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On graded polynomial identities with an antiautomorphism

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Abstract

Let G be a commutative monoid with cancellation and let $\mathcal{R}$ be a strongly G-graded associative algebra with finite G-grading and with antiautomorphism. Suppose that $\mathcal{R}$ satisfies a graded polynomial identity with antiautomorphism. We show that $\mathcal{R}$ is a PI algebra.

1. Introduction

Throughout this paper all rings and algebras are associative. The reader is referred to [7,16] for basic concepts and results on rings with (generalized) polynomial identities. Let G be a monoid with unity e and cancellation. Let $\mathcal{F}$ be a commutative ring with 1, and $\mathcal{R}$ an $\mathcal{F}$-algebra. We say that $\mathcal{R}$ is *almost G-graded* if there are $\mathcal{F}$-submodules $\mathcal{R}_g \subseteq \mathcal{R}$, $g \in G$, such that $\mathcal{R} = \sum_{g\in G} \mathcal{R}_g$ and $\mathcal{R}_g\mathcal{R}_h \subseteq \mathcal{R}_{gh}$ for all $g, h \in G$. If $\sum_{g\in G} \mathcal{R}_g$ is direct (i.e., $\sum_{g\in G} \mathcal{R}_g = \bigoplus_{g\in G} \mathcal{R}_g$), then we say that $\mathcal{R}$ is *G-graded*. Further, set $\mathrm{supp}(\mathcal{R}) = \{g \in G \mid \mathcal{R}_g \neq 0\}$. The G-grading is said to be *finite* if $|\mathrm{supp}(\mathcal{R})| < \infty$. A G-graded algebra $\mathcal{R}$ is called *strongly G-graded* if

(1) $\mathrm{supp}(\mathcal{R})$ consists of invertible elements,
(2) $\mathcal{R}$ has an identity 1, and

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(3) $1 \in \mathcal{R}_g\mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in \operatorname{supp}(\mathcal{R})$.

When G is the group of order 2, a G-graded algebra is called a *superalgebra*.

Let $U(\mathcal{F})$ be the group of invertible elements of $\mathcal{F}$, and let $\mathcal{R}$ be a G-graded algebra. Assume that G is commutative. An automorphism $\phi : \mathcal{R} \to \mathcal{R}$ of the $\mathcal{F}$-module $\mathcal{R}$ is called an *antiautomorphism of the G-graded algebra $\mathcal{R}$* if $\mathcal{R}_g^\phi = \mathcal{R}_g$ for all $g \in G$ and there exists a map $\nu : G \times G \to U(\mathcal{F})$ such that $\nu(e, p) = 1 = \nu(p, e)$ and $(ab)^\phi = \nu(p, q)b^\phi a^\phi$ for all $a \in \mathcal{R}_p$, $b \in \mathcal{R}_q$, and $p, q \in G$. In the case when $\mathcal{R}$ is a superalgebra with $G = \{e, g\}$ and $\nu(g, g) = -1$, the antiautomorphism ϕ is called a *superinvolution* provided that $\phi^2 = 1$.

Throughout the rest of the paper, we assume the following conditions:

(1) G is a commutative monoid with cancellation,
(2) $\mathcal{F}$ is an associative ring,
(3) $\mathcal{R}$ is an associative $\mathcal{F}$-algebra with a finite G-grading, and
(4) $\phi : \mathcal{R} \to \mathcal{R}$ is an antiautomorphism of the G-graded algebra $\mathcal{R}$.

Let $X = \bigcup_{g\in G} X_g$ be a disjoint union of infinite sets X_g, $g \in G$, and let $\mathcal{F}\langle X\rangle$ be the free $\mathcal{F}$-algebra on X. Let $\mathcal{A}$ be an almost G-graded $\mathcal{F}$-algebra. An element $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X\rangle$ is said to be a *G-graded polynomial identity on $\mathcal{A}$* provided that $\psi(f) = 0$ for all algebra homomorphisms $\psi : \mathcal{F}\langle X\rangle \to \mathcal{A}$ with $\psi(X_g) \subseteq \mathcal{A}_g$ for all $g \in G$.

We denote the set $\{x^\phi \mid x \in X\}$ as X^ϕ, and define a map $\delta : X \cup X^\phi \to G$ by the rule $\delta(x) = g = \delta(x^\phi)$ for all $x \in X_g$, $g \in G$. Next, given a monomial $M = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n} \in \mathcal{F}\langle X \cup X^\phi\rangle$, where each $\varepsilon_i \in \{1, \phi\}$, we set $\delta(M) = \delta(x_1)\delta(x_2)\dots\delta(x_n)$. An element $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi\rangle$ is said to be *a G-graded polynomial identity with ϕ on $\mathcal{R}$* provided that $\psi(f) = 0$ for all algebra homomorphisms $\psi : \mathcal{F}\langle X \cup X^\phi\rangle \to \mathcal{R}$ with $\psi(X_g) \subseteq \mathcal{R}_g$ and $\psi(x^\phi) = \psi(x)^\phi$ for all $x \in X_g$, $g \in G$.

Let $h(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi\rangle$ with at least one of it coefficients is equal to 1. It is easy to see that if h is a G-graded polynomial identity with antiautomorphism for $\mathcal{R}$, then $\mathcal{R}$ satisfies a multilinear G-graded polynomial identity $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi\rangle$ with at least one of the coefficients of f is 1. In this case, we may assume, without loss of generality, that the monomial $x_1x_2\dots x_n$ is involved in f with 1 as the coefficient, and that there exists $g \in G$ such that $\delta(N) = g$ for all monomials N involved in f. We set

$$G_f = \{\delta(x_1), \delta(x_2), \dots, \delta(x_n)\} \subseteq G.$$

In 1986 Bergen and Cohen [8] proved that $\mathcal{R}$ is PI provided that G is a finite group, $\mathcal{F}$ is a field, and $\mathcal{R}_e$ is a PI algebra. This result was extended to algebras over arbitrary commutative rings by Kelarev [11]. Bahturin and Zaicev [3] obtained an analogous result for algebras over a field with finite G-grading where G is any monoid with cancellation. Sehgal and Zaicev [17] proved that if H

is a normal subgroup of a group G with finite index and the group algebra $\mathcal{F}[G]$, considered as G/H-graded algebra, satisfies a G/H-graded polynomial identity, then $\mathcal{F}[G]$ is a PI algebra. Note, that in this case $\mathcal{F}[G]$ is a strongly G/H-graded algebra. Recently Beidar and Chebotar obtained the following generalization of their result.

