In this paper we extend a (scalarized) generalized type-I invexity into a vector invexity (V-type I). A number of sufficiency results are established using Lagrange multiplier conditions and under various types of generalized V-type I requirements. Weak, strong, and converse duality theorems are proved in the generalized V-invexity type I setting. © 2001 Academic Press

1. INTRODUCTION

The field of multiobjective programming, also known as vector programming, has grown remarkably in different directions in the settings of optimality conditions and duality theory since the 1980s. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions, and in
the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality, variational problems, etc. A new reader may like to consult Aggarwal [2], Pandian [23], Pini and Singh [24] for relatively more exhaustive references on the subject. More specifically, some of the recent work in the area of nonsmooth setting can be found in Abudouni and Thibault [1], Brandao et al. [8], and Mishra and Mukherjee [19]. Along the same lines, second order optimality conditions are the subject of the investigations in Aghezzaf [3], Aghezzaf and Hachini [4], Bolintineanu and Maghri [7], and Wang [25]. The setting of the investigation in Mukerjee and Mishra [21] is in semilocal convexity while in Osuna-Gomez et al. [22] it is under invexity. Multiobjective linear programming under a fuzzy environment is discussed in Wang and Wang [26]. Regularity conditions and constraint qualifications are the subject matter of Bigi and Pappalardo [5] and Maeda [17]. Benson-type proper efficiency under set-valued maps is treated in Li [16]. Multiobjective in infinite dimensions is the setting in Abdouni and Thibault [1], Brandao et al. [8] and in Jeyakumar and Zaffaroni [14]. The basic concepts in all these developments are the weak vector minimum, the efficient point, and the properly efficient point. Relatively fewer articles have appeared in the literature dealing with weak vector minima compared to the ones that deal with efficient points and proper efficient points. Finally, monotonicity of the compromising set was the subject matter of Blaso et al. [6].

Parallel to the above development in multiobjective programming there has been a very popular growth and application of invexity theory which was originated by Hanson [11] but so named by Craven [9]. Later Hanson and Mond [12] introduced type-I and type-II invexities which have been further generalized by many researchers and applied to nonlinear programming problems in different settings.

In this paper we introduce vector type invexity along the lines of Jeyakumar and Mond [13] extending the pseudo, quasi, quasi-pseudo, pseudo-quasi type-I invexity of Kaul et al. [15]. In Section 2, we introduce some preliminaries. Some sufficiency results are established in Section 3. A number of duality theorems in the Mond–Weir setting [20] are shown to hold in Section 4. In their paper, Kaul et al. [15] established duality results both in the Mond–Weir setting [20] and in the Wolfe setting [27]. It seems to be an open question to study Wolfe duality in our setting. This point seems to have its own merit in the sense that so far, since the introduction of Mond–Weir duality, most of the results that have appeared in the literature in recent years hold for both types of duals.
To compare vectors along the lines of Mangasarian [8], we will distinguish between \( \leq \) and \( \geq \) or between \( \leq \) and \( \geq \). Specifically,
\[
x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad x \leq y \iff x_i \leq y_i \ \forall \ i = 1, \ldots, n, \ x \neq y
\]
\[
x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad x \leq y \iff x_i \leq y_i \ \forall \ i = 1, \ldots, n.
\]
Similar notations are applied to distinguish between \( \geq \) and \( \leq \).

We consider the multiobjective optimization problem
\[\text{(VP)} \quad \text{V-minimize} \quad f(x) = (f_1(x), \ldots, f_p(x)) \text{ subject to } g(x) \leq 0,\]
where \( f: X \to \mathbb{R}^p \) and \( g: X \to \mathbb{R}^m \) are differentiable functions and \( X \subseteq \mathbb{R}^n \) is an open set. Here the symbol “V-minimize” stands for finding the collection of (properly) efficient points defined below.

Let \( X_0 \) be the set of feasible solutions of (VP). To refresh the reader’s memory we reproduce some of the definitions and also give some new ones.

**Definition 2.1.** A point \( a \in X_0 \) is said to be an efficient point (Pareto) of the problem (VP) if there exists no \( x \in X_0 \) such that \( f(x) \leq f(a) \).

**Definition 2.2 (Geoffrion [10]).** An efficient point \( a \) of (VP) is said to be properly efficient if there exists a positive real number \( M \) such that for each \( x \in X_0 \) and for each pair of indices \( j, i \) such that \( f_j(a) - f_j(x) > 0 \), then \( f_j(a) - f_j(x) \leq M(f_i(x) - f_i(a)) \) whenever \( f_i(x) - f_i(a) > 0 \).

Following Hanson [11], Jeyakumar and Mond [13], and Kaul et al. [15] we define vector type-I problems.

