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Temperature quantization from the TBA equations

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ABSTRACT

We analyze the Thermodynamic Bethe Ansatz equations for the mirror model which determine the ground state energy of the light-cone $AdS_5 \times S^5$ superstring living on a cylinder. The light-cone momentum of string is equal to the circumference of the cylinder, and is identified with the inverse temperature of the mirror model. We show that the natural requirement of the analyticity of the Y-functions leads to the quantization of the temperature of the mirror model which has never been observed in any other models.

1. Introduction

An effective way to analyze exact finite-size spectrum of two-dimensional integrable field theory models is provided by the Thermodynamic Bethe Ansatz (TBA) approach originally developed for relativistic models [1]. Its application to nonrelativistic models such as the light-cone $AdS_5 \times S^5$ superstring sigma model, for a review see [2], requires studying the thermodynamic properties of a so-called mirror model obtained from the original one by a double Wick rotation. The inverse temperature of the mirror model is identified with the circumference of the cylinder the original one lives on. Then, the ground state energy of the original model is related to the free energy (or for periodic fermions to Witten's index [3]) of the mirror model. Moreover, it has been shown that the TBA approach is also capable of accounting for the excited states [4–6], see [7–17] for further results and different approaches.

In the $AdS_5 \times S^5$ case the mirror model was introduced and studied in detail in [18].[2] In particular, the mirror model S-matrix was obtained from the $AdS_5 \times S^5$ world-sheet S-matrix [20–23] by means of a proper analytic continuation, and shown to be unitary. It was then used to derive the Bethe–Yang (BY) equations for elementary particles of the mirror theory which appeared to differ in a subtle but important way from the BY equations for the light-cone $AdS_5 \times S^5$ superstring and $\mathcal{N} = 4$ SYM [24–26]. The asymptotic spectrum of the mirror model was shown to consist of the elementary particles and Q-particle bound states which comprise into the tensor product of two $4Q$-dimensional atypical totally anti-symmetric multiplets of the centrally extended $\mathfrak{su}(2|2)$ algebra. This is in contrast to the light-cone string model where the bound states belong to the tensor product of two $4Q$-dimensional atypical totally symmetric multiplets [22,27], and it played an important role in the computation of the four-loop anomalous dimension of the Konishi operator [28].

The next step is to count all asymptotic states of the mirror model, and to determine its free energy. String hypothesis enables such a computation in practice [29], and it is the most important step towards realizing the TBA approach because TBA equations are then easily derived following a textbook route [30].

Recently, the results obtained in [18] were used to formulate a string hypothesis for the mirror model [31]. The derivation of the corresponding TBA equations was then performed in [32–34],[3] where the TBA equations were also used to analyze the existence of the associated Y-system [35–38]. It was shown in [32] that the Y-system for the planar AdS/CFT correspondence [39] conjectured in [40] followed from the TBA equations only for the values of the rapidity variable u from the interval $[-2, 2]$. It appeared that for other values of u on the real axis one had to impose additional conditions on the Y-functions. We will show in this Letter that these conditions, however, are not compatible with the ground state energy solution of the TBA equations, and, therefore, an *analytic* Y-*system* does not exist. In contrast to relativistic models, the Y-system if it exists would be defined on an infinite genus Riemann surface, and it is unclear how such a Y-system could be used for analyzing the string spectrum along the lines of [7–10,17].

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[2] The importance of the TBA approach for understanding the exact spectrum of the light-cone $AdS_5 \times S^5$ superstring was stressed in [19] where it was used to explain wrapping effects in gauge theory.

[3] All the three sets of TBA equations should yield the same answer provided the same string hypothesis has been used. Checking this however is not easy due to different notations and conventions used in the papers.