Theorem 1.1 [5, Theorem 1.1]. *Let G be a monoid with unity e and cancellation, let $\mathcal{F}$ be a commutative ring with* 1*, and let $\mathcal{R}$ be an almost G-graded $\mathcal{F}$-algebra with finite G-grading satisfying a G-graded multilinear polynomial identity $f(x_1, x_2, \ldots, x_n)$. Suppose that the monomial $x_1x_2\ldots x_n$ is involved in f with coefficient* 1*, $\delta(N) = \delta(x_1x_2\ldots x_n)$ for all monomials N involved in f and $G_f \subseteq \mathrm{supp}(\mathcal{R})$. Then*:

- (i) *If $\mathcal{R}$ is a prime ring and $|\mathrm{supp}(\mathcal{R})| = 2$, then the ring $\mathcal{R}_e$ contains a nonzero ideal satisfying the standard identity St_{2n-2} of degree $2n-2$, and the ring $\mathcal{R}$ satisfies a nontrivial generalized polynomial identity. If in addition $\mathcal{R}$ is a simple ring with* 1*, then $\mathcal{R}$ is a PI algebra.*
- (ii) *If both $\mathcal{R}$ and $\mathcal{R}_e$ are prime rings, then $\mathcal{R}_e$ satisfies St_{2n-2} and $\mathcal{R}$ is a PI algebra.*
- (iii) *If $\mathcal{R}$ has an identity $1 \in \mathcal{R}_e$, G_f consists of invertible elements, and $\mathcal{R}_g\mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in G_f$, then $\mathcal{R}$ is a PI algebra.*

On the other hand, in 1969 Amitsur [2] proved that a ring $\mathcal{A}$ satisfying a polynomial identity with involution is PI (see [1,9,13] for earlier results). Motivated by the aforesaid results we prove the following theorem.

Theorem 1.2. *Let G be a commutative monoid with unity e and cancellation, let $\mathcal{F}$ be a commutative ring with* 1*, and let $\mathcal{R}$ be a G-graded $\mathcal{F}$-algebra with an antiautomorphism ϕ. Suppose that $|\mathrm{supp}(\mathcal{R})| < \infty$, and that $\mathcal{R}$ satisfies a G-graded multilinear polynomial identity $f(x_1, x_2, \ldots, x_n)$ with antiautomorphism such that the monomial $x_1x_2\ldots x_n$ is involved in f with coefficient* 1*, $\delta(N) = \delta(x_1x_2\ldots x_n)$ for all monomials N involved in f, and $G_f \subseteq \mathrm{supp}(\mathcal{R})$. Then*:

- (i) *If $\mathcal{R}$ is a prime ring and $|\mathrm{supp}(\mathcal{R})| = 2$, then the ring $\mathcal{R}_e$ contains a nonzero ideal satisfying the standard identity St_{4n-2} of degree $4n-2$, and the ring $\mathcal{R}$ satisfies a nontrivial generalized polynomial identity. If in addition $\mathcal{R}$ is a simple ring with* 1*, then $\mathcal{R}$ is a PI algebra.*
- (ii) *If $\mathcal{R}$ and $\mathcal{R}_e$ are both prime rings, then $\mathcal{R}_e$ satisfies St_{4n-2} and $\mathcal{R}$ is a PI algebra.*
- (iii) *If $\mathcal{R}$ has an identity $1 \in \mathcal{R}_e$, G_f consists of invertible elements, and $\mathcal{R}_g\mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in G_f$, then $\mathcal{R}$ is a PI algebra.*

We now give the following examples to justify the necessity of the conditions set in Theorem 1.2. These examples are modification of Examples 1–3 from [5].

Example 1.3. Let $G = \langle a \rangle$ be a cyclic group of order 3. There exists a G-graded algebra $\mathcal{R}$ over a field with an antiautomorphism ϕ such that $\mathcal{R}$ is a simple Artinian ring not satisfying a (generalized) polynomial identity, $\mathcal{R}_e$ is a direct sum of two skew fields and $\mathcal{R}$ satisfies a G-graded polynomial identity with antiautomorphism $f(x, y) = xy^{\phi}$, $x, y, \in \mathcal{R}_a$, such that $G_f \subseteq \operatorname{supp}(\mathcal{R})$.

Indeed, let $\mathcal{D}$ be a skew field with an antiautomorphism ψ which is not a PI ring (for instance, $\mathcal{D}$ may be the classical ring of quotients of the Weyl algebra A_1 over the rational number field with involution $x_1^{\psi} = y_1$ and $y_1^{\psi} = -x_1$ [15]). Let $\mathcal{F} = Z(\mathcal{D})$ be the center of $\mathcal{D}$, let $\mathcal{R} = M_2(\mathcal{D})$ be the $\mathcal{F}$-algebra of 2×2 matrices over $\mathcal{D}$ and let $\{e_{ij} \mid 1 \leqslant i, j \leqslant 2\}$ be a system of matrix units of $\mathcal{R}$. Further, set $u = e_{11}$, $v = e_{22}$, and

$$\mathcal{R}_e = u\mathcal{R}u + v\mathcal{R}v, \qquad \mathcal{R}_a = u\mathcal{R}v \quad \text{and} \quad \mathcal{R}_{a^2} = v\mathcal{R}u.$$

Define an antiautomorphism ϕ of $\mathcal{R}$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\phi} = \begin{pmatrix} d^{\psi} & -b^{\psi} \\ -c^{\psi} & a^{\psi} \end{pmatrix} \quad \text{for all } a, b, c, d \in \mathcal{D}$$

and note that $\mathcal{R}$ is a G-graded algebra satisfying a G-graded polynomial identity with antiautomorphism $f(x, y) = xy^{\phi}$, $x, y \in \mathcal{R}_a$.

Hence the first statement of the theorem does not hold in general if $|\operatorname{supp}(\mathcal{R})| = 3$. Next, the second statement does not hold in general if $\mathcal{R}_e$ is not prime even if $\mathcal{R}$ is a simple Artinian ring and $\mathcal{R}_e$ is a direct sum of two skew fields.

Example 1.4. Let $G = \{e, g\}$ be a cyclic group of order 2. There exists a G-graded algebra $\mathcal{R}$ over a field with an antiautomorphism $*$ such that $\mathcal{R}$ is a simple ring (without identity) satisfying a generalized polynomial identity, $\mathcal{R}$ is not a PI algebra, $\mathcal{R}$ satisfies a G-graded polynomial identity

$$f(x_1, \dots, x_5) = [x_1, x_2]x_3[x_4, x_5], \quad x_1, x_2, x_4, x_5 \in \mathcal{R}_e,\ x_3 \in \mathcal{R}_g,$$

and $G_f = \operatorname{supp}(\mathcal{R}) = G$ (see Theorem 1.2(i)).