**Definition 2.3.** We say the problem (VP) is of \( V \)-type I at \( a \in X_0 \) if there exist positive real-valued functions \( \alpha_i \) and \( \beta_i \) defined on \( X \times X \) and an \( n \)-dimensional vector-valued function \( \eta: X \times X \to \mathbb{R}^n \) such that
\[
f_i(x) - f_i(a) \geq \alpha_i(x, a) \nabla f_i(a) \eta(x, a) \tag{2.1}
\]
and
\[
-g_j(a) \geq \beta_j(x, a) \nabla g_j(a) \eta(x, a), \tag{2.2}
\]
for every \( x \in X_0 \) and for all \( i = 1, \ldots, p \), and \( j = 1, \ldots, m \).

If (VP) is of \( V \)-type I at each \( a \in X \), we say (VP) is of \( V \)-type I on \( X \). If strict inequality holds in (2.1) (whenever \( x \neq a \)) we say that (VP) is of semi strictly \( V \)-type I at \( a \) or on \( X \) as the case may be.
DEFINITION 2.4. We say the problem (VP) is of quasi V-type I at \( a \in X_0 \) if there exist positive real-valued functions \( \alpha_i \) and \( \beta_j \) defined on \( X \times X \) and an \( n \)-dimensional vector-valued function \( \eta: X \times X \to \mathbb{R}^n \) such that for some vectors \( \tau \in \mathbb{R}^p, \tau \geq 0 \), and \( \lambda \in \mathbb{R}^m, \lambda \geq 0 \),

\[
\sum_i \tau_i \alpha_i(x, a)(f_i(x) - f_i(a)) \leq 0 \Rightarrow \sum_i \tau_i \eta(x, a) \nabla f_i(a) \leq 0 \forall x \in X \quad (2.3)
\]

and

\[
\sum_j \lambda_j \beta_j(x, a) g_j(a) \geq 0 \Rightarrow \sum_j \lambda_j \eta(x, a) \nabla g_j(a) \leq 0 \forall x \in X. \quad (2.4)
\]

If (VP) is of quasi V-type I at each \( a \in X \), we say (VP) is of quasi V-type I on \( X \). If the second (implied) inequality in (2.3) is strict \((x \neq a)\) we say that (VP) is semi strictly quasi V-type I at \( a \) or on \( X \) as the case may be.

DEFINITION 2.5. We say the problem (VP) is of pseudo V-type I at \( a \in X_0 \) if there exist positive real-valued functions \( \alpha_i \) and \( \beta_j \) defined on \( X \times X \) and an \( n \)-dimensional vector-valued function \( \eta: X \times X \to \mathbb{R}^n \), such that for some \( \tau \in \mathbb{R}^p, \tau \geq 0 \), and \( \lambda \in \mathbb{R}^m, \lambda \geq 0 \), the implications

\[
\sum_i \tau_i \eta(x, a) \nabla f_i(a) \geq 0 \Rightarrow \sum_i \tau_i \alpha_i(x, a)(f_i(x) - f_i(a)) \geq 0 \quad \forall x \in X \quad (2.5)
\]

and

\[
\sum_j \lambda_j \eta(x, a) \nabla g_j(a) \geq 0 \Rightarrow \sum_j \lambda_j \beta_j(x, a) g_j(a) \leq 0 \quad \forall x \in X. \quad (2.6)
\]

hold. If (VP) is of pseudo V-type I at each \( a \in X \), we say (VP) is of pseudo V-type I on \( X \). If the second (implied) inequality in (2.5) (Eq. (2.6)) is strict, we say that (VP) is semi strictly pseudo V-type I in \( f \) (in \( g \)) at \( a \) or on \( X \) as the case may be. If the second (implied) inequalities in (2.5) and (2.6) are both strict we say that (VP) is strictly pseudo V-type I at \( a \) or on \( X \) as the case may be.

DEFINITION 2.6. We say that the problem (VP) is of quasi pseudo V-type I at \( a \in X_0 \) if there exist positive real-valued functions \( \alpha_i \) and \( \beta_j \) defined on \( X \times X \) and an \( n \)-dimensional vector-valued function \( \eta: X \times X \to \mathbb{R}^n \), such that for some \( \tau \in \mathbb{R}^p, \tau \geq 0 \), and \( \lambda \in \mathbb{R}^m, \lambda \geq 0 \), the implications

\[
\sum_i \tau_i \alpha_i(x, a)(f_i(x) - f_i(a)) \leq 0 \Rightarrow \sum_i \tau_i \eta(x, a) \nabla f_i(a) \leq 0 \quad \forall x \in X \quad (2.7)
\]

and

\[
\sum_j \lambda_j \eta(x, a) \nabla g_j(a) \geq 0 \Rightarrow \sum_j \lambda_j \beta_j(x, a) g_j(a) \leq 0 \quad \forall x \in X. \quad (2.8)
\]