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Besides its potential application to determining the full spectrum of the light-cone $AdS_5 \times S^5$ superstring, the mirror model has a new interesting feature as a two-dimensional quantum field theory. In the temporal gauge (see [2]) the light-cone momentum P_+ carried by the string is equal to the charge J corresponding to one of the $U(1)$ isometries of S^5. Since the superstring lives on a cylinder of circumference $L = P_+$, the mirror theory temperature is equal to $T = 1/J$. In quantum theory the charge J is quantized and can take only integer or half-integer values. Consistency of the mirror model would then require quantization of the inverse temperature.[4] Temperature (and length) quantization, should it happen, seems to be a new phenomenon never seen before in any other model.

In this Letter we investigate the ground state energy of the light-cone string theory by using the TBA equations of [32]. In the sector with periodic fermions the ground state is BPS and its energy should not receive quantum corrections. In this case, however, we encounter a singularity in the TBA equations, and regularize it through the chemical potential for fermions [41]. We then solve the TBA equations in the vicinity of the BPS vacuum, and find that the inverse temperature is quantized at integer or half-integer values if one imposes a natural requirement of the analyticity of the Y-functions on the z-torus.

2. Ground state energy

In this section we use the TBA equations of [32] to compute the energy of the ground state. To save the space we do not list here all the equations, and do not provide explicit expressions for the kernels involved. The reader should consult the corresponding equations and definitions in [32].

The energy of the ground state of the light-cone string theory depends on L which in the mirror theory is equal to the inverse temperature, and in the string theory is identified with the total light-cone momentum $P_+ = L$. In the temporal gauge $P_+ = J$ where J is the angular momentum carried by the string rotating about the equator of S^5. The ground state energy also depends on the parameter h that allows one to interpolate between the even-winding number sector, $h = 0$, of the light-cone string theory with periodic fermions and supersymmetric vacuum, and the odd-winding number sector, $h = \pi$, with anti-periodic fermions and nonsupersymmetric vacuum. The energy is given by the following universal expression which has no explicit dependence on h

$$E_h(L) = -\int_{-\infty}^{\infty} \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\tilde{p}_Q}{du} \log(1 + Y_Q), \quad Y_Q \equiv e^{-\epsilon_Q}, \tag{2.1}$$

where $\tilde{p}^Q$ and ϵ_Q are the momentum and pseudo-energy of a mirror Q-particle.

For the supersymmetric vacuum we expect the ground state energy to vanish $E_{h=0} = 0$. According to (2.1), the condition $E_h = 0$ requires $Y_Q = 0$. We see that the whole set of TBA equations is solved by

$$\begin{aligned} &Y_Q = 0, \qquad Y_+^{(\alpha)} = Y_-^{(\alpha)} = 1, \\ &Y_{M|vw}^{(\alpha)} = Y_{M|w}^{(\alpha)} \neq 0, \qquad e^{ih_\alpha} = 1. \end{aligned} \tag{2.2}$$

A subtle point here is that the TBA equation for Q-particles is singular at $Y_Q = 0$. To regularize this singularity in the next subsection we consider the general case with $h \neq 0$ and take the limit $h \to 0$.

2.1. Small h expansion

If h is small any Y-function can be expanded in a series in h. The leading small h behavior can be understood from the TBA equation for Q-particles

$$\begin{aligned} -\log Y_Q = {} & L\tilde{\mathcal{E}}_Q - \log(1 + Y_{Q'}) \star K_{\mathfrak{sl}(2)}^{Q'Q} \\ & - \log\left(1 + \frac{1}{Y_{M|vw}^{(\alpha)}}\right) \star K_{vwx}^{MQ} - \frac{1}{2} \log \frac{1 - \frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1 - \frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}} \star K_Q \\ & - \frac{1}{2} \log\left(1 - \frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right)\left(1 - \frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right) \star K_{yQ}, \end{aligned} \tag{2.3}$$

where

$$\tilde{\mathcal{E}}_Q = 2 \operatorname{arcsinh}\left(\frac{\sqrt{Q^2 + \tilde{p}^2}}{2g}\right), \tag{2.4}$$

is the energy of a Q-particle, $h_\alpha = (-1)^\alpha h$, the summation over $\alpha = 1, 2$ is understood, and the string tension g is related to the 't Hooft coupling λ as $g = \frac{\sqrt{\lambda}}{2\pi}$.