Indeed, let $\mathcal{F}$ be a field, let $\mathcal{R}$ be the $\mathcal{F}$-algebra of infinite matrices with finite number of nonzero entries and let u be the matrix whose $(1, 1)$ entry is equal to 1 and all the other ones are equal to 0. Obviously $uxuyu - uyuxu$ is a generalized polynomial identity on $\mathcal{R}$ and $\mathcal{R}$ is not a PI algebra. Further, set

$$\mathcal{R}_e = u\mathcal{R}u + (1-u)\mathcal{R}(1-u) \quad \text{and} \quad \mathcal{R}_g = u\mathcal{R}(1-u) + (1-u)\mathcal{R}u.$$

Clearly $\mathcal{R}$ is a G-graded algebra. Next the transpose map $*$ is an antiautomorphism of the G-graded algebra $\mathcal{R}$ and $\mathcal{R}$ satisfies the G-graded polynomial identity $f(x_1, \dots, x_5)$.

Example 1.5. Let $G = \{e, g\}$ be a cyclic group of order 2 and let $\mathcal{F}$ be a field. For any positive integer n the algebra $\mathcal{R} = M_n(\mathcal{F})$ admits a G-grading such that $\mathcal{R}$ is a strongly G-graded algebra with antiautomorphism satisfying the G-graded polynomial identity $f(x_1, \ldots, x_5)$ (see Example 2).

Indeed, let $u = e_{11}$. As above set $\mathcal{R}_e = u\mathcal{R}u + (1-u)\mathcal{R}(1-u)$ and $\mathcal{R}_g = u\mathcal{R}(1-u) + (1-u)\mathcal{R}u$. Obviously $\mathcal{R}_g^2 = \mathcal{R}_e$ and so $\mathcal{R}$ is strongly G-graded. We already know that $f(x_1, \ldots, x_5)$ is a G-graded polynomial identity on $\mathcal{R}$ and the transpose map is an antiautomorphism of the G-graded algebra $\mathcal{R}$. On the other hand, the minimal degree of a polynomial identity on $\mathcal{R}$ is $2n$ [16, Lemma 1.4.3]. Therefore there exists no function $m = m(\deg(f))$ such that a simple algebra with 1 satisfying the G-graded polynomial identity f satisfies a polynomial identity of degree m even if $\mathcal{R}$ is a strongly G-graded simple finite-dimensional algebra (see Theorem 1.2(iii)).

The following two corollaries are special cases of the above theorem.

Corollary 1.6. *Let $\mathcal{R}$ be a strongly G-graded algebra with identity and having an antiautomorphism. Suppose that $|\mathrm{supp}(\mathcal{R})| < \infty$, and $\mathcal{R}$ satisfies a G-graded multilinear polynomial identity $f(x_1, x_2, \ldots, x_n)$ with antiautomorphism such that the monomial $x_1x_2 \ldots x_n$ is involved in f with coefficient 1. Then $\mathcal{R}$ is a PI algebra.*

Corollary 1.7. *Let $\mathcal{R}$ be a superalgebra with superinvolution. Suppose that $\mathcal{R}$ satisfies a graded multilinear polynomial identity $f(x_1, x_2, \ldots, x_n)$ with superinvolution such that the monomial $x_1x_2 \ldots x_n$ is involved in f with coefficient 1. Further, assume that $\mathcal{R}$ is a prime ring. Then $\mathcal{R}$ satisfies a nonzero generalized polynomial identity. If in addition $\mathcal{R}$ is a simple ring with 1, then $\mathcal{R}$ is a PI algebra.*

We also obtain the following generalization of Amitsur's result [2] on algebras with polynomial identities with involution.

Corollary 1.8. *Let $\mathcal{F}$ be a commutative ring with 1, and $\mathcal{R}$ an $\mathcal{F}$-algebra with antiautomorphism ϕ. Suppose that $\mathcal{R}$ satisfies a polynomial identity with ϕ, and at least one of the coefficients of the polynomial is equal to 1. Then $\mathcal{R}$ is a PI algebra.*

Proof. Let $f(x_1, x_2, \ldots, x_n) \in \mathcal{F}\langle X \cup X^{\phi}\rangle$ be a multilinear polynomial identity with ϕ on $\mathcal{R}$ such that at least one coefficient of f is equal to 1. Let $\mathcal{R}^{\#}$ be the ring $\mathcal{R}$ with 1 adjoined. Clearly, $\mathcal{R}^{\#}$ satisfies $f([x_1, y_1], \ldots, [x_n, y_n])$. Set $G = \{e\}$ and $\mathcal{R}_e^{\#} = \mathcal{R}^{\#}$. Then $\mathcal{R}^{\#}$ is a strongly G-graded algebra. The result now follows from Corollary 1.6. $\square$

2. Proof of main theorem

We first set some further notation in place and obtain some preliminary results for rings.

Let $\mathcal{A}$ be a ring. Given right $\mathcal{A}$-modules $\mathcal{U}$ and $\mathcal{V}$ and a module map $h:\mathcal{U}_{\mathcal{A}} \to \mathcal{V}_{\mathcal{A}}$, we denote by hx the image of $x \in \mathcal{U}$ under h. If $\mathcal{I}$ is a nonempty subset of $\mathcal{A}$, we set

$$\ell(\mathcal{U};\mathcal{I}) = \{x \in \mathcal{U} \mid x\mathcal{I} = 0\}.$$

Let n be a positive integer and let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$, $\mathcal{M}$ be right $\mathcal{A}$-modules. We shall use $\boldsymbol{u}_n$ to denote the element $(u_1, u_2, \ldots, u_n) \in \prod_{k=1}^{n} \mathcal{L}_k$, and use $\hat{\boldsymbol{u}}_n^i$ to denote the element

$$(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \in \prod_{\substack{k=1,\\ k\neq i}}^{n} \mathcal{L}_k \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

Let $a \in \mathcal{A}$ be fixed. For nonnegative integers s and t with $t \leqslant n$, let

$$E_{ij} : \prod_{\substack{k=1,\\ k\neq i}}^{n-t} \mathcal{L}_k \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (1 \leqslant i \leqslant n-t \text{ and } 0 \leqslant j \leqslant s+t)$$

be maps having the property that

$$\sum_{i=1}^{n-t}\sum_{j=0}^{s+t} E_{ij}\big(\hat{\boldsymbol{u}}_{n-t}^i\big)u_i a^j = 0 \quad \text{for all } \boldsymbol{u}_{n-t} \in \prod_{k=1}^{n-t} \mathcal{L}_k.$$

If $E_{ij} = 0$ for all i and j, they certainly have the above property. On the other hand, under certain conditions, the converse is also true.