hold. If (VP) is of quasi pseudo V-type I at each \( a \in X \), we say (VP) is of quasi pseudo V-type I on \( X \). If the second (implied) inequality in (2.8) is strict, we say that (VP) is quasi strictly pseudo V-type I at \( a \) or on \( X \) as the case may be.
Definition 2.7. We say problem (VP) is of pseudo quasi V-type I at 
\(a \in X_0\) if there exist positive real-valued functions \(\alpha_i\) and \(\beta_j\) defined on 
\(X \times X\) and an \(n\)-dimensional vector-valued function \(\eta: X \times X \to \mathbb{R}^n\), such 
that for some \(\tau \in \mathbb{R}^p, \tau \geq 0\), and \(\lambda \in \mathbb{R}^m, \lambda \geq 0\), the implications

\[
\sum_i \tau_i \eta(x, a) \nabla f_i(a) \geq 0 \implies \sum_i \tau_i \alpha_i(x, a)(f_i(x) - f_i(a)) \geq 0 \quad \forall x \in X \quad (2.9)
\]

and

\[
\sum_j \lambda_j \beta_j(x, a) g_j(a) \geq 0 \implies \sum_j \lambda_j \eta(x, a) \nabla g_j(a) \leq 0 \quad \forall x \in X \quad (2.10)
\]

hold. If (VP) is of pseudo quasi V-type I at each \(a \in X\), we say (VP) is of 
pseudo quasi V-type I on \(X\). If the second (implied) inequality in (2.9) is 
strict, we say (VP) is strictly pseudo quasi V-type I at \(a\) or on \(X\) as the case 
may be.

Remark 2.1. For clarity’s sake, whenever necessary we will specifically 
state the choice of \(\lambda\) and \(\tau\) with respect to which a particular problem is of 
any of the above kinds of generalized type I.

3. Optimality Conditions

In this section we establish some sufficient conditions for an \(a \in X_0\) to 
be an efficient solution of problem (VP) under various generalized type-I 
conditions specified in the definitions given above.

Theorem 3.1 (Sufficiency). Suppose that

(i) \(a \in X_0\);

(ii) there exist \(\tau^0 \in \mathbb{R}^p, \tau^0 \geq 0\), and \(\lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0\), such that

(a) \(\sum_i \tau^0_i \nabla f_i(a) + \sum_j \lambda^0_j \nabla g_j(a) = 0\),

(b) \(\lambda^0 g(a) = 0\);

(iii) the problem (VP) is quasi strictly pseudo V-type I at \(a\) with respect to 
\(\tau^0, \lambda^0\) and for some positive functions \(\alpha_i, \beta_j\), for \(i = 1, \ldots, p, j = 1, \ldots, m\).

Then \(a\) is an efficient solution for (VP).

Proof. Suppose \(a\) is not an efficient solution of (VP). Then there exists 
an \(x \in X_0\) such that \(f(x) \leq f(a)\) which implies that

\[
\sum_i \tau^0_i \alpha_i(x, a)(f_i(x) - f_i(a)) \leq 0.
\]
From the above inequality and the hypothesis (iii) (in view of Definition 2.6), it follows that
\[ \sum_i \tau_i^0 \eta(x, a) \nabla f_i(a) \leq 0. \] (3.1)
By the inequality (3.1) and Hypothesis (ii)(a) we have
\[ \sum_j \lambda_j^0 \eta(x, a) \nabla g_j(a) \geq 0. \]
From the above inequality and Hypothesis (iii) it follows that
\[ \sum_j \lambda_j^0 \eta(x, a) \nabla g_j(a) < 0. \] (3.2)
Now by Hypotheses (i) and (ii)(b) it follows that \( \lambda_j^0 g_j(a) = 0 \) for every \( j \), which further implies that
\[ \sum_j \lambda_j^0 \beta_j(x, a) g_j(a) = 0. \]
The last equation contradicts the inequality (3.2) and hence the conclusion follows.

**Theorem 3.2 (Sufficiency).** Suppose that

(i) \( a \in X_0 \);
(ii) there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \) and \( \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that
   \[ \sum_i \tau_i^0 \nabla f_i(a) + \sum_j \lambda_j^0 \nabla g_j(a) = 0, \]
   \[ \lambda^0 g(a) = 0; \]
(iii) the problem (VP) is pseudo quasi \( V \)-type I at \( a \) with respect to \( \tau^0, \lambda^0 \) and for some positive functions \( \alpha_i, \beta_j \), for \( i = 1, \ldots, p, j = 1, \ldots, m. \)

Then \( a \) is an efficient solution for (VP). If, further, there exist positive real numbers \( n_i, m_i \) such that \( n_i < \alpha_i(x, a) < m_i \), for all \( x \in X_0 \) and for all \( i = 1, \ldots, p, \) then \( a \) is properly efficient for (VP).