The last term in Eq. (2.3) shows that for small values of h, the functions $Y_\pm^{(\alpha)}$ should have an expansion of the form

$$Y_\pm^{(\alpha)} = 1 + hA_\pm^{(\alpha)} + \cdots. \tag{2.5}$$

Then the last term in Eq. (2.3) obviously behaves as $\log h$ for small h, and we get

$$-\log Y_Q = -2 \log h \star K_{yQ} + \text{finite terms}. \tag{2.6}$$

Taking into account that $1 \star K_{yQ} = 1$, we conclude that Y_Q behaves as h^2

$$Y_Q = h^2 B_Q + \cdots, \tag{2.7}$$

and, therefore, the ground state energy has the following small h expansion

$$E_h(L) = -h^2 \int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\tilde{p}^Q}{du} B_Q + \mathcal{O}(h^3). \tag{2.8}$$

Expanding all the Y-functions around the naïve solution (2.2)

$$\begin{aligned} &Y_Q \approx h^2 B_Q, \qquad Y_\pm^{(\alpha)} \approx 1 + hA_\pm^{(\alpha)} + h^2 B_\pm^{(\alpha)}, \\ &Y_{M|vw}^{(\alpha)} \approx A_M^{(\alpha)} + hB_{M|vw}^{(\alpha)}, \qquad Y_{M|w}^{(\alpha)} \approx A_M^{(\alpha)} + hB_{M|w}^{(\alpha)}, \end{aligned} \tag{2.9}$$

one can derive equations for the coefficients A's and B's by substituting the expansions into the TBA equations.

It turns out that the following conditions are consistent with the series expansion of the TBA equations up to the first order in h

$$B_{M|w}^{(\alpha)} = B_{M|vw}^{(\alpha)} \quad \Leftrightarrow \quad A_-^{(a)} = A_+^{(\alpha)} = 0. \tag{2.10}$$

Then, the TBA equations for B_Q (Q-particles), and $A_M^{(\alpha)}$ (w-strings) close within themselves, and take the following simple form[5]

[4] Even though this argument is based on the temporal gauge, the temperature quantization should happen also in an arbitrary light-cone gauge. The quantization condition, however, would take different forms in different gauges.

[5] We use the simplified equations for vw- and w-strings from Appendix 6.3 of [32].

$$-\log B_Q = L\widetilde{\mathcal{E}}_Q - \log\left(1+\frac{1}{A_M^{(\alpha)}}\right)\star K_{vwx}^{MQ}, \tag{2.11}$$

$$\log A_M^{(\alpha)} = \log\big(1+A_{M-1}^{(\alpha)}\big)\big(1+A_{M+1}^{(\alpha)}\big)\star s. \tag{2.12}$$

These equations admit a solution with all $A_M^{(\alpha)}$ being constants independent of u. In this case, taking into account that $1\star s=\frac{1}{2}$, the second equation reduces to the following form

$$\big(A_M^{(\alpha)}\big)^2 = \big(1+A_{M-1}^{(\alpha)}\big)\big(1+A_{M+1}^{(\alpha)}\big) \quad \text{for } M\geqslant 1,\ A_0^{(\alpha)}=0. \tag{2.13}$$

It has the following regular solution[6]

$$A_{M-1}^{(\alpha)} = M^2-1 \quad (M\geqslant 1), \tag{2.14}$$

which coincides with the constant solution of a Y-system discussed in [8,9].

