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# Theory of Annihilation Games—I*

A. S. FRAENKEL AND Y. YESHA†

*Department of Applied Mathematics,*
*The Weizmann Institute of Science, Rehovot 76100, Israel*

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Place tokens on distinct vertices of an arbitrary finite digraph with $n$ vertices which may contain cycles or loops. Each of two players alternately selects a token and moves it from its present position $u$ to a neighboring vertex $v$ along a directed edge which may be a loop. If $v$ is occupied, and $u \neq v$, both tokens get annihilated and phase out of the game. The player first unable to move is the loser, the other the winner. If there is no last move, the outcome is declared a draw. An $O(n^6)$ algorithm for computing the previous-player-winning, next-player-winning and draw positions of the game is given. Furthermore, an algorithm is given for computing a best strategy in $O(n^6)$ steps and winning—starting from a next-player-winning position—in $O(n^5)$ moves.

## 1. INTRODUCTION

Throughout $R$ is a finite digraph with vertex set $V = V(R)$, $|V(R)| = n$ and edge set $E = E(R)$, which may contain cycles or loops. A two-person game is defined as in the abstract. The pupose of this work is to give a complete strategy for the game, the computation time of which is polynomial in $n$. The problem was communicated to us by John Conway. The results were announced in [10]. The present paper contains detailed proofs. Further results and ramifications will be given elsewhere. Examples of actual annihilation games—without the underlying theory—are given in [9].

In Section 2 we review the necessary tools from game theory, and prove a result on $D$-morphisms. Computational complexity results of games are reviewed briefly in Section 3. In Section 4 the notion of an abstract $\mathfrak{C}$-graph is developed, which is essentially a digraph $R$ such that $V(R)$ forms a vector

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† Present address: Department of Computer Science, University of Toronto, Toronto, Ontario, Canada.



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space under addition over $GF(2)$. The discussion is specialized to annihilation games in Section 5, which contains an $O(n^6)$ algorithm for determining the nature of each position (losing, winning or draw). In the final Section 6 we give a polynomial algorithm for computing a best strategy of play to win the game in $O(n^5)$ moves when starting from a next-player-winning position.

## 2. GAME THEORY BACKGROUND

We consider two-person perfect-information games without chance moves which are impartial (i.e., the possible moves from any position do not depend on which player is about to play). Any such game $M$ can be represented by a digraph called a *game-graph* $R$, whose vertices $V(R)$ are the game's positions, and $(u, v) \in E(R)$ (directed from $u$ to $v$) if and only if there is a move from position $u$ to position $v$. Conversely, given any digraph $R$, we can define on it a game by placing a token on one of its vertices. Each player at his turn moves the token to a neighboring vertex along a directed edge. Because of this duality, we occasionally identify $M$ with $R$, and the positions and moves of a game with the vertices and edges of its corresponding game-graph, using them interchangeably.

We shall restrict attention to those games in which the player first unable to move is the loser, the other the winner. If there is no last move, the outcome is declared a (dynamic) draw.

For every $u \in V(R)$, define its set of *followers* $F(u)$ by $F(u) = \{v \in V(R) : (u, v) \in E(R)\}$. If $F(u) = \varnothing$, $u$ is called a *sink*. For every $v \in V(R)$, define its set of *ancestors* by $F^{-1}(v) = \{u \in V(R) : v \in F(u)\}$. In particular, for a loop $(u, u)$, $u$ is both a follower and an ancestor of itself. We also define $F'(u) = F(u) - \{u\}$. In a loopless digraph $R$ of course $F'(u) = F(u)$ for all $u \in V(R)$. As customary, we denote by $N$ the set of all positions for each of which there is a move by which the *next* player can force a win, no matter what his opponent may do. By $P$ we denote the set of all positions such that if the previous player leaves his opponent in one of them, the *previous* player can force a win no matter what his opponent may do. For example, all sinks are $P$-positions. Finally, we denote by $T$ the set of all vertices which are draw positions, i.e., positions from which no player can force a win (and therefore each player can avoid losing). Clearly $u \in P$ if and only if $F(u) \subseteq N$; $u \in N$ if and only if $F(u) \cap P \neq \varnothing$; and $u \in T$ if and only if $F(u) \cap P = \varnothing$ and $F(U) \cap T \neq \varnothing$.

The classical Sprague–Grundy function $g$ is a mapping $g\colon V(R) \to J^0$, where $J^0$ is the set of nonnegative integers. See, e.g., Berge [3, Chap. 14]. The generalized Sprague–Grundy (GSG)-function is a mapping $G\colon V(R) \to J^0 \cup \{\infty\}$. If $\eta$ is any mapping $\eta\colon V(R) \to J^0 \cup \{\infty\}$, we let $\eta(F(u)) =$

$\{\eta(v) < \infty: \ v \in F(u)\}$. If $G(u) = \infty$, $G(F(u)) = K$, we also write $G(u) = \infty(K)$. If $G(u) = k$, $G(v) = l$, then $G(u) = G(v)$ if and only if one of the following holds: (a) $k = l < \infty$; (b) $k = \infty(K)$, $l = \infty(L)$ and $K = L$. Let $L$ be any finite set of nonnegative integers, mex $L$ the smallest nonnegative integer not in $L$. We use the notation $V^f(R) = \{u \in V(R): \ G(u) < \infty\}$, $V^\infty(R) = V(R) \oplus V^f(R)$, where here and below, $\oplus$ denotes the *symmetric difference* of sets: $S_1 \oplus S_2 = S_1 \cup S_2 - S_1 \cap S_2$. Finally, for any nonnegative integer $j$, let $V_j(R) = \{u \in V(R): \ G(u) = j\}$.