**Proof.** Suppose \( a \) is not an efficient solution of (VP). Then there exists an \( x \in X_0 \) such that \( f(x) \leq f(a) \) which implies that
\[ \sum_i \tau_i^0 \alpha_i(x, a) (f_i(x) - f_i(a)) < 0. \] (3.3)
Next, by the hypotheses (i) and (ii)(b), we have
\[ \sum_j \lambda_j^0 \beta_j(x, a) g_j(a) = 0. \]
From the above equality and the hypothesis (iii) (in view of Definition 2.7), it follows that
\[ \sum_{j} \lambda_{j}^{0} \eta(x, a) \nabla g_{j}(a) \leq 0. \]  
(3.4)

Now by (3.4) and the hypothesis (ii)(a), we have
\[ \sum_{i} \tau_{i}^{0} \eta(x, a) \nabla f_{i}(a) \geq 0. \]  
(3.5)

Finally, by (3.5) and the hypothesis (iii), we have, for all \( x \in X \),
\[ \sum_{i} \tau_{i}^{0} \alpha_{i}(x, a)(f_{i}(x) - f_{i}(a)) \geq 0. \]  
(3.6)

Since (3.5) and (3.6) contradict each other, we have the conclusion that \( a \) is an efficient solution of (VP).

We assume that \( p \geq 2 \). Next let
\[ M = (p - 1) \max_{i,j} \{(m, \tau_{j})/(n, \tau_{i})\} \quad \forall i \neq j; 1 \leq i, j \leq p. \]

Suppose \( a \) is not properly efficient for (VP). Then there exists an \( x_{0} \in X_{0} \) such that for some \( i \) with \( f_{i}(a) - f_{i}(x_{0}) > 0 \),
\[ f_{i}(a) - f_{i}(x_{0}) > M(f_{j}(x_{0}) - f_{j}(a)) \]
\[ \forall j \text{ such that } f_{j}(x_{0}) - f_{j}(a) > 0. \]  
(3.7)

From (3.7) it follows that
\[ f_{i}(a) - f_{i}(x_{0}) > (p - 1)((m, \tau_{j})/(n, \tau_{i}))(f_{j}(x_{0}) - f_{j}(a)) \quad \forall j \neq i, \]
which implies that
\[ f_{i}(a) - f_{i}(x_{0}) > (((p - 1)\tau_{j}\alpha_{j}(x_{0}, a)) \]
\[ /((\tau_{i}\alpha_{i}(x_{0}, a))(f_{j}(x_{0}) - f_{j}(a)) \quad \forall j \neq i, \]
which further implies that
\[ ((\tau_{j}\alpha_{j}(x_{0}, a))/(p - 1))(f_{i}(a) - f_{i}(x_{0})) > \tau_{j}\alpha_{j}(x_{0}, a)(f_{j}(x_{0}) - f_{j}(a)). \]  
(3.8)

Summing (3.8) with respect to \( j \), we have that
\[ (\tau_{i}\alpha_{i}(x_{0}, a))(f_{i}(a) - f_{i}(x_{0})) > \sum_{j \neq i} \tau_{j}\alpha_{j}(x_{0}, a)(f_{j}(x_{0}) - f_{j}(a)), \]
that is,
\[ \sum_{j} \tau_{j}\alpha_{j}(x_{0}, a)(f_{j}(x_{0}) - f_{j}(a)) < 0. \]  
(3.9)

Now (3.9) contradicts (3.6) and hence \( a \) is a properly efficient solution for (VP).
**Theorem 3.3 (Sufficiency).** Suppose that

(i) \( a \in X_0 \);
(ii) there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 \geq 0, \) and \( \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that

\[
\sum_i \tau^0_i \nabla f_i(a) + \sum_j \lambda^0_j \nabla g_j(a) = 0,
\]

(b) \( \lambda^0 g(a) = 0; \)

(iii) the problem (VP) is semi strictly quasi V-type I at \( a \) with respect to \( \tau^0, \lambda^0 \) and for some positive functions \( \alpha_i, \beta_j \) for \( i = 1, \ldots, p, \) \( j = 1, \ldots, m. \)

Then \( a \) is an efficient solution of (VP).