We can now find B_Q from Eq. (2.11). For constant $A_M^{(\alpha)}$ the convolution terms in (2.11) can be computed by using that $1\star K_{vwx}^{MQ}=n_{vwx}^{M,Q}$, where the integers $n_{vwx}^{M,Q}$ satisfy $n_{vwx}^{M,Q}=M-1$ for $M<Q-1$, and $n_{vwx}^{M,Q}=Q$ for $M\geqslant Q-1$. A simple computation then gives

$$\begin{aligned}\log&\left(1+\frac{1}{A_M^{(\alpha)}}\right)\star K_{vwx}^{MQ}\\ &=\sum_{M=2}^{Q-1} M\log\left(1+\frac{1}{M^2-1}\right)+Q\sum_{M=Q}^{\infty}\log\left(1+\frac{1}{M^2-1}\right)\\ &=\log\left(\frac{2(Q-1)^Q}{Q^{\,Q-1}}\right)+Q\log\left(\frac{Q}{Q-1}\right)=\log 2Q,\end{aligned} \tag{2.15}$$

for each $\alpha=1,2$, and therefore

$$Y_Q = 4h^2Q^2e^{-L\widetilde{\mathcal{E}}_Q}+\mathcal{O}\big(h^3\big). \tag{2.16}$$

Taking into account that the energy of a mirror Q-particle can be written in the form

$$\widetilde{\mathcal{E}}_Q = \log\frac{x^{Q-}}{x^{Q+}}, \quad x^{Q\pm}(u)=x\left(u\pm\frac{i}{g}Q\right), \tag{2.17}$$

the Y_Q-functions acquire the form

$$Y_Q = 4h^2Q^2\left(\frac{x^{Q+}}{x^{Q-}}\right)^L+\mathcal{O}\big(h^3\big). \tag{2.18}$$

Since the variables $x^{Q\pm}$ are expressed in terms of the function $x(u)=\frac{1}{2}(u-i\sqrt{4-u^2})$, the Y_Q-functions are not analytic on the u-plane for any value of L, and have there two cuts. On the other hand it is known that the dispersion relation for Q-particles is uniformized in terms of the z-torus rapidity variable [42], and the ratio x^{Q+}/x^{Q-} is given by $(\operatorname{cn} z+i\operatorname{sn} z)^2$, which is real when z is on the real axis of mirror region.[7] We conclude, therefore, that the Y_Q-functions are meromorphic on the z-torus if

$$L=\frac{1}{T} \quad \text{is integer or half-integer.} \tag{2.19}$$

We do not expect Y_Q to be analytic on the z-torus for finite values of h because then the dressing factor [43–45] would start contribute to the equations for Y_Q, and it is known that the dressing factor has infinitely many cuts on the z-torus. Nevertheless, as was recently shown in [46], the dressing factor[8] is holomorphic in the union of the physical regions of the string and mirror models, and, therefore, it is natural to require the Y_Q-functions to be meromorphic there too.

As we have shown above, an unusual consequence of this requirement is that the circumference of the circle the light-cone string theory lives on, and the temperature of the mirror theory are quantized. Let us also mention that the charge quantization is the first step to understand the full $\mathfrak{psu}(2,2|4)$ symmetry of the string spectrum from the TBA approach.[9]

Finally, the ground state energy at the leading order in h and arbitrary L is given by

$$\begin{aligned}E_h(L)&\approx -h^2\int\frac{du}{2\pi}\sum_{Q=1}^{\infty}\frac{d\tilde{p}^Q}{du}4Q^2e^{-L\widetilde{\mathcal{E}}_Q}\\ &=-h^2\sum_{Q=1}^{\infty}\int\frac{d\tilde{p}^Q}{2\pi}4Q^2e^{-L\widetilde{\mathcal{E}}_Q}.\end{aligned} \tag{2.20}$$

By using Eq. (2.4) for the energy of a Q-particle, one can show that the sum is convergent for $L>2$. For $L=2$ the series in Q diverges as $\frac{1}{Q}$. We do not understand the reason for the divergency. Note that $L=2$ is the lowest value the total light-cone momentum can have.

2.2. General h at large L

It is also of interest to consider the large L asymptotics of the ground state energy with h fixed. In this case we expect that the finite-size corrections to the energy of the ground state can be also computed by introducing a twist in the generalized Lüscher formula [28,48–50]:

$$\begin{aligned}E_{\mathrm{gL}}(L)=&-\int\frac{du}{2\pi}\sum_{Q=1}^{\infty}\frac{d\tilde{p}^Q}{du}e^{-L\widetilde{\mathcal{E}}_Q}\,\mathrm{tr}_Q\,e^{i(\pi+h)F}\\ &+\mathcal{O}\big(e^{-2L\widetilde{\mathcal{E}}_Q}\big).\end{aligned} \tag{2.21}$$

Here the trace runs through all $16Q^2$ polarizations of a Q-particle state, and F is the fermion number operator which in our case is equal to the difference F_1-F_2 where F_α is equal to the number of $y^{(\alpha)}$-particles.