DEFINITION 1. A function $G: V(R) \to J^0 \cup \{\infty\}$ is a GSG-function with counter function $c: V^f(R) \to J$, where $J$ is any well-ordered set, if the following conditions hold:

**A.** If $u$ is a vertex with finite $G$, then $G(u) = \text{mex } G(F(u))$.

**B.** If $v$ is a follower of $u$ with larger $G$ than $u$, then there exists a follower $w$ of $v$ satisfying $G(w) = G(u)$ and $c(w) < c(u)$.

**C.** If $u$ has infinite $G$, then $u$ has a follower $v$ with infinite $G$ such that mex $G(F(u)) \notin G(F(v))$.

For the existence and uniqueness proof of the GSG-function and the proof of Theorems 1 and 2 below, see Smith [19] and Fraenkel and Perl [8].

THEOREM 1. *For every digraph $R$,*

$$P = V_0, \qquad T = \{u \in V: \ G(u) = \infty(K),\ 0 \notin K\}, \qquad N = V \oplus P \oplus T.$$

DEFINITION 2. A counter function $c$ such that $G(u) < G(v) < \infty \Rightarrow c(u) < c(v)$ is called a *monotonic* counter function.

Since Algorithm A below for computing the GSG-function produces a monotonic counter function $c$, we shall assume below that $c$ is monotonic.

A *disjunctive compound* is a game $M$ consisting of $m \geqslant 1$ disjoint two-player games. There are two players playing alternately, each selecting at his turn a move in any simgle game at will. The player first unable to make a move loses. The other is the winner. If there is no last move, the outcome is defined to be a draw. For analyzing $M$ we define:

(i) *The game-graph $\bar{R} = \mathfrak{D}(R)$ (disjunctive compound) of $M$.* Let $R_1, \ldots, R_m$ be the digraphs of the constituent games. Then $\bar{u} \in V(\bar{R})$ if $\bar{u} = (u_1, \ldots, u_m)$, $u_i \in V(R_i)$ $(1 \leqslant i \leqslant m)$. If $\bar{v} = (v_1, \ldots, v_m) \in V(\bar{R})$, then $(\bar{u}, \bar{v}) \in E(\bar{R})$ if $v_j \in F(u_j)$ in $R_j$ for some $1 \leqslant j \leqslant m$, and $v_i = u_i$ for all $i \neq j$.

(ii) *The (generalized) nim-sum.* The nim-sum of two nonnegative integers $a_1, a_2$ is the symmetric difference of their binary representations, i.e., $a_1 \oplus a_2 = \sum_i d_i 2^i$, where $a_1 = \sum_i b_i 2^i$, $a_2 = \sum_i c_i 2^i$ are the binary representations of $a_1, a_2$ and $d_i \equiv b_i + c_i \pmod 2$, $d_i = 0$ or $1$. The nim-sum of

$\infty(K_1)$, $\infty(K_2)$ is defined by $\infty(K_1) \oplus \infty(K_2) = \infty(\varnothing)$, and the nim-sum of a nonnegative integer $a$ and $\infty(K)$ is defined by $\infty(K) \oplus a = a \oplus \infty(K) = \infty(K \oplus a)$, where $K \oplus a = \{k \oplus a\colon\ k \in K\}$. The nim-sum of $m \geqslant 2$ summands is $\sum_{i=1}^{\prime m} a_i = a_1 \oplus \cdots \oplus a_m$, which is well-defined, since it is clearly associative.

Let $\bar{u} = (u_1,..., u_m) \in V(\bar{R})$, $u_i \in V(R_i)$ $(1 \leqslant i \leqslant m)$. Define $\sigma(\bar{u}) = \sum_{i=1}^{\prime m} G(u_i)$, $c(\bar{u}) = \sum_{i=1}^{m} c_i(u_i)$, where $c_i$ is monotonic $(1 \leqslant i \leqslant m)$.

THEOREM 2. *The GSG-function $G$ of the disjunctive compound is given by $G(\bar{u}) = \sigma(\bar{u})$ with counter function $c(\bar{u})$.*

*Note.* Though $c_i$ is monotonic $(1 \leqslant i \leqslant m)$, $c$ is not necessarily monotonic. But it can easily be made monotonic by letting $\tilde{c}(u) = (G(u), c(u))$, ordered lexicographically (where $(a_i, b_i) < (a_j, b_j)$ if either $a_i < a_j$ or $(a_i = a_j$ and $b_i < b_j))$.

We now formulate a winning strategy for the disjunctive compound (which may of course consist of a single game only).

*Notation.* For every $u$ in the set $A_j(R) = \{u \in V(R)\colon F(u) \cap V_j(R) \neq \varnothing\}$, let $cm_j(u) = \min\{c(v)\colon\ v \in F(u) \cap V_j\}$. In particular for every $u \in N$, $cm_0(u) = \min\{c(v)\colon v \in F(u) \cap P\}$.

DEFINITION 3. Let $\bar{R}$ be a digraph which depends on graphs $R_1,..., R_m$. Suppose that there exists a move in $R_i$ to a vertex in $V_j(R_i)$ leading to a win in $\bar{R}$. Then a function $\delta_j\colon A_j(R_i) \to V_j(R_i)$ is called a *winning j-strategy in the narrow sense* with respect to $\bar{R}$ (see [19, p. 52]), if the move $(u, v)$—where $u \in A_j(R_i)$, $v = \delta_j(u) \in V_j(R_i)$—leads to a win in $\bar{R}$, no matter what the opponent does.