Lemma 2.1 [5, Lemma 2.1]. *Suppose that the following conditions are satisfied*:

(i) *For any $0 \leqslant r \leqslant n+s-1$ there exist a positive integer $m = m(r)$ and elements $b_{rq}, c_{rq} \in \mathcal{A}$, $q = 1, 2, \ldots, m$, such that*

$$d_r = \sum_{q=1}^{m} b_{rq} a^r c_{rq} \neq 0 \quad \textit{and}$$

$$\mathcal{M}\sum_{q=1}^{m} b_{rq} a^p c_{rq} = 0 \quad \textit{for all } p = 0, 1, \ldots, n+s-1,\ p \neq r.$$

(ii) $\ell(\mathcal{M}; \mathcal{A}d_r) = 0$ *for all* $r = 0, 1, \ldots, n+s-1$.

Then $E_{ij} = 0$ *for all* i *and* j.

Now let ϕ be a antiautomorphism of $\mathcal{A}$, and assume that $\mathcal{L}_k$, $1 \leqslant k \leqslant n$, are $\mathcal{A}$-$\mathcal{A}$-bimodules with the unary operation ϕ such that $\phi : \mathcal{L}_k \to \mathcal{L}_k$ is a bijective map, $(xc)^\phi = c^\phi x^\phi$ and $(cx)^\phi = x^\phi c^\phi$ for all $c \in \mathcal{A}$ and $x \in \mathcal{L}_k$, $1 \leqslant k \leqslant n$. Further, let $J \subseteq \{1, 2, \ldots, n\}$ with $|J| = n - t$ and let

$$F_{ip} : \prod_{\substack{k=1,\\ k\neq i}}^{n} \mathcal{L}_k \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (1 \leqslant i \leqslant n \text{ and } 0 \leqslant p \leqslant t),$$

$$E_{jq} : \prod_{\substack{k=1,\\ k\neq j}}^{n} \mathcal{L}_k \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (j \in J \text{ and } 0 \leqslant q \leqslant t)$$

be maps such that

$$\sum_{i=1}^{n} \sum_{p=0}^{t} F_{ip}\big(\hat{\boldsymbol{u}}_n^i\big) u_i a^p + \sum_{j\in J} \sum_{q=0}^{t} E_{jq}\big(\hat{\boldsymbol{u}}_n^j\big) u_j^\phi a^q = 0 \tag{1}$$

for all $\boldsymbol{u}_n \in \prod_{k=1}^n \mathcal{L}_k$.

The following result, which we shall need in the sequel, is a generalization of both Lemma 2.1 and [6, Theorem 3.3].

Lemma 2.2. *Suppose that the following conditions are satisfied*:

(i) *For any* $0 \leqslant r \leqslant 2n - 1$ *there exist a positive integer* $m = m(r)$ *and elements* $b_{rk}, c_{rk} \in \mathcal{A}$, $k = 1, 2, \ldots, m$, *such that*

$$d_r = \sum_{k=1}^{m} b_{rk} a^r c_{rk} \neq 0 \quad \textit{and}$$

$$\mathcal{M} \sum_{k=1}^{m} b_{rk} a^s c_{rk} = 0 \quad \textit{for all } s = 0, 1, \ldots, 2n - 1 \textit{ with } s \neq r,$$

(ii) $\ell(\mathcal{M}; \mathcal{A}d_r) = 0$ *for all* $r = 0, 1, \ldots, 2n - 1$.

Then $F_{ip} = 0$ *and* $E_{jq} = 0$ *for all* i, j, p, *and* q.

Proof. We proceed by induction on $n - t$. If $n - t = 0$, then (1) reads

$$\sum_{i=1}^{n} \sum_{p=0}^{n} F_{ip}\big(\hat{\boldsymbol{u}}_n^i\big) u_i a^p = 0 \quad \text{for all } \boldsymbol{u}_n \in \prod_{k=1}^{n} \mathcal{L}_k,$$

and the result follows from Lemma 2.1.

In the inductive case $n - t > 0$, we may assume without loss of generality that $n \in J$. Set $J' = J \setminus \{n\}$. Substituting $a^{\phi^{-1}} u_n$ for u_n in (1), and using the notation $(\hat{\boldsymbol{u}}_{n-1}^{i}, a^{\phi^{-1}}u_n)$ for $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n-1}, a^{\phi^{-1}}u_n)$ and the likes, we obtain

$$\sum_{i=1}^{n-1}\sum_{p=0}^{t} F_{ip}\bigl(\hat{\boldsymbol{u}}_{n-1}^{i}, a^{\phi^{-1}}u_n\bigr)u_i a^p + \sum_{p=0}^{t} F_{np}(\boldsymbol{u}_{n-1})a^{\phi^{-1}}u_n a^p$$
$$+ \sum_{j\in J'}\sum_{q=0}^{t} E_{jq}\bigl(\hat{\boldsymbol{u}}_{n-1}^{j}, a^{\phi^{-1}}u_n\bigr)u_j^{\phi}a^q + \sum_{q=0}^{t} E_{nq}(\boldsymbol{u}_{n-1})u_n^{\phi}a^{q+1} = 0 \tag{2}$$

for all $\boldsymbol{u}_n \in \prod_{k=1}^{n}\mathcal{L}_k$. Multiplying (1) by a from the right and subtracting the resulting expression from (2), we see that

$$\sum_{i=1}^{n}\sum_{p=0}^{t+1} \widetilde{F}_{ip}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr)u_i a^p + \sum_{j\in J'}\sum_{q=0}^{t+1} \widetilde{E}_{jq}\bigl(\hat{\boldsymbol{u}}_{n}^{j}\bigr)u_j^{\phi}a^q = 0 \tag{3}$$