Computing the trace in (2.21)

$$\begin{aligned}\mathrm{tr}_Q\,e^{i(\pi+h)F}&\equiv\mathrm{tr}_Q\,e^{i(\pi+h)(F_1-F_2)}\\ &=2Q\big(1-e^{ih}\big)\cdot 2Q\big(1-e^{-ih}\big),\end{aligned} \tag{2.22}$$

and substituting the result back into (2.21), we obtain

$$\begin{aligned}E_{\mathrm{gL}}(L)=&-\int\frac{du}{2\pi}\sum_{Q=1}^{\infty}\frac{d\tilde{p}^Q}{du}16Q^2\sin^2\frac{h}{2}e^{-L\widetilde{\mathcal{E}}_Q}\\ &+\mathcal{O}\big(e^{-2L\widetilde{\mathcal{E}}_Q}\big).\end{aligned} \tag{2.23}$$

At small values of h the formula obviously agrees with (2.20). We will see in a moment that it matches precisely the large L asymptotics of the ground state energy with h fixed computed by using the TBA equations.

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[6] One can also check that Eq. (2.14) solves the original TBA equations for vw- and w-strings, by using $n_{xv}^{QM}\equiv 1\star K_{xv}^{QM}=M-1$ for $M-1<Q$ and $n_{xv}^{QM}=Q$ for $M-1\geqslant Q$, together with $n_{MM'}\equiv 1\star K_{MM'}=2M$ for $M<M'$ and $n_{MM'}=2M'-\delta_{MM'}$ for $M\geqslant M'$.

[7] The elliptic modulus $k=-4g^2/Q^2$ of the Jacobi functions depends on Q, and, therefore, the periods of the torus depend on Q too.

[8] The BES dressing factor [45] was proven [47] to be the minimal solution of crossing equations [42].

[9] Note that the $\mathfrak{psu}(2|2)^2$ charges of the string theory and the mirror theory are not physically equivalent due to double Wick rotation.

The series expansion of Y-functions in terms of $e^{-L\tilde{\mathcal{E}}_Q}$ can be performed almost in the same manner as the small h expansion. We can find a consistent solution at the leading order assuming the following expansion of the Y-functions

$$Y_Q \approx B_Q e^{-L\tilde{\mathcal{E}}_Q}, \qquad Y_\pm^{(\alpha)} \approx A_\pm^{(\alpha)},$$
$$Y_{M|w}^{(\alpha)} \approx A_{M|w}^{(\alpha)}, \qquad Y_{M|vw}^{(\alpha)} \approx A_{M|vw}^{(\alpha)}, \tag{2.24}$$

where B_Q and A's are independent of u.

Then, $A_{M|w}^{(\alpha)} = A_{M|vw}^{(\alpha)}$ given by the same constant ansatz of (2.14) solve the equations for vw- and w-strings if $A_\pm^{(\alpha)} = 1$. The TBA equation for Q-particles (2.3) then takes the same form as for the small h case with the only change $h^2 \to 4\sin^2\frac{h}{2}$, and, therefore, its solution is given by

$$B_Q = 16Q^2\sin^2\frac{h}{2}.$$

Thus, the energy of the ground state becomes

$$E_h(L) = -\int\frac{du}{2\pi}\sum_{Q=1}^{\infty}\frac{d\tilde{p}^Q}{du}16Q^2\sin^2\frac{h}{2}e^{-L\tilde{\mathcal{E}}_Q} + \mathcal{O}\big(e^{-2L\tilde{\mathcal{E}}_Q}\big), \tag{2.25}$$

which completely agrees with the generalized Lüscher formula (2.23). It is worth stressing that the contribution of the vw-strings is crucial for the agreement.[10]

The energy, as well as the generalized Lüscher formula, obviously diverges again logarithmically for $L = 2$.[11] Note that the corresponding formula in the computation of the anomalous dimension of the Konishi operator [28] contains additional factor of $e^{-2\tilde{\mathcal{E}}_Q}$ coming from the transfer matrix, which renders the series convergent.