If $\delta_j$ depends on $u \in A_j(R_i)$ and on a nonempty subset of the possible ancestors of $u$ as well, then $\delta_j$ is called a *winning j-strategy in the wide sense* with respect to $\bar{R}$.

Let $\bar{u} = (u_1, u_2,..., u_m) \in P$, $\bar{v} \in F(\bar{u})$. Then $\bar{v} \in N$. We may assume $\bar{v} = (v_1, u_2,..., u_m)$, $G(u_1) = j$. Consider two cases:

I. $G(v_1) > j$. By **B** there exists $w_1' \in F(v_1)$ such that $G(w_1') = j$, $c(w_1') < c(u_1)$. Thus if we let $\delta_j(v_1) = w_1$, where $c(w_1) = cm_j(v_1)$, then $\delta_j$ is a winning $j$-strategy in the narrow sense, since for $\bar{w} = (w_1, u_2,..., u_m)$ we have $G(\bar{w}) = G(\bar{u})$, $c(\bar{w}) < c(\bar{u})$.

II. $G(v_1) < j$. Then $c(v_1) < c(u_1)$. By the fundamental property of nim-sums (see, e.g., Lemma 2 in [8, p. 685]), there exists $i \neq 1$, say $i = 2$, and $w_2' \in F(u_2)$ with $l = G(w_2') < G(v_2)$, $c(w_2') < c(v_2)$, such that $G(v_1, w_2', u_3,..., u_m) = G(\bar{u})$. In this case we let $w_2 = \delta_l(v_2)$ (where $c(w_2) = cm_l(v_2))$, which is a winning $l$-strategy in the narrow sense.

We present now an algorithm for computing the GSG-function.

*Algorithm* A *for the GSG-Function*

1. (Initialize.) Put 0 into $i$ and $m$, and $\nu$ into $G(u)$ and $\delta_j(u)$ $(0 \leqslant j < |F(u)|)$ for all $u \in V(R)$. (Here $\nu$ denotes "unlabeled.")

2. (Label.) If there exists $u \in V_\nu(R)$ such that no follower of $u$ is labeled $i$ and every follower of $u$ which is either unlabeled or labeled $\infty$ has a follower labeled $i$, then do the following:

(a) put $i \to G(u)$, $m \to c(u)$, $m + 1 \to m$,

(b) for every $v \in F^{-1}(u) \cap (V_\nu \cup V^\infty)$ for which $\delta_i(v) = \nu$, put $u \to \delta_i(v)$.

Repeat 2.

3. ($\infty$-label.) For every $u \in V_\nu(R)$ which has no follower labeled $i$ put $\infty \to G(u)$.

4. (Increase label.) If $V_\nu(R) \neq \varnothing$, put $i + 1 \to i$ and return to 2; otherwise end.

If the digraph is stored as an *adjacency list* (see, e.g., [1, pp. 51–52]), linked by both rows and columns in the fashion of a sparse matrix (see [13, pp. 298–302]), then:

(i) The number of steps of each iteration is $O(|V| + |E|)$. Letting $\mathrm{GMAX} = \max_{u \in V^f} G(u) \leqslant \max_{u \in V^f} |F(u)| < |V|$, the algorithm is bounded by $O((|V| + |E|)\,\mathrm{GMAX}$ or by $O(|E| \cdot \mathrm{GMAX})$ for a connected digraph.

(ii) After the first iteration, $N = V_\nu$, $P = V_0$, $T = V^\infty$. Thus the $N$, $P$, $T$ classification of $R$ can be determined in only $O(|V| + |E|)$ steps, or $O(|E|)$ for a connected digraph.

(iii) The $\delta_j(u)$ provide winning $j$-strategies in the narrow sense (for example, for the disjunctive compound).

THEOREM 3. *The function G assigned by Algorithm A is a GSG-function with monotonic counter function c for every digraph R.*

*Proof.* (i) Let $R$ be any digraph. Every $u \in V(R)$ gets exactly one label. For if in the $i$th iteration no $u$ gets labeled $i$, then all unlabeled vertices get assigned $\infty$ in step 3. It is also clear that $c$ is monotonic.

(ii) For every $u$ such that $G(u) = i < \infty$ and every $0 \leqslant j < i$, there exists $v \in F(u)$ such that $G(v) = j$. For suppose that there exists $u$ such that $G(u) = i$ for which the claim does not hold. Then there exists $0 \leqslant j < i$ such that $G(v) = j$ for no $v \in F(u)$. During the $j$th iteration of the algorithm, $u$ was clearly not labeled by $j$. Hence in step 3 of the $k$th iteration $u$ was labeled $\infty$ for some $k \leqslant j$, a contradiction.

(iii) $G(u) = i < \infty$ and $v \in F(u) \Rightarrow G(v) \neq G(u)$. This follows immediately from step 2 of the algorithm. This demonstrates **A**.

(iv) $G(u)=i$, $v \in F(u)$ and $G(v) > G(u) \Rightarrow$ there exists $w \in F(v)$ such that $G(w)=i$, $c(w) < c(u)$. For in the $i$th iteration, $v$ is either unlabeled or $G(v)=\infty$. Since $u$ gets labeled in the $i$th iteration, step 2 implies existence of some $w \in F(v)$ which was already labeled by $i$, and so $c(w) < c(u)$. This demonstrates **B**.