**Proof.** Suppose that there exists an \( x \in X_0, x \neq a \) such that \( f(x) \leq f(a); \) this implies that

\[
\sum_i \tau^0_i \alpha_i(x, a)(f_i(x) - f_i(a)) \leq 0. \tag{3.10}
\]

From inequality (3.10) and the hypothesis (iii) (in view of Definition 2.4), it follows that

\[
\sum_i \tau^0_i \eta(x, a) \nabla f_i(a) < 0. \tag{3.11}
\]

Since \( \lambda^0 g(a) = 0 \) implies that \( \lambda^0_j g_j(a) = 0 \) for all \( j \) and \( \beta_j > 0 \) for all \( j, \) we have

\[
\sum_j \lambda^0_j \beta_j(x, a) g_j(a) = 0. \tag{3.12}
\]

Now (3.12) and the hypothesis (iii) imply that

\[
\sum_j \lambda^0_j \eta(x, a) \nabla g_j(a) \leq 0. \tag{3.13}
\]

Adding (3.11) and (3.13) we see that the hypothesis (ii)(a) is contradicted. Hence the conclusion follows.

**Theorem 3.4 (Sufficiency).** Suppose that

(i) \( a \in X_0; \)
(ii) there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 \geq 0, \) and \( \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that

(a) \( \sum_i \tau^0_i \nabla f_i(a) + \sum_j \lambda^0_j \nabla g_j(a) = 0, \)

(b) \( \lambda^0 g(a) = 0; \)
(iii) problem (VP) is strictly pseudo $V$-type I at $a$ with respect to $\tau^0, \lambda^0$ and for some positive functions $\alpha_i, \beta_j$ for $i = 1, \ldots, p$, $j = 1, \ldots, m$.

Then $a$ is an efficient solution of (VP). If further $\tau > 0$ and there exist positive real numbers $n_i, m_j$ such that $n_i < \alpha_i(x, a) < m_j$, for all $x \in X_0$ and for all $i = 1, \ldots, p$, then $a$ is properly efficient for (VP).

Proof. By hypothesis (ii)(b) it follows that

$$\sum_j \lambda_0^j \beta_j(x, a) g_j(a) = 0 \quad \forall \ x \in X,$$

which implies by the hypothesis (iii) (in view of the reverse implication in Definition 2.5) that

$$\sum_j \lambda_0^j \eta_j(x, a) \nabla g_j(a) < 0 \quad \forall \ x \in X,$$

which in turn implies by the hypothesis (ii)(a) that

$$\sum_i \tau_0^i \eta(x, a) \nabla f_i(a) > 0 \quad \forall \ x \in X.$$

(3.14)

Now from (3.14) and hypothesis (iii), we have

$$\sum_i \tau_0^i \alpha_i(x, a)(f_i(x) - f_i(a)) > 0 \quad \forall \ x \in X.$$

(3.15)

Next if $a$ is not an efficient solution of (VP), then there exists an $x^* \in X_0$ such that $f(x) \leq f(a)$ which implies that

$$\sum_i \tau_0^i \alpha_i(x, a)(f_i(x) - f_i(a)) \leq 0 \quad \forall \ x \in X.$$

(3.16)

Since (3.15) and (3.16) contradict each other, the conclusion follows.

To establish the proper efficiency of $a$ (VP), we follow the same argument as in the proof of Theorem 3.2 except in the end we appeal to the inequality (3.16) for a contradiction.

With a slight modification on the vector $\lambda_0$ (Kaul et al.’s $\mu^*$; see [15]) we state the following theorem for easy reference. Of course our Theorem 3.5 follows from Kaul et al.’s Theorem 3.7.

**Theorem 3.5** (Necessity). (See Theorem 3.7 in [15].) Suppose that

(i) $a$ is properly efficient for (VP);

(ii) there exists an $x^* \in X$ with $g_i(x^*) < 0$ where $I = \{i : g_i(a) = 0\}$ such that

$$-g_i(a) > \nabla g_i(a) \eta(x^*, a) \quad \forall \ i \in I.$$

Then there exist $\tau^0 \in R^p$, $\tau^0 > 0$, and $\lambda^0 \in R^l$, $\lambda^0 \geq 0$, such that

$$\sum_{i=1}^p \tau_i^0 \nabla f_i(a) + \sum_j \lambda^0_j \nabla g_j(a) = 0.$$
4. DUALITY THEORY (MOND–WEIR TYPE)

We consider a multiobjective dual to problem (VP),

\[(VD) \text{ V-maximize } f(y) = (f_1(y), \ldots, f_p(y)).\]

subject to

\[\sum_i \tau_i \nabla f_i(y) + \sum_j \lambda_j \nabla g_j(y) = 0,\]

\[\lambda_j g_j(y) = 0, j = 1, \ldots, m,\]

\[\tau \in \mathbb{R}^p, \tau \geq 0,\]

\[\lambda \in \mathbb{R}^m, \lambda \geq 0.\]

We let \(Y^0\) be the set of feasible solutions of problem (VD); i.e.,

\[Y^0 = \left\{ (y, \tau, \lambda) : \sum_i \tau_i \nabla f_i(y) + \sum_j \lambda_j \nabla g_j(y) = 0,\right\}

\[\lambda_j g_j(y) = 0, j = 1, \ldots, m, \quad \tau \in \mathbb{R}^p, \tau \geq 0, \quad \lambda \in \mathbb{R}^m, \lambda \geq 0 \right\}.\]

Efficient points and proper efficient points for (VD) are defined in a manner analogous to those of problem (VP) by simply reversing the inequalities in Definitions 2.1 and 2.2.