Let us finally mention that for $h = \pi$ the formula (2.25) should give the energy of the non-BPS ground state of the light-cone string theory in the sector with anti-periodic fermions, and through the AdS/CFT correspondence the scaling dimension of the dual $\mathcal{N} = 4$ SYM operator. It would be interesting to identify this operator and compute its perturbative scaling dimension.

2.3. Analyticity of Y-system

We recall that the simplified TBA equation for Y_1-function takes the form [32]

$$\log Y_1 = \log\frac{(1-\frac{e^{ih_1}}{Y_-^{(1)}})(1-\frac{e^{ih_2}}{Y_-^{(2)}})}{1+\frac{1}{Y_2}}\star s - \Delta\star s, \tag{2.26}$$

where

$$\Delta = \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\big(\theta(-u-2)+\theta(u-2)\big) + L\check{\mathcal{E}} - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right) \times\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K} - \log\left(1+\frac{1}{Y_{M|vw}^{(1)}}\right)\left(1+\frac{1}{Y_{M|vw}^{(2)}}\right)\star\check{K}_M + 2\log(1+Y_Q)\star\check{K}_Q^\Sigma, \tag{2.27}$$

is the obstruction to have the Y-system outside $u\in[-2,2]$. It has been shown in [32] that the TBA equations may lead to a usual Y-system only if Δ vanishes on any solution. Let us also stress that the vanishing of Δ is only one of the several necessary conditions the Y-functions should satisfy, see [32] for a detailed discussion. These conditions are trivially satisfied at the leading order in the small h expansion, but it is unclear how to verify all the necessary conditions for finite h.

To compute Δ for the ground state solution we use that $1\star\check{K} = \frac{1}{2}(\theta(-u-2)+\theta(u-2))$, and $1\star\check{K}_M = 0$, and in the both small h and large L cases we get the following leading term

$$\Delta = L\check{\mathcal{E}} = L\log\frac{x(u-i0)}{x(u+i0)} \neq 0 \quad \text{for } u\in(-\infty,-2)\cup(2,\infty). \tag{2.28}$$

Since Δ does not vanish, the Y-functions are not analytic in the complex u-plane,[12] and the TBA equations do not lead to an analytic Y-system. It still might be possible to define (but not to derive) the Y-system on an infinite genus Riemann surface,[13] and require the validity of the Y-system equations on its particular sheet.

To see how this might work, let us recall that the Y-equation is obtained from (2.26) by applying to it an operator s^{-1}, and has the following form [32]

$$e^{\Delta(u)}Y_1\left(u+\frac{i}{g}-i0\right)Y_1\left(u-\frac{i}{g}+i0\right) = \frac{(1-\frac{e^{ih_1}}{Y_-^{(1)}})(1-\frac{e^{ih_2}}{Y_-^{(2)}})}{1+\frac{1}{Y_2}}. \tag{2.29}$$

The explicit ground state solution (2.18) and (2.28) then show that the jump discontinuity of $\log Y_1(u\pm\frac{i}{g})$ across the real u-line is given by $\pm\Delta(u)$, and, therefore, for the ground state solution (2.29) can be written in the form

$$Y_1\left(u+\frac{i}{g}\pm i0\right)Y_1\left(u-\frac{i}{g}\pm i0\right) = \frac{(1-\frac{e^{ih_1}}{Y_-^{(1)}})(1-\frac{e^{ih_2}}{Y_-^{(2)}})}{1+\frac{1}{Y_2}}. \tag{2.30}$$

Thus, we conclude that the Y-system equation for Y_1 might hold on the u-plane with the cuts running from $\pm2\pm\frac{i}{g}$ to infinity along the horizontal lines if the shifts upward and downward are defined with the infinitesimal parts of the same sign. The equations for Y_Q ($Q\geqslant2$) would then induce infinitely many cuts on the u-plane with the branch points located at $\pm2\pm\frac{i}{g}Q$.