(v) $G(u)=\infty \Rightarrow$ there exists $v \in F(u)$ such that $G(v)=\infty(K)$, $\operatorname{mex} G(F(u)) \notin K$. For let $\operatorname{mex} G(F(u))=i$. For every $0 \leqslant j < i$, there exists $v \in F(u)$ such that $G(v)=j$. Hence $u$ is not labeled $\infty$ in the $j$th iteration. Since $v \in F(u) \Rightarrow G(v) \neq i$, $u$ is labeled $\infty$ in step 3 of the $i$th iteration. Moreover, since $u$ was not labeled $i$ in step 2 of the $i$th iteration, there exists $v \in F(u)$ with $v$ unlabeled or $G(v)=\infty$, such that $w \in F(v) \Rightarrow G(w) \neq i$. If such a $v$ is unlabeled, it got labeled $\infty$ in step 3 of the $i$th iteration. This demonstrates **C**. ∎

DEFINITION 4. Let $R$, $\bar{R}$ be digraphs. A mapping $\lambda\colon V(R) \to V(\bar{R})$ is called a *D-morphism* if for every $u \in V(R)$,

$$F_{\bar{R}}(\lambda(u)) \subseteq \lambda(F_R(u)), \tag{1}$$

$$\lambda(F'_R(u)) \subseteq F_{\bar{R}}(\lambda(u)) \cup F_{\bar{R}}^{-1}(\lambda(u)), \tag{2}$$

where for any set $S$, $\lambda(S)=\{\lambda(s)\colon s \in S\}$; hence $\lambda(S_1 \cup S_2)=\lambda(S_1) \cup \lambda(S_2)$, and the subscripts $R$, $\bar{R}$ indicate the graph to which the corresponding followers belong.

If $R$ has no loops, then Definition 4 coincides with Banerji's definition of a $D$-morphism. For acyclic and loopless digraphs, Banerji [2, Theorem 2] proved that if $\bar{R}$ has a classical Sprague–Grundy function $g$, then the function $g'(u)=g(\lambda(u))$ is a $g$-function on $R$.

If $R$, $\bar{R}$ have neighter cycles nor loops, then $\bar{R}$ has a $g$-function, and the $g$-function on the game-graph $R$ determines a winning strategy there. Thus $\lambda$ relates the winning strategy of $R$ to that of $\bar{R}$. If either $R$ or $\bar{R}$ has cycles or loops, $R$ or $\bar{R}$ may not have a $g$-function. In the cyclic case, even if $R$ or $\bar{R}$ has a $g$-function, it does not necessarily determine a winning strategy.

The next theorem relates the GSG-function of $R$ (which always exists and always determines a strategy) to that of $\bar{R}$ under a $D$-morphism.

THEOREM 4. *Let $R$, $\bar{R}$ be digraphs, and $\lambda\colon V(R) \to V(\bar{R})$ a D-morphism. Then $G(\lambda(u))=G(u)$ for every $u \in V^f(R)$.*

*Proof.* Let $K=\{u \in V^f(R)\colon\ G(\lambda(u)) \neq G(u)\}$, $k=\min_{u \in K}(G(\lambda(u)), G(u))$. If there exists $v \in K$ such that $G(\lambda(v))=k$, then $k < G(v) < \infty$, and so there exists $u \in F'_R(v)$ such that $G(u)=k$. By (2), $\lambda(u) \in F_{\bar{R}}(\lambda(v)) \cup F_{\bar{R}}^{-1}(\lambda(v))$. Hence by **A** of Definition 1, $G(\lambda(u)) > k$, and so $u \in K$. Let $U=$

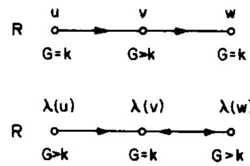


FIG. 1. An impossible situation.

$\{u \in K\colon G(u)=k\}$. Then $K \neq \varnothing \Rightarrow U \neq \varnothing$. Pick $u \in U$ with $c(u)$ minimal. Then $G(u)=k$, $G(\lambda(u))>k$. Suppose there exists $v' \in F_{\bar{R}}(\lambda(u))$ such that $G(v')=k$. By (1), there exists $v \in F_R(u)$ such that $\lambda(v)=v'$. Thus $G(\lambda(v))=k$, $G(v)>k$ (see Fig. 1). Hence there exists $w \in F'_R(v)$ such that $G(w)=k$, $c(w)<c(u)$. Now $\lambda(w) \in F_{\bar{R}}(\lambda(v)) \cup F_{\bar{R}}^{-1}(\lambda(v))$. Hence $G(\lambda(w))>k$ and so $w \in U$, contradicting the minimality of $c(u)$.

If $v' \in F_{\bar{R}}(\lambda(u)) \Rightarrow G(v') \neq k$, then $G(\lambda(u))=\infty$. For every $0 \leqslant j < k$, there exists $v \in F'_R(u)$ such that $G(v)=j$. By the minimality of $k$, $G(\lambda(v))=j$. By (2), either $\lambda(u) \in F_{\bar{R}}(\lambda(v))$, in which case **B** implies $j \in G(F(\lambda(u)))$, or $\lambda(v) \in F_{\bar{R}}(\lambda(u))$. Thus in any case $j \in G(F(\lambda(u)))$. Hence $\operatorname{mex} G(F(\lambda(u)))=k$. By **C**, there exists $v' \in F_{\bar{R}}(\lambda(u))$ such that $G(v')=\infty(K)$, $k \notin K$. By (1) there exists $v \in F_R(u)$ such that $\lambda(v)=v'$. Now $G(v)>k$ and so there exists $w \in F'_R(v)$ such that $G(w)=k$, $c(w)<c(u)$. Since $\lambda(w) \in F_{\bar{R}}(\lambda(v)) \cup F_{\bar{R}}^{-1}(\lambda(v))$, we have $G(\lambda(w))>k$ by **B**; hence $w \in U$, contradicting the minimality of $c(u)$. ∎

