel section $\sigma : G/\Gamma \to G$ define the following cocycle:

$$
\alpha : G \times G/\Gamma \to \Gamma
$$

$$
\alpha(g, x) = \sigma(gx)^{-1}g\sigma(x)
$$

and consider the associated representation of $G$:

$$
\tau_\alpha : G \to L^2(G/\Gamma, \mathcal{H}, \tilde{\mu})
$$

$$
\tau_\alpha(g)X(x) = \tau(\alpha(g, x))X(xg),
$$

where $\tilde{\mu}$ is the push-forward of the Haar measure $\mu$ under the natural projection $\pi : G \to G/\Gamma$. It is well known (see [27]) that $\tau_\alpha$ is isomorphic to $\text{ind}_G^\Gamma \tau$. Remark that the cocycle $\alpha$ depends on the choice of the section $\sigma$ but if we choose another section $\sigma'$, the associated cocycle $\alpha(\sigma')$ is cohomologous to the first one and then the two representations are isomorphic. For any $\eta \in \mathcal{H}$ define the element $X_\eta \in L^2(G/\Gamma, \mathcal{H}, \tilde{\mu})$ as $X_\eta(x) = \eta, \forall x \in G/\Gamma$. It follows that $\|X_\eta\|_{L^2(G/\Gamma, \mathcal{H}, \tilde{\mu})}^2 = \int_{G/\Gamma} \|\eta\|_{\mathcal{H}}^2 d\tilde{\mu}(x) = \|\eta\|_{\mathcal{H}}^2$. So we compute:

$$
\langle \text{ind}_G^\Gamma \tau(g)X_\xi, X_\eta \rangle_{L^2(G/\Gamma, \mathcal{H}, \tilde{\mu})} = \int_{G/\Gamma} \langle \tau(\alpha(g, x))\xi, \eta \rangle_{\mathcal{H}} d\tilde{\mu}(x)
$$

$$
= \sum_{\gamma \in \Gamma} \tilde{\mu}(x \in G/\Gamma : \alpha(g, x) = \gamma) \langle \tau(\gamma)\xi, \eta \rangle_{\mathcal{H}}
$$

$$
= \sum_{\gamma \in \Gamma} \varphi(\gamma, g) \langle \tau(\gamma)\xi, \eta \rangle_{\mathcal{H}}.
$$

Finally, if $F \in C_c(G)$ we have:
\[
\langle \text{ind}_G \tau(F) X_\xi, X_\eta \rangle_{L^2(G/\Gamma, \mathcal{H}, \hat{\mu})} = \int_G F(g) \langle \text{ind}_G \tau(g) X_\xi, X_\eta \rangle_{L^2(G/\Gamma, \mathcal{H}, \hat{\mu})} d\mu(g) \\
= \int_G F(g) \sum_{\gamma \in \Gamma} \varphi(\gamma, g) \langle \tau(\gamma) \xi, \eta \rangle_{\mathcal{H}} d\mu(g) \\
= \sum_{\gamma \in \Gamma} \Psi(F)(\gamma) \langle \tau(\gamma) \xi, \eta \rangle_{\mathcal{H}} \\
= \langle \tau(\Psi(F)) \xi, \eta \rangle_{\mathcal{H}}. \]

\textbf{Remark 2.} Let \( \Sigma_G \) be the set of all classes of unitary representations of \( G \); a subset \( S \) of \( \Sigma_G \) is closed (in the sense of Fell) if \( \tau \prec \sigma \) and \( \sigma \in S \) implies that \( \tau \in S \). Fell shows that if \( H \) is a closed subgroup of \( G \), induction from the representations of \( H \) to the representations of \( G \) is closed, that is \( \text{ind}_H^G S \) is closed whenever \( S \) is closed. Let \( C^*_S(G) \) denote the \( C^* \)-algebra associated to the closed subset \( S \) of \( \Sigma_G \) and let \( B_S(G) \) denote its dual Banach space (see [8]). If \( \Gamma \leq G \) is a uniform lattice the theorem above asserts that for all closed subset \( S \subset \Sigma_{\Gamma} \) the linear map \( \Psi \) extends to a continuous map

\[ \Psi_S : C^*_{\text{ind}_G^\Gamma S}(G) \rightarrow C^*_S(\Gamma). \]