for all $\boldsymbol{u}_n \in \prod_{k=1}^{n}\mathcal{L}_k$, where

$$\widetilde{F}_{i0}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr) = F_{i0}\bigl(\hat{\boldsymbol{u}}_{n-1}^{i}, a^{\phi^{-1}}u_n\bigr), \qquad 1 \leqslant i \leqslant n-1,$$
$$\widetilde{F}_{ip}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr) = F_{ip}\bigl(\hat{\boldsymbol{u}}_{n-1}^{i}, a^{\phi^{-1}}u_n\bigr) - F_{i,p-1}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr), \quad 1 \leqslant i \leqslant n-1,\ 1 \leqslant p \leqslant t, \tag{4}$$
$$\widetilde{F}_{i,t+1}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr) = -F_{it}\bigl(\hat{\boldsymbol{u}}_{n}^{i}\bigr), \qquad 1 \leqslant i \leqslant n-1, \tag{5}$$
$$\widetilde{F}_{n0}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr) = F_{n0}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr)a^{\phi^{-1}},$$
$$\widetilde{F}_{np}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr) = F_{np}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr)a^{\phi^{-1}} - F_{n,p-1}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr), \qquad 1 \leqslant p \leqslant t, \tag{6}$$
$$\widetilde{F}_{n,t+1}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr) = -F_{nt}\bigl(\hat{\boldsymbol{u}}_{n}^{n}\bigr), \tag{7}$$

and the maps $\widetilde{E}_{jq}$ ($j \in J$ and $0 \leqslant q \leqslant t+1$) are defined similarly. Applying the induction assumption on (3), we see that, in particular, $\widetilde{F}_{ip} = 0$ for $1 \leqslant i \leqslant n$ and $0 \leqslant p \leqslant t+1$.

Now, (5) implies that $F_{it} = 0$ for $1 \leqslant i \leqslant n-1$. From (4) we infer that $F_{ip} = 0$ for all $1 \leqslant i \leqslant n-1$ and $0 \leqslant p \leqslant t$. Analogously, (6) and (7) together yield that $F_{np} = 0$ for $0 \leqslant p \leqslant t$. Taking these into account, the identity (1) becomes

$$\sum_{j\in J}\sum_{q=0}^{t} E_{jq}\bigl(\hat{\boldsymbol{u}}_{n}^{j}\bigr)u_j^{\phi}a^q = 0 \quad \text{for all } \boldsymbol{u}_n \in \prod_{k=1}^{n}\mathcal{L}_k.$$

Substituting $u_k^{\phi^{-1}}$ for u_k ($k = 1, 2, \ldots, n$) in the above equation, we obtain

$$\sum_{j\in J}\sum_{q=0}^{t} E_{jq}\bigl(u_1^{\phi^{-1}}, \ldots, u_{j-1}^{\phi^{-1}}, u_{j+1}^{\phi^{-1}}, \ldots, u_n^{\phi^{-1}}\bigr)u_j a^q = 0,$$

and the result follows at once from Lemma 2.1. $\square$

As a special case of Lemma 2.2, we have the following corollary.

Corollary 2.3. *Let* $F_i, E_i : \prod_{\substack{k=1,\\ k\neq i}}^{n} \mathcal{L}_k \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M})$, $i = 1, 2, \ldots, n$, *be maps such that*

$$\sum_{i=1}^{n} F_i\big(\hat{\boldsymbol{u}}_n^i\big)u_i + \sum_{i=1}^{n} E_i\big(\hat{\boldsymbol{u}}_n^i\big)u_i^{\phi} = 0 \quad \text{for all } \boldsymbol{u}_n \in \prod_{k=1}^{n} \mathcal{L}_k.$$

Suppose that there exists an element $a \in \mathcal{A}$ *such that the following conditions are satisfied*:

(i) *There exist a positive integer* m *and elements* $b_{rk}, c_{rk} \in \mathcal{A}$, $0 \leqslant r \leqslant 2n-1$, $1 \leqslant k \leqslant m$, *such that for all* $r = 0, 1, \ldots, 2n-1$,

$$d_r = \sum_{k=1}^{m} b_{rk} a^r c_{rk} \neq 0 \quad \text{and}$$

$$\mathcal{M} \sum_{k=1}^{m} b_{rk} a^s c_{rk} = 0 \quad \text{for all } s = 0, 1, \ldots, 2n-1 \text{ with } s \neq r.$$

(ii) $\ell(\mathcal{M}; \mathcal{A}d_r) = 0$ *for all* $r = 0, 1, \ldots, 2n-1$.

Then $F_i = E_i = 0$ *for all* i.

Before we can prove Theorem 2.1, some more results about G-graded $\mathcal{F}$-algebras have to be stated.

Proposition 2.4 [5, Proposition 2.3]. *Let* G *be a monoid with cancellation and let* $\mathcal{R}$ *be an almost* G*-graded algebra with finite* G*-grading. Let* $n = |\mathrm{supp}(\mathcal{R})|$, *let* m *be a positive integer, let* $\mathcal{L} = \sum_{g\in G} \mathcal{L}_g$ *be a* G*-graded subring of* $\mathcal{R}$ *(i.e.,* $\mathcal{L}_g \subseteq \mathcal{R}_g$ *is a subgroup and* $\mathcal{L}_g\mathcal{L}_h \subseteq \mathcal{L}_{gh}$ *for all* $g, h \in G$*) and let* $\mathcal{I}$ *be a right ideal of* $\mathcal{R}_e$*. Further, let*

$$H = \big\{ g \in \mathrm{supp}(\mathcal{R}) \;\big|\; g \text{ is not invertible in } G \text{ or } \mathcal{R}_g\mathcal{R}_{g^{-1}} = 0 \big\}$$

and let $\mathcal{U}$ *be the ideal of* $\mathcal{R}$ *generated by* $\sum_{h\in H} \mathcal{R}_h$*. Then*:

(i) *If* $(\mathcal{L}_e)^m = 0$, *then* $\mathcal{L}^{nm} = 0$.
(ii) *If* $\mathcal{I}^m = 0$, *then* $(\mathcal{I}\mathcal{R})^{nm} = 0$.
(iii) $\mathcal{U}$ *is a nilpotent ideal of* $\mathcal{R}$.

Proposition 2.5 [5, Proposition 2.4]. *Let* G *be a monoid with cancellation,* $\mathcal{R}$ *be an almost* G*-graded algebra and* $n = |\mathrm{supp}(\mathcal{R})| < \infty$*. Suppose that* $\mathcal{R}$ *is a prime algebra. Then* $\mathcal{R}_e$ *is a semiprime algebra containing nonzero ideals* $\mathcal{I}_1, \ldots, \mathcal{I}_m$, *such that*:

(a) $\mathcal{I}_i \cap \mathcal{I}_j = 0$ *for all* $i \neq j$;

(b) $\mathcal{I}=\bigoplus_{k=1}^{m}\mathcal{I}_k$ *is an essential ideal of* $\mathcal{R}_e$;
(c) *each* $\mathcal{I}_k$, $k=1,2,\dots,m$, *is a prime ring*;
(d) $m\leqslant n$;
(e) *if* $d_k\in\mathcal{I}_k\setminus\{0\}$ $(k=1,2,\dots,m)$ *and* $d=\sum_{k=1}^{m}d_k$, *then* $\ell(\mathcal{R}_g,\mathcal{R}_e d)=0$ *for all* $g\in G$.