COROLLARY 1. *Let $R_1, R_2, \bar{R}$ be digraphs, and $\eta\colon V(R_1) \times V(R_2) \to V(\bar{R})$ a mapping such that for every $(u_1,u_2) \in V(R_1) \times V(R_2)$,*

$$F_{\bar{R}}(\eta(u_1,u_2)) \subseteq \eta((u_1, F_{R_2}(u_2)) \cup (F_{R_1}(u_1), u_2)), \tag{3}$$

$$\eta((u_1, F'_{R_2}(u_2)) \cup (F'_{R_1}(u_1), u_2)) \subseteq F_{\bar{R}}(\eta(u_1,u_2)) \cup F_{\bar{R}}^{-1}(\eta(u_1,u_2)). \tag{4}$$

*Then $G(\eta(u_1,u_2)) = G(u_1) \oplus G(u_2)$ for all $(u_1,u_2) \in V^f(R_1) \times V^f(R_2)$.*

*Proof.* Let $R = \mathfrak{D}(R_1,R_2)$ be the disjunctive compound of $R_1$ and $R_2$. For every $(u_1,u_2) \in V(\mathfrak{D}(R_1,R_2)) = V(R_1) \times V(R_2)$, define $\lambda\colon R \to \bar{R}$ by $\lambda(u_1,u_2)=\eta(u_1,u_2)$. Since $F'_R(u_1,u_2)=(u_1,F'_{R_2}(u_2)) \cup (F'_{R_1}(u_1),u_2)$, (3) and (4) (with $\eta$ replaced by $\lambda$) imply (1) and (2). Thus $G(\lambda(u_1,u_2))=G(u_1,u_2)=G(u_1) \oplus G(u_2)$ for all $(u_1,u_2) \in V^f(R_1) \times V^f(R_2)$ by Theorems 4 and 2. Since $\eta(u_1,u_2)=\lambda(u_1,u_2)$, the result follows. ∎

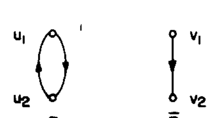


FIG. 2. A *D*-morphism which does not preserve *G*.

The following example shows that if $u \in V^{\infty}(R)$, then a $D$-morphism does not preserve $G$ in general.

EXAMPLE 1. The mapping $\lambda\colon V(R) \to V(\bar{R})$ defined by $\lambda(u_i) = v_i$ $(i = 1, 2)$ (Fig. 2), is clearly a $D$-morphism, yet $G(u_i) = \infty$, $G(v_i) < \infty$ $(i = 1, 2)$.

## 3. Computational Complexity Background

One of the main issues of this work is the formulation of a polynomial algorithm to solve the provlem: Given a digraph $R$ and a position in the annihilation game on $R$ find the GSG-function value of this position, and a next move if this position is in $N$ or in $T$. By "polynomial" we mean that the "running time" (= number of steps) of the algorithm is bounded from above by a polynomial in the size of its input. All relevant definitions and results from computational complexity theory can be found, say, in Aho *et al.* [1].

Since the input to our problem is an arbitrary finite digraph $R = (V, E)$, the input size cannot be less than $O(|E|)$, where for any set $S$, the cardinality of $S$ is denoted by $|S|$. In order to show that our algorithm is polynomial, it thus suffices to prove the existence of a polynomial $p$ such that the running time of the algorithm is bounded by $p(|E|)$.

We point out that the problem posed above is only just polynomial, in the sense that slight perturbations in various directions are already $NP$-hard. See Fraenkel and Yesha [11]. In this connection we mention briefly other recent results which show that certain interesting games are *transpolynomial* (that is, $NP$-hard, Pspace-hard, Exptime-hard, etc.—see Garey and Johnson [12] for these concepts). Shannon's switching game was proven Pspace-complete by Even and Tarjan [4], and node kayles and a large number of Boolean type games were prove Pspace-complete by Schaéfer [18]. A number of Boolean type games were proven Exptime-complete by Stockmeyer and Chandra [20]. As for board games, checkers, go, gobang and hex were proven Pspace-hard on $n \times n$ boards [6, 14, 16, 17] and chess was proven Exptime-complete on an $n \times n$ board [7].

## 4. Abstract $\mathfrak{C}$-Graphs

If the $m$ constituent games of a disjunctive compound have identical graphs $R_i = R$ $(1 \leqslant i \leqslant m)$, the game can be thought of as a single graph $R$ on which $m$ tokens are placed on $m$—not necessarily distinct—vertices. Each player at his turn selects one token and moves it to a neighboring vertex along a directed edge. Of course $u$ is its own neighbor if the graph contains a loop $(u, u)$.

An annihilation game on $R$ is played in the same way, but with two differences:

(i) On every vertex there is at most one token.

(ii) If a token is moved from $u$ to $v$ with $u \neq v$ and $v$ is occupied, both tokens are removed from the game (*annihilation* move).

In a finite acyclic digraph without loops, annihilation moves do not affect the game's final outcome. For when two tokens meet, but without annihilation, any move of one of these tokens by one player can be countered by the other player making exactly the same move with the other token. After finitely many such "pursuit" moves, both tokens get in effect removed from the game, since both reach a sink. A slightly modified argument shows that the same conclusion holds for any finite digraph all of whose $G$-values are finite (see also the special case $\bar{u}_\infty = \varnothing$ of Theorem 7(v) below).