Moreover it is known that \( B_S(G) \) can be realized as a space of bounded functions on \( G \), any element \( u \in B_S(G) \) can be written in the form \( u(g) = \langle \tau(g) \xi, \eta \rangle \) with \( \|u\|_{B_S(G)} = \|\xi\| \|\eta\| \) for some representation \( \tau \in S \) and for some \( \xi, \eta \in \mathcal{H} \) (see once more [8]). Then the above theorem gives us an explicit formula for the adjoint map:

\[ \Psi^*_S : B_S(\Gamma) \rightarrow B_{\text{ind}_G^\Gamma S}(G) \]

\[ (\gamma \mapsto \langle \tau(\gamma) \xi, \eta \rangle) \mapsto (g \mapsto \langle \text{ind}_G^\Gamma \tau(g) X_\xi, X_\eta \rangle). \]

\section{Admissible set}

\textbf{3.1. Operator norm for coarse admissible family}

We introduce in this section some tools from ergodic theory that we need in what follows. We refer to [9] for an exhaustive discussion about coarse admissible families, including several important examples.

\textbf{Definition 3.1.} Let \( G \) be an lcsc group with left Haar measure \( \mu \). An increasing family of bounded Borel subsets \( \{G_t\}_{t \in \mathbb{R}^+} \) is said to be coarsely admissible if

\begin{enumerate}
    \item[(1)] for every compact subset \( K \subset G \), there exists \( c = c(K) > 0 \) such that \( KG_tK \subset G_{t+c} \forall t \gg 1 \),
    \item[(2)] for every \( c > 0 \) there exists \( D > 0 \) such that \( \mu(G_{t+c}) \leq D \mu(G_t) \forall t \gg 1 \).
\end{enumerate}
The following proposition is a special case of Theorem (6.4)(I) of [9, p. 65]. Since we consider only uniform lattices, the proof is elementary and it does not need the tools developed in [9].

**Proposition 3.2.** Let $G$ be an lcsc group and let $Γ ⊆ G$ be a cocompact lattice in $G$, let $G_1$ be a coarse admissible family in $G$ and $Γ_1 := Γ ∩ G_1$. Then there exists $C > 0$ such that:

$$C^{-1} μ(G_1) \leqslant |Γ_1| \leqslant C μ(G_1) \quad ∀ t \gg 1.$$  \hspace{1cm} (16)

**Proof.** For the inequality $|Γ_1| \leqslant C μ(G_1)$ see the proof of Lemma (6.5)(I) of [9, p. 66]. Conversely, let $B ⊆ G$ be a Borel relatively compact set such that $∀ x ∈ G$ we have $Γ ∩ xB \neq ∅$ (such a set exists because $Γ$ is uniform). Hence for any $x ∈ G$ we can find $γ(x) ∈ Γ ∩ xB$; defining the Borel retraction:

$$ρ : G → Γ \quad x ↦ γ(x).$$

By coarse admissibility there exists $c = c(B)$ such that

$$ρ(G_t) ⊆ Γ ∩ G_1 B ⊆ Γ ∩ G_{t+c} = Γ_{t+c}, \quad t \gg 1.$$  

In other words

$$G_1 ⊆ \bigcup_{γ ∈ Γ_{t+c}} ρ^{-1}(γ) ⊆ \bigcup_{γ ∈ Γ_{t+c}} γ B^{-1}.$$  

Always by coarse admissibility we obtain

$$μ(G_1) \leqslant D μ(G_{t+c}) \leqslant D μ(B^{-1}) |Γ_1|, \quad t \gg 1. \quad \square$$

**Theorem 3.3.** Let $G$ be an lcsc group, let $Γ < G$ be a cocompact lattice and let $τ : Γ → U(ℋ)$ a positive unitary representation. Suppose that $∃ c, k > 0$ such that:

$$\left\| \text{ind}_τ^G τ(1_{Γ_1}) \right\|_{op} \leqslant c(1 + t)^k \|1_{Γ_1}\|_2 \quad ∀ t \gg 1.$$  