Let us stress again that the Y-system equations can have the canonical form only on a particular sheet of the infinite genus Riemann surface, and probably would take different forms on other sheets. In this respect it is similar to the $AdS_5\times S^5$ crossing equations [42]. It would be interesting (and necessary) to understand the corresponding transformation properties of the Y-system. This is in contrast to relativistic models, and it is unclear to us if such

[10] There is simple generalization of this agreement. If we generalize the solution (2.13) to $A_{M-1} = \sin^2(Mz/2)/\sin^2(z/2) - 1$, the Y_Q-functions agree with a so-called elliptic genus, $\mathrm{tr}_Q\, e^{-L\tilde{\mathcal{E}}_Q + i(\pi+h)F + izJ_3}$ with $J_3 = \sigma_3\cdot\mathbb{L} + \sigma_3\cdot\mathbb{R}$ in the notation of [51].

[11] Eq. (2.25) was derived for large L. It nevertheless is valid for finite L and small g because in this case Y_Q is decreasing as g^{2L}.

[12] We have seen this already from the explicit solution (2.18).

[13] We thank Pedro Vieira for a discussion of this possibility.

a Y-system would be useful for analyzing the spectrum along the lines of [7–10,17].

3. Conclusions

In this Letter we have analyzed the TBA equations for the ground state energy of the light-cone superstring on the $AdS_5 \times S^5$ background. We have shown that the natural condition of the analyticity of the solution of the TBA equations on the union of the physical regions of the string and mirror $AdS_5 \times S^5$ models leads to the quantization of the circumference of the cylinder the string theory lives on, or, equivalently, to the temperature quantization of the mirror model.

The temperature quantization is a new phenomenon never seen before, and it is the simplest manifestation of the $\mathfrak{psu}(2,2|4)$ symmetry of the superstring and $\mathcal{N}=4$ SYM spectrum. The full $\mathfrak{psu}(2,2|4)$ symmetry can been seen only in the TBA equations for excited states, and it is of utmost importance to prove it.

We also have analyzed the TBA equations for large L and finite h, and have shown that the ground state energy completely agrees with the twisted Lüscher formula. We have observed that the energy is logarithmically divergent for $L=2$. It would be interesting to understand the origin of the divergency.

The TBA equations describe the spectrum of light-cone superstring on $AdS_5 \times S^5$ only for $h=0$ or $h=\pi$. The general h case should describe something which goes beyond the usual correspondence between $AdS_5 \times S^5$ superstring and $\mathcal{N}=4$ super-Yang–Mills.

On string theory side, one can introduce magnetic flux coupled to worldsheet fermions. A fermion acquires an extra phase $\chi \to e^{ih}\chi$ when it goes around the worldsheet cylinder. This magnetic flux is topological (or Aharanov–Bohm type) in the sense that the phase of a worldsheet fermion remains unchanged for any contractible cycle on the cylindrical worldsheet. The magnetic field must be a spacetime singlet, because the extra phase does not change the spacetime index of fermions. The magnetic field can couple to worldsheet fermions by replacing $\partial_a\chi$ with $(\partial_a - iA_a)\chi$ in the light-cone superstring sigma model.

At weak coupling, the TBA energy is a quantity of order $\mathcal{O}(g^{2L})$ for any length L operator and for any h. Since the effect of non-zero h can be observed only beyond the wrapping order, the chemical potential does not modify the local structure of the dilatation operator of $\mathcal{N}=4$ super-Yang–Mills. Thus, on the gauge theory side, we expect that there is a way to interpret the chemical potential as a modification of local operators, e.g. as in orbifold models, see [52], rather than a deformation of the Lagrangian. A convincing interpretation is not known to us, and it would be interesting to clarify this.

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