The next result is a special case of [3, Theorem 3].

Theorem 2.6. *Let G be a monoid with cancellation, let $\mathcal{F}$ be a field and let $\mathcal{R}$ be an almost G-graded algebra with finite G-grading. If $\mathcal{R}_e$ is a PI algebra, then so is $\mathcal{R}$.*

Theorem 2.7 [12, Theorem 3]. *Let $\mathcal{E}$ be a class of rings which is closed under direct powers and homomorphic images. If every prime ring in $\mathcal{E}$ satisfies a generalized polynomial identity, then $\mathcal{E}$ consists of PI rings.*

Now, we are ready to prove the main theorem.

Proof of Theorem 1.2. Let $\mathcal{P}$ be a prime ideal of $\mathcal{R}$. Set $\overline{\mathcal{R}}=\mathcal{R}/\mathcal{P}$, and for $g\in G$, set $\overline{\mathcal{R}}_g=(\mathcal{R}_g+\mathcal{P})/\mathcal{P}$. It is clear that $\overline{\mathcal{R}}$ is an almost G-graded $\mathcal{F}$-algebra. Given $a\in R$, we denote $\bar{a}=a+\mathcal{P}\in\overline{\mathcal{R}}$.

It follows from Proposition 2.4(iii) that $\operatorname{supp}(\overline{\mathcal{R}})$ consists of invertible elements and that

$$\overline{\mathcal{R}}_g\overline{\mathcal{R}}_{g^{-1}}\neq 0\quad\text{for all } g\in\operatorname{supp}(\overline{\mathcal{R}}).\tag{8}$$

Next, Proposition 2.5 implies that $\overline{\mathcal{R}}_e$ is a semiprime ring. Write

$$f(x_1,\dots,x_n)=f(\boldsymbol{x}_n)=\sum_{i=1}^{n}f_i\big(\hat{\boldsymbol{x}}_n^i\big)x_i+\sum_{i=1}^{n}g_i\big(\hat{\boldsymbol{x}}_n^i\big)x_i^{\phi},$$

where all $f_i(\hat{\boldsymbol{x}}_n^i)$ and $g_i(\hat{\boldsymbol{x}}_n^i)$ are multilinear polynomials in $x_1,x_1^{\phi},\dots,x_{i-1},x_{i-1}^{\phi}$, $x_{i+1},x_{i+1}^{\phi},\dots,x_n,x_n^{\phi}$. Let $\mathcal{I}_1,\mathcal{I}_2,\dots,\mathcal{I}_w$ be ideals of $\overline{\mathcal{R}}_e$ as in Proposition 2.5. Assume that each $\mathcal{I}_l$, $l=1,2,\dots,w$, does not satisfy St_{4n-2}. We claim that

$$\overline{f_i\big(\hat{\boldsymbol{u}}_n^i\big)u_i}=0\quad\text{for all } i\in\{1,2,\dots,n\}\text{ and }\boldsymbol{u}_n\in\prod_{k=1}^{n}\mathcal{R}_{\delta(x_k)}.\tag{9}$$

Indeed, fix $1\leqslant l\leqslant w$ and recall that $\mathcal{I}_l$ is a prime ring. Let $\mathcal{E}_l$ be the Martindale (extended) centroid of $\mathcal{I}_l$. If every elements of $\mathcal{I}_l$ is algebraic of degree $\leqslant 2n-1$ over $\mathcal{E}_l$, then $\mathcal{I}_l$ is a subring of the ring of $(2n-1)\times(2n-1)$ matrices over the algebraic closure of $\mathcal{E}_l$ (see [4, p. 3928]), and so Amitsur–Levitzki theorem [16] implies that $\mathcal{I}_l$ satisfies St_{4n-2}, a contradiction. Therefore $\mathcal{I}_l$ contains an element $\overline{a_l}$ which is not algebraic of degree $2n-1$ over $\mathcal{E}_l$, which means that

$1, \overline{a_l}, \overline{a_l}^2, \ldots, \overline{a_l}^{2n-1}$ are linearly independent over $\mathcal{E}_l$. By [7, Theorem 2.3.3], we see that for any r, $0 \leqslant r \leqslant 2n-1$, there exist a positive integer $m = m(l,r)$ and elements $\overline{b_{lrk}}, \overline{c_{lrk}} \in \mathcal{I}_l$, $k = 1, 2, \ldots, m$, such that

$$\overline{d_{lr}} = \sum_{k=1}^{m} \overline{b_{lrk} a_l^r c_{lrk}} \neq 0 \quad \text{and}$$
$$\sum_{k=1}^{m} \overline{b_{lrk} a_l^s c_{lrk}} = 0 \quad \text{for all } s = 0, 1, \ldots, 2n-1,\ s \neq r.$$

We may assume without loss of generality that m does not depend on both l and r. Now, set

$$\bar{a} = \sum_{l=1}^{w} \overline{a_l},$$
$$\overline{b_{rk}} = \sum_{l=1}^{w} \overline{b_{lrk}}, \quad \overline{c_{rk}} = \sum_{l=1}^{w} \overline{c_{lrk}} \quad (1 \leqslant r \leqslant 2n-1 \text{ and } 1 \leqslant k \leqslant m),$$

and put

$$\overline{d_r} = \sum_{k=1}^{m} \overline{b_{rk} a^r c_{rk}} \quad (1 \leqslant r \leqslant 2n-1).$$

Then we have

$$\overline{d_r} = \sum_{l=1}^{m} \overline{d_{lr}} \neq 0 \quad \text{and}$$
$$\sum_{k=1}^{m} \overline{b_{rk} a^s c_{rk}} = 0 \quad \text{for } s = 1, 2, \ldots, 2n-1 \text{ with } s \neq r, \tag{10}$$

for all $r = 1, 2, \ldots, 2n-1$. It follows from Proposition 2.5(e) that

$$\ell\big(\overline{\mathcal{R}}_g, \overline{\mathcal{R}}_e \overline{d_r}\big) = 0 \quad \text{for all } g \in G \text{ and } r = 1, 2, \ldots, 2n-1. \tag{11}$$