Then there exist $c′, k′ > 0$ such that:

$$\left\| τ(1_{Γ_1}) \right\|_{op} \leqslant c′(1 + t)^{k′} \|1_{Γ_1}\|_2 \quad ∀ t \gg 1.$$  

**Proof.** We observe that if $γ ∈ Γ_1$ then the support of the function $g ↦ ϕ(γ, g)$ is contained in $D Γ_1 D^{-1}$, where $D$ denotes the fundamental domain as in Section 2. Then there exists $C = C(D ∪ D^{-1}) > 0$ such that $\text{supp}(g ↦ ϕ(γ, g)) ⊆ G_{t+c}$. It follows that if $γ ∈ Γ_1$ then:

$$Ψ(1_{Γ_{t+c}})(γ) = \int_{Γ_{t+c}} ϕ(γ, g) dμ(g) = 1.$$
On the other hand, by construction $\Psi(I_{G_t+C}) \geq 0$, $\forall \gamma \in \Gamma$, and thus

$$I_{\Gamma_t} \leq \Psi(I_{G_t+C}) \quad \forall t \gg 1.$$  

Applying Proposition 3.2, we see that there exists some $C_1 > 0$ such that

$$\|I_{G_t+C}\|_2 = \mu(G_t+C)^{1/2} \leq C_1 \#\Gamma_t^{1/2} = C_1 \|I_{\Gamma_t}\|_2.$$  

Applying Lemma 2.7 and Theorem 2.11 we have, for $t \gg 1$

$$\|\tau(I_{\Gamma_t})\|_\text{op} \leq \|\tau(\Psi(I_{G_t+C}))\|_\text{op}$$

$$\leq \|\text{ind}_{\Gamma}^G \tau(I_{G_t+C})\|_\text{op}$$

$$\leq c(1 + t + C)^k \|I_{G_t+C}\|_2$$

$$\leq c'(1 + t)^k' \|I_{\Gamma_t}\|_2.$$  

**Remark 3.** As illustrated in the next section, Theorem 3.3 is useful to deduce upper bounds on operator norm on $\Gamma$ from upper bounds on operator norm on $G$. This is why this result looks very close to some result in [9] with a substantial difference: here we are not interested on averages as in ergodic theory but on the existence of some closed $*$-subspace of the reduced $C^*$-algebra of the group $\Gamma$. The way from $\Gamma$ to $G$ seems easier as illustrated in theorem of [15] and the following elementary proposition.

**Proposition 3.4.** Let $G$ be an lcsc group and let $\Gamma < G$ be a uniform lattice.

Let $d : G \times G \to \mathbb{R}_+$ a left-invariant distance on $G$ such that the balls $B(t) = \{ x \in G \mid d(1_G, x) \leq t \}$ form a coarse admissible family. Set $\beta(t) = \{ x \in \Gamma \mid d(1_G, x) \leq t \}$. Let $\tau : G \to \mathcal{U}(\mathcal{H})$ a positive unitary representation and suppose that $\exists c, k > 0$ such that:

$$\|\tau(I_{\beta(t)})\|_\text{op} \leq c(1 + t)^k \quad \forall t \gg 1.$$  

Then there exist $c', k' > 0$ such that:

$$\|\tau(I_{\beta(t)})\|_\text{op} \leq c'(1 + t)^{k'} \quad \forall t \gg 1.$$  

**Proof.** Let $d := \min\{ n \in \mathbb{N} \mid D \subset B(n) \}$. Then if $x \in \gamma D$, where $D$ denotes the fundamental domain, with $d(1_G, \gamma) \geq t$ we have $d(1_G, x) \geq t - d$. This implies that $B(t) \subset \bigcup_{\gamma \in B(t+d+1) \cap \Gamma} \gamma D$ and hence

$$I_{B(t)} \leq I_{B(t+d+1) \cap \Gamma} * I_D$$

and using Lemma 2.7 and Proposition 3.2:
\[ \| \tau(I_{B(t)}) \|_{op} \leq \| \tau(I_{B(t+d+1) \cap \Gamma}) \|_{op} \| \tau(I_D) \|_{op} \leq Cc(2 + t + d)^k \| I_{B(t+d+1) \cap \Gamma} \|_2 \leq c'(1 + t)^k \| I_{B(t)} \|_2, \]

where all the inequalities hold for \( t \) sufficiently large. \( \square \)

Here we are more interested in the case of the left regular representation, so we summarize the previous results in the following corollary.