The game-graph of an annihilation game on $R$ is called the *contrajunctive compound* of $R$, denoted by $\mathfrak{C}(R)$. We proceed to express this concept more formally. If $s_1, s_2, ..., s_m$ are $n$-tuples over a field $K$ with an addition operation $\oplus$, we use the notation $\sum_{i=1}^{\prime m} s_i = s_1 \oplus s_2 \oplus \cdots \oplus s_m$. (A special case of this is the nim-sum (Section 2).)

We now order $V(R)$, say,

$$V(R) = (z_1, z_2, ..., z_n).$$

Then a position in the annihilation game can be described as $\bar{u} = (u^1, ..., u^n)$ or $\bar{u}_i = (u_i^1, ..., u_i^n)$ over $GF(2)$, where $u^k = 1$ $(u_i^k = 1)$ if and only if there is a token on $z_k$ $(1 \leqslant k \leqslant n)$. If $u^k = 1$ $(u_i^k = 1)$ and $z_l \in F(z_k)$, let $\bar{v} = \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l$ $(\bar{v}_i = \bar{u}_i \oplus \bar{z}_k \oplus \bar{z}_l)$, where $\bar{z}_j$ is an $n$-tuple over $GF(2)$ with components $z_j^i = 1$ if $i = j$; $z_j^i = 0$ if $i \neq j$ $(i, j = 1, ..., n)$. The relation $\bar{v} = \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l$ with $u^k = 1$, and $z_l \in F(z_k)$ holds if and only if $\bar{v} \in F(\bar{u})$, and is denoted by $\bar{v} = F_{kl}(\bar{u})$. Note that the move $(\bar{u}, \bar{v})$ involves annihilation ("annihilation move") if and only if $u^l = 1$ and $k \neq l$. If $\bar{v} = F_{kl}(u)$, then $F_{lk}(\bar{u})$ is not always defined. But if both are defined, then: (i) $F_{kl}(\bar{u}) = F_{lk}(\bar{u})$; (ii) the move from $\bar{u}$ to $F_{kl}(\bar{u}) = F_{lk}(\bar{u})$ is an annihilation move if $k \neq l$; and a non-annihilation move along a loop if $k = l$.

DEFINITION 5. For any digraph $R$ with $|V(R)| = n$ define the graph $\mathfrak{C}(R)$ (*contrajunctive compound* of $R$) by:

(i) $V(\mathfrak{C}(R))$ is the set of all $n$-tuples over $GF(2)$.

(ii) $(\bar{u}, \bar{v}) \in E(\mathfrak{C}(R))$ if $\bar{v} = F_{kl}(\bar{u})$ for some $1 \leqslant k,\ l \leqslant n$, i.e., if $\bar{v} = \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l$, where $u^k = 1$, $z_l \in F(z_k)$.

*Notation.* Let $R$ be any digraph. For every $W(R) \subseteq V(\mathfrak{C}(R))$, denote the "even vertex set of $W$" by $W^{(0)}(R) = \{\bar{u} \in W(R)\colon \sum_{k=1}^{n} u^k \equiv 0 \pmod 2\}$ and the "odd vertex set" by $W^{(1)}(R) = \{\bar{u} \in W(R)\colon \sum_{k=1}^{n} u^k \equiv 1 \pmod 2\}$. Further, let $\mathfrak{C}^{(0)}(R)$ and $\mathfrak{C}^{(1)}(R)$ denote the vertex subgraphs of $\mathfrak{C}(R)$ whose vertex sets are $V^{(0)}(\mathfrak{C}(R))$ and $V^{(1)}(\mathfrak{C}(R))$, respectively.

*Notes.* (i) The set of all followers of $\bar{u} \in V(\mathfrak{C}(R))$ is given by

$$F(\bar{u}) = \bigcup_{u^k=1} \bigcup_{z_l \in F(z_k)} F_{kl}(\bar{u}).$$

(ii) The set $V(\mathfrak{C}(R))$ is an Abelian group under $\oplus$ with identity $\Phi = (0,\dots,0)$ which is a sink of $V(\mathfrak{C}(R))$. Every element is its own inverse. Moreover, $V(\mathfrak{C}(R))$ is a vector space over $GF(2)$ satisfying $1 \cdot \bar{u} = \bar{u}$, $0 \cdot \bar{u} = \Phi$ for every $\bar{u} \in V(\mathfrak{C}(R))$.

(iii) Define the set $Z_1(\mathfrak{C}(R)) = (\bar{z}_1,\dots,\bar{z}_n)$ of unit vectors by $\bar{z}_j = (z_j^1,\dots,z_j^n)$, where $z_j^i = 1$ if $i = j$; $z_j^i = 0$ if $i \neq j$ $(i, j = 1,\dots,n)$. The vertex subgraph of $\mathfrak{C}(R)$ whose vertex set is $Z_1(\mathfrak{C}(R))$ is isomorphic to $V(R)$ under the mapping $\bar{z}_j \to z_j$. Thus $R$ can be imbedded in $\mathfrak{C}(R)$. In the sequel it will often be convenient to consider $R$ to be thus imbedded.

(iv) Let $R$ be a connected digraph. Then $\mathfrak{C}(R)$ splits into the two connected components $\mathfrak{C}^{(0)}(R)$, $\mathfrak{C}^{(1)}(R)$, each of which contains $2^{n-1}$ vertices.