**Theorem 4.1 (Weak Duality).** Suppose that

(i) \(x \in X_0;\)

(ii) \((y, \tau, \lambda) \in Y^0 \text{ and } \tau > 0;\)

(iii) the problem (VP) is pseudo quasi V-type I at \(y\) with respect to \(\tau, \lambda\) and for some positive functions \(\alpha_i, \beta_j\) for \(i = 1, \ldots, p, j = 1, \ldots, m.\)

Then \(f(x) \leq f(y).\)

**Proof.** By the hypothesis (ii) we have \(\lambda_j g_j(y) = 0,\) for all \(j = 1, \ldots, m,\) which implies that

\[\sum_j \lambda_j \beta_j(x, y) g_j(y) = 0.\]  \hspace{1cm} (4.1)

By the hypothesis (iii) (in view of Definition 2.7) and (4.1) it follows that

\[\sum_j \lambda_j \eta(x, y) \nabla g_j(y) \leq 0.\] \hspace{1cm} (4.2)

Using the inequality (4.2) and the hypothesis (ii) we have

\[\sum_i \tau_i \eta(x, y) \nabla f_i(y) \geq 0.\]  \hspace{1cm} (4.3)
Hypothesis (iii) and (4.3) give

\[ \sum_{i} \tau_{i} \alpha_{i}(x, y)(f_{i}(x) - f_{i}(y)) \geq 0. \] (4.4)

Now suppose to the contrary that \( f(x) \leq f(y) \). Then since each \( \alpha_{i} > 0 \) and \( \tau > 0 \), we have

\[ \sum_{i} \tau_{i} \alpha_{i}(x, y)(f_{i}(x) - f_{i}(y)) < 0, \]

which contradicts (4.4). Hence the conclusion follows.

**Theorem 4.2 (Weak Duality).** Suppose that

1. \( x \in X_{0} \);
2. \( (y, \tau, \lambda) \in Y_{0}^{0} \);
3. Problem (VP) is semi strictly V-type I at \( y \) for some positive functions \( \alpha_{i}^{*}, \beta_{j}^{*} \) for \( i = 1, \ldots, p; j = 1, \ldots, m \).

Then \( f(x) \not\leq f(y) \).

**Proof.** By the hypothesis (ii) we have \( \lambda_{j} g_{j}(y) = 0 \), for all \( j = 1, \ldots, m \), which implies that

\[ \sum_{j} \lambda_{j} \beta_{j}^{*}(x, y) g_{j}(y) = 0. \] (4.5)

By (4.5) and the hypothesis (iii) (with \( 1/\beta_{j}(x, y) \) in Definition 2.3 replaced with \( \beta_{j}^{*}(x, y) \)) it follows that

\[ \sum_{j} \lambda_{j} \eta(x, y) \nabla g_{j}(y) \leq 0. \] (4.6)

Using the inequality (4.6) and the hypothesis (ii) we have

\[ \sum_{i} \tau_{i} \eta(x, y) \nabla f_{i}(y) \geq 0. \] (4.7)

By (4.7) and the hypothesis (iii) (with \( 1/\alpha_{i}(x, y) \) in Definition 2.3 replaced with \( \alpha_{i}^{*}(x, y) \)) we have

\[ \sum_{i} \tau_{i} \alpha_{i}^{*}(x, y)(f_{i}(x) - f_{i}(y)) > 0. \] (4.8)

Now suppose to the contrary that \( f(x) \leq f(y) \). Then since each \( \alpha_{i}^{*} > 0 \) and \( \tau \geq 0 \), we have

\[ \sum_{i} \tau_{i} \alpha_{i}^{*}(x, y)(f_{i}(x) - f_{i}(y)) \leq 0, \]

which contradicts (4.8).
Theorem 4.3 (Strong Duality). Suppose that

(1) \( a \) is a properly efficient solution of problem (VP);
(2) the hypothesis (ii) of Theorem 3.5 is satisfied.

Then there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \) and \( \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that \( (a, \tau^0, \lambda^0) \in Y^0 \) and the objective of (VP) and (VD) have the same values at \( a \) and \( (a, \tau^0, \lambda^0) \), respectively. If, further, the problem (VP) is pseudo quasi V-type I at all feasible solutions of (VD) then \( (a, \tau^0, \lambda^0) \in Y^0 \) is an efficient solution of (VD).