Without loss of generality, we may assume that $a, b_{rk}, c_{rk}, d_r \in \mathcal{R}_e$ for all r and k. Now, regard each $\overline{R}_g$ as a right $\mathcal{R}_e$-module. Then, from (10), we obtain, for $r \in \{1, 2, \ldots, 2n-1\}$ and $g \in G$, that

$$d_r = \sum_{k=1}^{m} b_{rk} a^r c_{rk} \neq 0 \quad \text{and}$$
$$\overline{\mathcal{R}}_g \sum_{k=1}^{m} b_{rk} a^s c_{rk} = 0 \quad \text{for } s = 1, 2, \ldots, 2n-1 \text{ with } s \neq r. \tag{12}$$

Moreover, (11) yields

$$\ell\big(\overline{\mathcal{R}}_g, \mathcal{R}_e d_r\big) = 0 \quad \text{for all } g \in G \text{ and } r = 1, 2, \ldots, 2n-1. \tag{13}$$

The left multiplications (induced by the right $\mathcal{R}$-module structure on $\overline{\mathcal{R}}$) by $\overline{f_i(\hat{\boldsymbol{u}}_n^i)}$ and by $\overline{g_i(\hat{\boldsymbol{u}}_n^i)}$ $(i = 1, 2, \ldots, 2n-1$ and $\boldsymbol{u}_n \in \mathcal{R}^n)$ can be viewed as elements of $\operatorname{Hom}_{\mathcal{R}_e}(\mathcal{R}_{\delta(x_i)}, \overline{\mathcal{R}_h})$ where $h = \delta(x_1 x_2 \ldots x_n)$. Since $\overline{f_i(\hat{\boldsymbol{u}}_n^i)u_i} = \overline{f_i(\hat{\boldsymbol{u}}_n^i)}u_i$, we see from Corollary 2.3 that (9) is fulfilled. Note that (9) is true for any multilinear polynomial $f(\boldsymbol{x}_n)$ with antiautomorphism ϕ such that $\overline{f(\boldsymbol{u}_n)} = 0$ for all $\boldsymbol{u}_n \in \prod_{k=1}^n \mathcal{R}_{\delta(x_k)}$.

(i) Assume that $\mathcal{R}$ is prime and $|\operatorname{supp}(\mathcal{R})| = 2$. Replacing $\mathcal{F}$ by $\mathcal{F}/\ell(\mathcal{F};\mathcal{R})$ we may assume that $\mathcal{F}$ is an integral domain. Setting $S = \mathcal{F} \setminus \{0\}$ and considering the $S^{-1}\mathcal{F}$-algebra $S^{-1}\mathcal{R}$, we reduce the proof to the case when $\mathcal{F}$ is a field.

Since $\mathcal{R}$ is prime, Proposition 2.4 implies that $e \in \operatorname{supp}(\mathcal{R})$. Let $g \in \operatorname{supp}(\mathcal{R})$ with $g \neq e$. Recalling that $g^{-1} \in \operatorname{supp}(\mathcal{R})$, we conclude that $g = g^{-1}$ and so $g^2 = e$.

We claim that $\ell(\mathcal{R}_e; \mathcal{R}_g) = \{a \in \mathcal{R}_e \mid a\mathcal{R}_g = 0\} = 0$. Indeed, let $b \in \ell(\mathcal{R}_e; \mathcal{R}_g)$. Then

$$b\mathcal{R}\mathcal{R}_g = b(\mathcal{R}_e\mathcal{R}_g) + (b\mathcal{R}_g)\mathcal{R}_g \subseteq b\mathcal{R}_g = 0$$

and so $b = 0$ because $\mathcal{R}$ is prime and $\mathcal{R}_g \neq 0$.

We now set $h = \delta(x_n) \in G_f \subseteq \operatorname{supp}(\mathcal{R}) = \{e, g\}$. It follows from the above result together with semiprimeness of $\mathcal{R}_e$ that $\ell(\mathcal{R}_e; \mathcal{R}_h) = 0$. Let $\mathcal{K} = \ell(\mathcal{R}; \mathcal{R}_h)$. Clearly $\mathcal{K}$ is a G-graded ring and a left ideal of $\mathcal{R}$. Next, $\mathcal{K}_e = \ell(\mathcal{R}_e; \mathcal{R}_h) = 0$ and so Proposition 2.4(i) implies that $\mathcal{K}^2 = 0$. As $\mathcal{R}$ is prime, $\mathcal{K} = 0$.

Assume that $\mathcal{R}_e$ has no nonzero ideals satisfying St_{4n-2}. Then (9) (with $\mathcal{P} = 0$) implies that $f_n(\boldsymbol{u}_{n-1}) \in \mathcal{K} = 0$, and so we conclude that $f_n(x_1, \ldots, x_{n-1})$ is a G-graded polynomial identity with antiautomorphism on $\mathcal{R}$. Making use of induction on $n = \deg(f)$, we get that $\mathcal{R}_e$ contains a nonzero ideal satisfying St_{4n-4} and so St_{4n-2}, a contradiction. Therefore $\mathcal{R}_e$ contains a nonzero ideal $\mathcal{I}$ satisfying St_{4n-2}. Now set $\mathcal{L}_p = \mathcal{I}\mathcal{R}_p$, $p \in G$, and $\mathcal{L} = \sum_{p \in G} \mathcal{L}_p$. Since $\mathcal{L}_e$ satisfies St_{4n-2}, Theorem 2.6 implies that $\mathcal{L}$ is a PI algebra. It now follows from [10] that $\mathcal{R}$ satisfies a nonzero generalized polynomial identity (see also [7, Theorem 6.3.20]). Suppose that in addition $\mathcal{R}$ is a simple ring with 1. Then the central closure of $\mathcal{R}$ is equal to $\mathcal{R}$. It now follows from Martindale theorem on prime rings with generalized polynomial identity [14] that $\mathcal{R}$ has a nonzero socle and the associated skew field is finite-dimensional over its center (see also [7, Theorem 6.1.6]). Since $\mathcal{R}$ is simple, it coincides with its socle. In particular, 1 is an idempotent of finite rank and so Litoff's theorem [7, Theorem 4.3.11] yields that $\mathcal{R}$ is a matrix ring over a skew field which is of finite dimensional over its center. Therefore $\mathcal{R}$ is a PI algebra and the first statement of the theorem is proved.