LEMMA 1. *Let $\bar{u}_1,\dots,\bar{u}_h \in V(\mathfrak{C}(R))$, $\bar{u} = \sum_{i=1}^{\prime h} \bar{u}_i$. Then:*

(i) *$F(\bar{u}) \subseteq \bigcup_{j=1}^{h} (F(\bar{u}_j) \oplus \sum'_{i\neq j} \bar{u}_i)$, where for any set $S$, $S \oplus \bar{u} = \{\bar{s} \oplus \bar{u}\colon \bar{s} \in S\}$.*

(ii) *$\bigcup_{j=1}^{h} (F'(\bar{u}_j) \oplus \sum'_{i\neq j} \bar{u}_i) \subseteq F(\bar{u}) \cup F^{-1}(\bar{u})$.*

(iii) *Let $\bar{v}_j = F_{kl}(\bar{u}_j)$, $k \neq l$, $\bar{v} = \bar{v}_j \oplus \sum'_{i\neq j} \bar{u}_i$. Then $\bar{u} \in F'(\bar{v})$ if and only if either*

(a) *$u^k = 0$, or*

(b) *$u^l = 0$, $z_k \in F(z_l)$.*

*Proof.* Let $\bar{v} \in F(\bar{u})$. Then $\bar{v} = F_{kl}(\bar{u})$, $u^k = 1$, $z_l \in F(z_k)$ for some $1 \leqslant k, l \leqslant n$. Hence $u_j^k = 1$ for some $1 \leqslant j \leqslant h$, and so

$$\begin{aligned}\bar{v} = F_{kl}(\bar{u}) \Leftrightarrow \bar{v} &= \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l \\ &= \sum_{i=1}^{h}{}' \bar{u}_i \oplus \bar{z}_k \oplus \bar{z}_l = (\bar{u}_j \oplus \bar{z}_k \oplus \bar{z}_l) \oplus \sum_{i\neq j}{}' \bar{u}_i \\ &= F_{kl}(\bar{u}_j) \oplus \sum_{i\neq j}{}' \bar{u}_i \in F(\bar{u}_j) \oplus \sum_{i\neq j}{}' \bar{u}_i,\end{aligned}$$

proving (i).

Now let $\bar{v} \in \bigcup_{j=1}^{h} (F'(\bar{u}_j) \oplus \sum'_{i \neq j} \bar{u}_i)$. Then for some $1 \leqslant j \leqslant h$, $1 \leqslant k$, $l \leqslant n$, $k \neq l$, $\bar{v} = F_{kl}(\bar{u}_j) \oplus \sum'_{i \neq j} \bar{u}_i$. Adding $\Phi = \bar{u}_j \oplus \bar{u}_j$ to the two summands on the right,

$$\bar{v} = \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l \qquad (u_j^k = 1,\ z_l \in F(z_k)). \tag{5}$$

If $u^k = 1$, then $\bar{v} \in F(\bar{u})$. If $u^k = 0$, then $v^k = 1$. Since (5) implies

$$\bar{u} = \bar{v} \oplus \bar{z}_k \oplus \bar{z}_l, \tag{6}$$

$\bar{u} \in F(\bar{v})$ in this case, proving (ii).

Note that (5) and (6) are valid also for part (iii). Suppose first that (a) holds. Then $v^k = 1$, and so $\bar{u} = F_{kl}(\bar{v})$. If (b) holds, then $v^l = 1$. This together with $z_k \in F(z_l)$ implies $\bar{u} = F_{lk}(\bar{v})$.

Conversely, suppose that $\bar{u} \in F'(\bar{v})$, say $\bar{u} = F_{st}(\bar{v})$. By (6),

$$\bar{u} = \bar{v} \oplus \bar{z}_s \oplus \bar{z}_t = \bar{v} \oplus \bar{z}_k \oplus \bar{z}_l \Rightarrow \bar{z}_s \oplus \bar{z}_t = \bar{z}_k \oplus \bar{z}_l \Rightarrow \bar{u} = F_{kl}(\bar{v})$$

or

$$\bar{u} = F_{lk}(\bar{v}).$$

In the former case, $v^k = 1$ and so $u^k = 0$. In the latter case $v^l = 1$ (hence $u^l = 0$), and $z_k \in F(z_l)$. ∎

COROLLARY 2. *For $\bar{u}, \bar{v} \in V(\mathfrak{C}(R))$,*

$$F(\bar{u} \oplus \bar{v}) \subseteq \bar{u} \oplus F(\bar{v}) \cup F(\bar{u}) \oplus \bar{v}, \tag{7}$$

$$\bar{u} \oplus F'(\bar{v}) \cup F'(\bar{u}) \oplus \bar{v} \subseteq F(\bar{u} \oplus \bar{v}) \cup F^-(\bar{u} \oplus \bar{v}). \tag{8}$$

EXAMPLE 2. For the digraph $R$ given in Fig. 3, the connected component $\mathfrak{C}^{(0)}(R)$ of $\mathfrak{C}(R)$ is given in Fig. 4. The vertices of $R$ and $\mathfrak{C}(R)$