**Corollary 3.5.** Let \( G, \Gamma, B(t) \) and \( \beta(t) \) be as before. Then there exist two constants \( C, c > 0 \) such that for \( t \) sufficiently large:

\[ C^{-1} \| \lambda_G(I_{B(t-c)}) \|_{op} \| I_{B(t-c)} \|_2 \leq \| \lambda_\Gamma(I_{\beta(t)}) \|_{op} \| I_{\beta(t)} \|_2 \leq C \| \lambda_G(I_{B(t+c)}) \|_{op} \| I_{B(t+c)} \|_2. \]

**Proof.** This follows from Theorem 3.3 and Proposition 3.4, using the equivalences:

\[ \text{Ind}^G_{\Gamma} \lambda_\Gamma \cong \lambda_G \quad \text{and} \quad \lambda_G | \Gamma \cong [G : \Gamma] \cdot \lambda_\Gamma. \quad \square \]

### 3.2. Proofs of Theorems 1.3–1.5 and applications

Theorem 3.3 establishes property of radial rapid decay for cocompact lattices in large classes of examples.

**Proof of Theorem 1.3.** First observe that \( \{ \Omega^n \}_{n \geq 0} \) is a coarse admissible family; in fact if \( B \) is a bounded domain there exists some \( k = k(B) > 0 \) s.t. \( B \subset \Omega^k \) and hence \( B \Omega^n B \subset \Omega^{n+2k} \). As \( \Omega \) is compact there exists a finite set \( F \subset G \) such that \( \Omega^2 \subset \bigcup_{x \in F} x \Omega \) and consequently \( \Omega^{n+k} \subset \bigcup_{x \in F^k} x \Omega^n \). This implies \( \mu(Omega^{n+k}) \leq |F|^{k-1} \mu(Omega^n) \). Observe that the same argument is valid for any proper length function with at most exponential growth. So Theorem 3.3 applies.

By definition \( BL | \Gamma (n) = \Omega^n \cap \Gamma \). Hence \( \| \tau(I_{BL | \Gamma (n)}) \|_{op} \leq P_1(n) \| I_{BL | \Gamma (n)} \|_2 \).

Using Proposition 5 of [25] we conclude the first part of the proof.

The second part of the theorem is a direct consequence of Proposition 3.4, together with the following lemma:

**Lemma 3.6.** Let \( G, \Gamma, \Omega \) be defined as before. Let \( \tau : G \to U(H) \) be a positive unitary representation of \( G \). Suppose that the triple \( (\tau | \Gamma, L | \Gamma, E | \Gamma) \) has RD then the triple \( (\tau, L, E) \) has RD.

**Proof.** Observe that any \( L \)-radial function \( F \in E_L \) with support in a ball of radius \( n \) is of the form \( F = \sum_{k=1}^n \hat{F}(k) S_k \), where \( S_n := \Omega^n - \Omega^{n-1} \). Remark that if \( G \) is amenable then the hypothesis implies that \( G \) is of polynomial growth with respect to \( L \) (see Corollary 2.9), and this is a quasi-isometric invariant and there is nothing to show; so we can suppose without restriction that \( G \) and \( \Gamma \) are both non-amenable. Because \( G \) is a locally compact compactly generated non-amenable group, the Følner condition (see [12]) on \( G \) ensures that there exists \( \epsilon > 0 \) such that

\[ \mu(Omega^n - Omega^{n-1}) \geq \epsilon \mu(Omega^n). \]
So we have

\[
\| \tau(S_n) \|_{op} \leq \| \tau(I_{\Omega_n}) \|_{op} \\
\leq P_2(n) \| I_{\Omega_n} \|_2 \\
\leq \epsilon P_2(n) \| S_n \|_2
\]

and finally

\[
\| \tau(F) \|_{op} \leq P_3(n) \| F \|_2.
\]