Proof. Let \( M = \{1, \ldots, m\} \). By Theorem 3.5, there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \) and \( \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that

\[
\sum_{i=1}^{p} \tau^0_i \nabla f_i(a) + \sum_{j} \lambda^0_j \nabla g_j(a) = 0.
\]

Since \( g_i(a) = 0 \) for all \( i \in I \), \( \lambda^0_i g_i(a) = 0 \) for all \( i \in I \).

Taking \( \lambda^0_i = 0 \) for all \( i \in M - I \), we have \( \lambda^0_i g_i(a) = 0 \) for all \( i \in M \). It also follows that

\[
\sum_{i=1}^{p} \tau^0_i \nabla f_i(a) + \sum_{j=1}^{m} \lambda^0_j \nabla g_j(a) = 0.
\]

Therefore \( (a, \tau^0, \lambda^0) \in Y^0 \). Trivially, the objective function values of (VP) and (VD) are equal.

Next suppose that \( (a, \tau^0, \lambda^0) \) is not an efficient solution of (VD). Then there exists a point \( (y^*, \tau^*, \lambda^*) \in Y^0 \) such that \( f(a) \leq f(y^*) \), which violates the weak duality Theorem 4.1. Hence \( (a, \tau^0, \lambda^0) \) is indeed an efficient solution of (VP).

The proof of the following theorem is very similar to the proof of Theorem 4.3, except that we appeal to the weak duality theorem 4.2 instead of Theorem 4.1.

Theorem 4.4 (Strong Duality). Suppose that

(i) \( a \) is properly efficient solution of problem (VP);
(ii) hypothesis (ii) of Theorem 3.5 is satisfied.

Then there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \lambda^0 \in \mathbb{R}^m, \lambda^0 \geq 0, \) such that \( (a, \tau^0, \lambda^0) \in Y^0 \) and the objective functions of (VP) and (VD) have the same values at \( a \) and \( (a, \tau^0, \lambda^0) \), respectively. If, further, problem (VP) is semi strictly V-type I at all feasible solution of (VD), then \( (a, \tau^0, \lambda^0) \in Y^0 \) is an efficient solution of (VD).
The following result is in the spirit of linear programming theory in the sense that whenever a feasible solution of the dual is a feasible solution of the primal, it is indeed an optimal solution of the primal.

**Theorem 4.5 (Converse Duality).** Suppose that

(i) \((y^0, x^0, \lambda^0) \in Y^0 \) with \(x^0 > 0\);

(ii) \(y^0 \in X_0\);

(iii) the problem \((VP)\) is \(V\)-type I at \(y^0\) for some positive functions \(\alpha_i, \beta_j\) for \(i = 1, \ldots, p\) and \(j = 1, \ldots, m\).

Then \(y^0\) is an efficient solution of \((VP)\). If, further, there exist positive real numbers \(n_i, m_i\) such that \(n_i < \alpha_i(x, a) < m_i\) for all \(x \in X_0\) and for all \(i = 1, \ldots, p\), then \(a\) is properly efficient for \((VP)\).

**Proof.** It follows by the hypotheses (i) and (ii) that

\[ \lambda_0^0 g_j(y^0) = 0, \quad \forall j = 1, \ldots, m. \tag{4.9} \]

By Hypothesis (iii) (in view of Definition 2.3), for any \(x \in X_0\), we have

\[ f_i(x) - f_i(y^0) \geq \alpha_i(x, y^0) \eta(x, y^0) \nabla f_i(y^0) \quad \forall i \tag{4.10} \]

\[ -g_j(y^0) \geq \beta_j(x, y^0) \eta(x, y^0) \nabla g_j(y^0) \quad \forall j \tag{4.11} \]

Now by the facts that \(\alpha_i > 0, \beta_j > 0 \forall i, j\) and \(x^0 > 0, \lambda^0 \geq 0\), it follows by (4.10) and (4.11) that

\[ \sum_i (\tau_i^0 / (\alpha_i(x, y^0)) (f_i(x) - f_i(y^0))) - \sum_j [\lambda_j^0 / \beta_j(x, y^0)] g_j(y^0) \]

\[ \geq \sum_i \tau_i^0 \eta(x, y^0) \nabla f_i(y^0) + \sum_j \lambda_j^0 \eta \nabla g_j(y^0) \tag{4.12} \]

\[ = 0. \]

From (4.9) and (4.12) it follows that

\[ \sum_i (\tau_i^0 / (\alpha_i(x, y^0)) (f_i(x) - f_i(y^0))) \geq 0 \quad \forall x \in X^0. \tag{4.13} \]

Now suppose that \(y^0\) is not an efficient solution of \((VP)\). Then there exists an \(x \in X_0\) such that \(f(x) \leq f(y^0)\) which implies that

\[ \sum_i (\tau_i^0 / (\alpha_i(x, y^0)) (f_i(x) - f_i(y^0))) < 0 \quad \forall x \in X^0. \tag{4.14} \]

Now (4.13) and (4.14) contradict each other. Hence the conclusion follows.
To establish the proper efficiency of $y^0$ for (VP), we define

$$M = (p - 1) \max_{i,j} \{ (m_i \tau_j) / (n_j \tau_i) \} \quad \forall \ i \neq j; 1 \leq i, j \leq p,$$

and appeal to (4.13) for a contradiction.