(ii) Now assume that both $\mathcal{R}$ and $\mathcal{R}_e$ are prime rings. As above we reduce the proof to the case when $\mathcal{F}$ is a field. If $\mathcal{R}_e$ has a nonzero ideal $\mathcal{I}$ satisfying St_{4n-2}, then $\mathcal{R}_e$ satisfies St_{4n-2} and so Theorem 2.6 implies that $\mathcal{R}$ is PI. Assume that $\mathcal{R}_e$ has no nonzero ideals satisfying St_{4n-2}. Then (9) implies that $f_n(\boldsymbol{u}_{n-1})u_n = 0$ for all $\boldsymbol{u}_n \in \prod_{k=1}^n \mathcal{R}_{\delta(x_k)}$. Setting

$$\mathcal{I} = \mathcal{R}_{\delta(x_n)}\mathcal{R}_{\delta(x_n)^{-1}}, \qquad \mathcal{K}_g = \{b \in \mathcal{R}_g \mid b\mathcal{I} = 0\}, \quad \text{and} \quad \mathcal{K} = \sum_{g \in G} \mathcal{K}_g,$$

we see that $\mathcal{K}$ is a left ideal of $\mathcal{R}$. Since $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}_e$ by (8), we conclude that $\mathcal{K}_e = 0$. Therefore, Proposition 2.4(i) yields that $\mathcal{K}$ is a nilpotent ideal of $\mathcal{R}$, forcing $\mathcal{K} = 0$. As $f_n(\boldsymbol{u}_{n-1}) \in \mathcal{K}$, we see that $f_n(\boldsymbol{x}_{n-1})$ is a G-graded polynomial identity on $\mathcal{R}$. The second statement of the theorem now follows from induction on $\deg(f)$.

(iii) Suppose that $1 \in \mathcal{R}$, G_f consists of invertible elements, and $1 \in \mathcal{R}_g\mathcal{R}_{g^{-1}}$ for all $g \in G_f$.

Let r be a positive integer and let $\mathcal{H}_r$ be the class of all homomorphic images of G-graded algebras $\mathcal{B}$ with finite G-grading, with antiautomorphism, satisfying multilinear G-graded polynomial identity f with antiautomorphism in which the monomial $x_1x_2\ldots x_n$ is involved with coefficient 1 and such that for any $g \in G_f$ there exist $u_1, u_2, \ldots, u_r \in \mathcal{B}_g$ and $v_1, v_2, \ldots, v_r \in \mathcal{B}_{g^{-1}}$ with $\sum_{i=1}^r u_iv_i = 1$. Clearly the class $\mathcal{H}_r$ is homomorphically closed and is closed under direct powers. Further, $\mathcal{R} \in \mathcal{H}_r$ for some integer r. In view of Theorem 2.7 it is enough to show that every prime homomorphic image $\overline{\mathcal{B}}$ of a G-graded algebra $\mathcal{B} \in \mathcal{H}_r$ satisfies a nonzero generalized polynomial identity.

If $\overline{\mathcal{B}}_e$ contains a nonzero ideal satisfying St_{4n-2}, then as in the proof of (i) we get that $\overline{\mathcal{B}}$ satisfies a nonzero generalized polynomial identity. Therefore we may assume without loss of generality that $\overline{\mathcal{B}}_e$ has no nonzero ideals satisfying St_{4n-2}. Set $g = \delta(x_n)$. It follows from (9) that $\overline{f_n(\boldsymbol{u}_{n-1})\mathcal{B}_g} = 0$ for all $\boldsymbol{u}_{n-1} \in \prod_{k=1}^{n-1} \mathcal{B}_{\delta(x_k)}$. Since $\bar{1} \in \overline{\mathcal{B}}_g\overline{\mathcal{B}}_{g^{-1}}$, we conclude that $\overline{f_n(\boldsymbol{u}_{n-1})} = 0$ for all $\boldsymbol{u}_{n-1} \in \prod_{k=1}^{n-1} \mathcal{B}_{\delta(x_k)}$. Proceeding inductively on $k = \deg(h)$, where h is a G-graded polynomial with antiautomorphism in which the monomial $x_1x_2\ldots x_m$ is involved with coefficient 1, such that $\overline{h(\boldsymbol{u}_m)} = 0$ for all $\boldsymbol{u}_m \in \prod_{k=1}^m \mathcal{B}_{\delta(x_k)}$, we see that $\overline{\mathcal{B}}$ satisfies a nonzero generalized polynomial identity. The proof is thereby complete. $\square$

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References

[1] S.A. Amitsur, Rings with involution, Israel J. Math. 6 (1968) 99–106.
[2] S.A. Amitsur, Identities in rings with involution, Israel J. Math. 7 (1969) 63–68.

[3] Yu.A. Bahturin, M.V. Zaicev, Identities of graded algebras, J. Algebra 205 (1998) 1–12.
[4] K.I. Beidar, M.A. Chebotar, On functional identities and d-free subsets of rings, I, Comm. Algebra 28 (2000) 2925–2951.
[5] K.I. Beidar and M.A. Chebotar, When is a graded PI algebra a PI algebra? Comm. Algebra, accepted.
[6] K.I. Beidar, W.S. Martindale 3rd, On functional identities in prime rings with involution, J. Algebra 203 (1998) 491–532.
[7] K.I. Beidar, W.S. Martindale 3rd, A.V. Mikhalev, Rings with Generalized Identities, Dekker, 1996.
[8] J. Bergen, M. Cohen, Actions of commutative Hopf algebras, Bull. London Math. Soc. 18 (1986) 159–164.
[9] I.N. Herstein, Special simple rings with involution, J. Algebra 6 (1967) 369–375.
[10] S.K. Jain, Prime rings having one-sided ideal with polynomial identity coincide with special Johnson rings, J. Algebra 19 (1971) 125–130.
[11] A.V. Kelarev, On semigroup graded PI algebras, Semigroup Forum 47 (1993) 294–298.
[12] M. Kępczyk, E. Puczyłowski, Rings which are sums of two subrings satisfying polynomial identities, Comm. Algebra 29 (2001) 2059–2065.
[13] W.S. Martindale 3rd, Rings with involution and polynomial identities, J. Algebra 11 (1969) 186–194.
[14] W.S. Martindale 3rd, Prime rings satisfying a generalized polynomial identity, J. Algebra 12 (1969) 576–584.
[15] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley, 1987.
[16] L.H. Rowen, Polynomial Identities in Ring Theory, Academic Press, 1980.
[17] S.K. Sehgal, M.V. Zaicev, Graded identities of group algebras, Comm. Algebra 30 (2002) 489–505.