This concludes the proof of Theorem 1.3. \( \square \)

In particular we obtain that if \( \Gamma \) is a uniform lattice in a connected semisimple Lie group \( G \) and \( \Omega \) is any neighborhood of the identity, then \( \Gamma \) has radial rapid decay property with respect to the restriction of the length function \( L_{\Omega} \) associated to \( \Omega \). In fact Theorem 0.1 of [3] shows that \( G \) has property RD with respect to the Riemannian metric and hence with respect to \( L_{\Omega} \) because \( d(x, 1_G) \leq d(\Omega)L_{\Omega}(x) \).

**Definition 3.7.** Let \( G \) be a semisimple Lie group and let \( d : G \times G \rightarrow \mathbb{R}_+ \) be a Riemannian metric. For any \( \epsilon > 0 \) and for any \( n \in \mathbb{N} \) consider the set \( S_{\epsilon}^n := \{ x \in G : n \epsilon \leq d(1_G, x) < (n + 1) \epsilon \} \). Let \( \Gamma \triangleleft G \) a lattice subgroup, we say that a function \( f \in \mathbb{C}[\Gamma] \) is \( \epsilon \)-radial if it is constant on the sets \( S_{\epsilon}^n \cap \Gamma \) for all \( n \in \mathbb{N} \) (i.e. \( f \) is of the form \( f = \sum_{k=0}^{N} f_k I_{S_{\epsilon}^k \cap \Gamma} \)).

**Corollary 3.8.** Let \( G \) be a semisimple Lie group with finite center, let \( \Gamma \triangleleft G \) be a irreducible uniform lattice. Then for any \( \epsilon > 0 \) the space of \( \epsilon \)-radial functions has property RD.

**Proof.** We have already seen that the group \( G \) has property RD with respect to the Riemannian metric. To prove the corollary is sufficient to observe the growth Riemannian balls. If \( \Gamma \) is an infinite irreducible lattice it is known (see [17] and [21]) that there exist constants \( \alpha > 1 \) and \( \beta > 0 \) such that:

\[
\#(B(T) \cap \Gamma) \sim T^{-\beta} e^{\alpha T}.
\] (17)

Remark that this growth rate can be used to show directly coarse admissibility of Riemannian balls. On the other hand this implies that \( \forall \epsilon > 0 : \)

\[
\frac{\#(S_{\epsilon}^n \cap \Gamma)}{\#(B(n\epsilon) \cap \Gamma)} = \frac{\#(B((n+1)\epsilon) \cap \Gamma)}{\#(B(n\epsilon) \cap \Gamma)} - \frac{\#(B(n\epsilon) \cap \Gamma)}{\#(B(n\epsilon) \cap \Gamma)} \xrightarrow{n \to \infty} e^{\alpha \epsilon} - 1 > 0.
\]

In other words there exists \( \delta = \delta(\epsilon) > 0 \) such that:

\[
\| I_{(S_{\epsilon}^n \cap \Gamma)} \|_2 \geq \delta \| I_{(B(n\epsilon) \cap \Gamma)} \|_2.
\] (18)

See [25] for definitions use in the theorem below.
Proof of Theorem 1.5. The group $\Gamma$ is a uniform lattice in a simple algebraic group $G$ over a non-archimedean local field $\mathbb{F}$ and it follows from Theorem 0.1 of [3] that $G$ has property RD. Proposition 3.13(2) of [9] asserts that the pre-image $G_n$ of the ball $B_n$ of radius $n$ and center $v_0$ forms a coarse admissible family inside $G$. We can apply Theorem 3.3. □

Definition 3.9. Let $G$ be an lcsc group and let $H < G$ be a closed subgroup. A length function $L : H \to \mathbb{R}_+$ is $G$-coherent if for any compact subset $K \subset G$ there exists a constant $C = C(K) > 0$ such that:

$$\sup_{h \in H \cap Kh_0K^{-1}} \left| L(h) - L(h_0) \right| \leq C \quad \forall h_0 \in H.$$  

Example 3.10. Suppose that $L = L'_H$, where $L' : G \to \mathbb{R}_+$ is a proper length function on $G$. Then $L$ is a proper $G$-coherent length function on $H$.