**Theorem 4.6 (Converse Duality).** Suppose that

(i) $(y^0, r^0, \lambda^0) \in Y^0$ with $r^0 > 0$;

(ii) $y^0 \in X_0$;

(iii) (VP) is semi strictly pseudo V-type I in $g$ at $y^0$ with respect to $r^0, \lambda^0$ and for some positive functions $\alpha_i, \beta_j$ for $i = 1, \ldots, p, j = 1, \ldots, m$.

Then $y^0$ is an efficient solution of (VP). If, further, there exist positive real numbers $n_i, m_i$ such that $n_i < \alpha_i(x, a) < m_i$ for all $i = 1, \ldots, p$, then $a$ is properly efficient for (VP).

**Proof.** It follows by hypotheses (i) and (ii) that $\lambda^0_i g_i(y^0) = 0 \ \forall \ j = 1, \ldots, m$, which implies that

$$\sum_j \lambda^0_i \beta_j(x, y^0) g_j(y^0) = 0.$$

Now using the contrapositive argument in (2.6) of Definition 2.5, we have

$$\sum_j \lambda^0_i \eta(x, y^0) \nabla g_j(y^0) < 0. \quad (4.15)$$

Using (4.15) and the hypothesis (i) we have

$$\sum_i \tau^0_i \eta(x, a) \nabla f_i(y^0) > 0,$$

which in turn implies by Definition 2.5 that

$$\sum_i \tau^0_i \alpha_i(x, y^0)(f_i(x) - f_i(y^0)) \geq 0 \quad \forall \ x \in X^0. \quad (4.16)$$

Comparing (4.16) with (4.13), the rest of the proof to establish efficiency follows along the lines of the proof of Theorem 4.5.

To prove that $y^0$ is properly efficient for (VP) we follow the argument given in the proof of Theorem 3.2 and in the end we appeal to the inequality (4.16) for contradiction.

**Theorem 4.7 (Converse Duality).** Suppose that

(i) $(y^0, r^0, \lambda^0) \in Y^0$;

(ii) $y^0 \in X_0$;
(iii) the problem (VP) is strictly pseudo quasi V-type I at $y^0$ with respect to $\tau^0, \lambda^0$, and for some positive functions $\alpha_i, \beta_j$ for $i = 1, \ldots, p$, $j = 1, \ldots, m$.

Then $y^0$ is an efficient solution of (VP). If, further, there exist positive real numbers $n_i, m_i$ such that $n_i < \alpha_i(x, a) < m_i$ for all $x \in X_0$ and for all $i = 1, \ldots, p$, then $a$ is properly efficient for (VP).

Proof. It follows by Hypotheses (i) and (ii) that $\lambda_j^0 g_j(y^0) = 0 \quad \forall \quad j = 1, \ldots, m$, which implies that

$$\sum_j \lambda_j^0 \beta_j(x, y^0) g_j(y^0) = 0 \quad \forall \quad x \in X.$$  

From the above equality and Definition 2.7, we have

$$\sum_j \lambda_j^0 \eta(x, y^0) \nabla g_j(y^0) \leq 0 \quad \forall \quad x \in X. \quad (4.17)$$

Combining (4.17) with Hypothesis (i) and appealing to Definition 2.7 again we have

$$\sum_j \tau_i^0 \alpha_i(x, y^0) (f_i(x) - f_i(y^0)) > 0 \quad \forall \quad x \in X. \quad (4.18)$$

Next, if $y^0$ is not an efficient solution of (VP), there exists an $x \in X^0$ such that $f(x) \leq f(y^0)$, which implies that

$$\sum_i \tau_i^0 \alpha_i(x, y^0) (f_i(x) - f_i(y^0)) \leq 0 \quad \forall \quad x \in X. \quad (4.19)$$

Since (4.18) and (4.19) contradict each other, it follows that $y^0$ is an efficient solution for (VP). To establish the proper efficiency of $y^0$ for (VP) we follow the same argument as that given in the proof of Theorem 3.2 and in the end appeal to the inequality (4.18) for the contradiction.

REFERENCES