Example 3.11. Let $G$ be an lcsc group and let $\Gamma < G$ be a uniform lattice. Let $L : \Gamma \to \mathbb{R}_+$ be a proper length function and let $\Omega$ be a compact subset of $G$ that contains the fundamental domain for the right action of $\Gamma$ on $G$. Suppose that there exists a constant $C > 0$ such that:

$$\sup_{\gamma \in \Gamma \cap \Omega \gamma \Omega^{-1}} \left| L(\gamma) - L(\gamma_0) \right| \leq C \quad \forall \gamma_0 \in \Gamma.$$  

Then $L$ is $G$-coherent.

Proof of Corollary 1.4. Let $L : \Gamma \to \mathbb{N}$ be a $G$-coherent word length function on $\Gamma$. Put $L_G : G \to \mathbb{R}_+$, $L_G(g) := \sum_{\gamma \in \Gamma} \varphi(\gamma, g)L(\gamma)$. The function $L_G$ is subadditive:

$$L_G(gh) = \sum_{\gamma \in \Gamma} \varphi(gh, \gamma)L(\gamma)$$

$$= \int_{G/\Gamma} L(\alpha(gh, x)) d\tilde{\mu}(x)$$

$$= \int_{G/\Gamma} L(\alpha(g, x)\alpha(h, g^{-1} \cdot x)) d\tilde{\mu}(x)$$

$$\leq \int_{G/\Gamma} L(\alpha(g, x)) d\tilde{\mu}(x) + \int_{G/\Gamma} L(\alpha(h, g^{-1} \cdot x)) d\tilde{\mu}(x)$$

$$= L_G(g) + L_G(h).$$

Let $\Omega = D \cup D^{-1}$, we claim that $L_G$ is quasi-isometric to $L_\Omega$. Indeed, given two elements $g \in G$ and $\gamma \in \Gamma$ such that $\varphi(\gamma, g) \geq 0$ one has for obvious reason that $L_\Omega(\gamma) \geq L_G(g) - 2$. On the other hand the word length function $L$ dominates the restriction of $L_\Omega$ to $\Gamma$ as well as $L_\Omega$ dominates $L_G$. So there exists a constant $a > 0$ such that $L_\Omega(\gamma) \leq aL(\gamma)$ for any $\gamma$ in $\Gamma$ and $L_G(g) \leq aL_\Omega(g)$ for any $g \in G$. We have that:
\[ aL_\Omega(g) \geq L_G(g) = \sum_{\gamma \in \Gamma} \varphi(\gamma, g)L(\gamma) \geq a \sum_{\gamma \in \Gamma} \varphi(\gamma, g)L_\Omega(\gamma) \geq a \sum_{\gamma \in \Gamma} \varphi(\gamma, g)(L_\Omega(g) - 2) = aL_\Omega(g) - 2a. \]

Hence the function \( L_G \) is a proper length function on \( G \) and defines a metric on \( G \) quasi-isometric to the Riemannian one. So \( G \) has property RD with respect to \( L_G \). On the other hand, as \( L \) is \( G \)-coherent, there exists a constant \( c > 0 \) such that

\[ B_{L_G}(n - c) \cap \Gamma \subset B_{L}(n) \subset B_{L_G}(n + c) \cap \Gamma, \]

where

\[ B_{L_G}(r) = \{ g \in G : L_G(g) \leq r \} \quad \text{and} \quad B_{L}(n) = \{ \gamma \in \Gamma : L(\gamma) \leq n \}. \]

So we may apply Theorem 3.3. \( \square \)

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