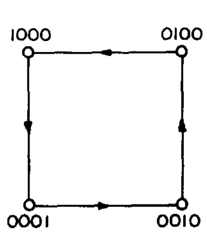


FIGURE 3

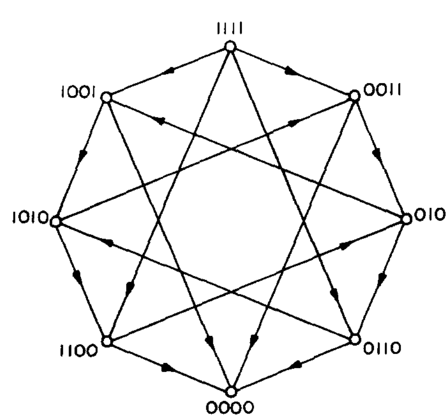


FIGURE 4

are denoted by 4-tuples over $GF(2)$, since $R$ contains $n=4$ vertices. Let $\bar{u}=0110$, $\bar{v}=0011$. Then $\bar{u}\oplus\bar{v}=0101$, and $F(\bar{u}\oplus\bar{v})=\{0110, 1001\}$, $F^{-1}(\bar{u}\oplus\bar{v})=\{1100, 0011\}$. Also $F(\bar{u})=\{0000, 1010\}$, $F(\bar{v})=\{0000, 0101\}$, $\bar{u}\oplus F(\bar{v})\cup F(\bar{u})\oplus\bar{v}=\{0110, 1001, 0011\}$, and so (7), (8) become $\{0110, 1001\}\subseteq\{0110, 1001, 0011\}\subseteq\{1100, 0110, 1001, 0011\}$. Also note that $0011=0110\oplus 0101\in\bar{u}\oplus F(\bar{v})\subseteq F^{-1}(\bar{u}\oplus\bar{v})=F^{-1}(0101)$.

The relations (7), (8) are a special case of a $D$-morphism (Section 2).

LEMMA 2. *Let $\lambda$: $V(R)\times V(R)\to V(R)$ be a mapping such that for every $(u_1, u_2)\in V(R)\times V(R)$,*

$$F(\lambda(u_1, u_2))\subseteq\lambda(u_1, F(u_2))\cup\lambda(F(u_1), u_2), \tag{9}$$

$$\lambda(u_1, F'(u_2))\cup\lambda(F'(u_1), u_2)\subseteq F(\lambda(u_1, u_2))\cup F^{-1}(\lambda(u_1, u_2)), \tag{10}$$

$$\lambda(\lambda(u_1, u_2), u_2)=u_1, \qquad \lambda(u_1, \lambda(u_1, u_2))=u_2. \tag{11}$$

*If $G(u_1)<\infty$ or $G(u_2)<\infty$, then $G(\lambda(u_1, u_2))=G(u_1)\oplus G(u_2)$, where $G$ is the GSG-function.*

*Proof.* By symmetry we may assume $G(u_1)<\infty$. If $G(u_2)<\infty$, the result follows directly from Corollary 1 (Section 2). So suppose $G(u_2)=\infty$. If $G(\lambda(u_1, u_2))<\infty$, then (11) and Corollary 1 imply

$$G(u_2)=G(\lambda(u_1, \lambda(u_1, u_2)))=G(u_1)\oplus G(\lambda(u_1, u_2))<\infty,$$

a contradiction. Hence $G(\lambda(u_1, u_2))=\infty$. Suppose that the statement is false. Then there exist $u_1, u_2$ with $G(u_1)<\infty$, $G(u_2)=\infty$, $c(u_1)$ minimal such that $G(\lambda(u_1, u_2))\neq G(u_1)\oplus G(u_2)$, i.e., $M\neq L\oplus G(u_1)$, where $G(u_2)=\infty(L)$, $G(\lambda(u_1, u_2))=\infty(M)$.

Let $d\in L\oplus G(u_1)$. Then $d_1\in L$, where $d_1=d\oplus G(u_1)$. Let $v_2\in F'(u_2)$ satisfy $G(v_2)=d_1$. Then $G(\lambda(u_1, v_2))=G(u_1)\oplus d_1=d$ by Corollary 1. By (10), $\lambda(u_1, v_2)\in F(\lambda(u_1, u_2))\cup F^{-1}(\lambda(u_1, u_2))$. Hence **B** of Definition 1 implies $d\in M$.

Now let $d\in M$, $v\in F(\lambda(u_1, u_2))$ such that $G(v)=d$. By (9), $v\in\lambda((u_1, F(u_2))\cup(F(u_1), u_2))$. There are two cases:

I. $v=\lambda(u_1, v_2)$, $v_2\in F(u_2)$. If $G(v_2)=\infty$, then $G(v)=\infty\neq d$. Hence $G(v_2)\in L$. By Corollary 1, $d=G(v)=G(u_1)\oplus G(v_2)\in G(u_1)\oplus L$.

II. $v=\lambda(v_1, u_2)$, $v_1\in F(u_1)$. If $G(v_1)<\infty$, then $G(v)=\infty$, since we showed already above that $G(u_1)<\infty$, $G(u_2)=\infty\Rightarrow G(\lambda(u_1, u_2))=\infty$. Hence $G(v_1)=\infty$. There exists $w_1\in F'(v_1)$ satisfying $G(w_1)=G(u_1)$, $c(w_1)<c(u_1)$. Let $w=\lambda(w_1, u_2)$. By the minimality of $c(u_1)$, $G(w)=\infty(L\oplus G(w_1))=\infty(L\oplus G(u_1))$. Moreover, $w\in F(v)\cup F^{-1}(v)$ by (10) and $G(v)=d$. Hence $d\in L\oplus G(u_1)$ by **B**, and so $M=L\oplus G(u_1)$, a contradiction. ∎

DEFINITION 6. An abstract $\mathfrak{C}$-graph is a digraph $C$ whose vertex set $V(C)$ forms a vector space under addition $\oplus$ over $GF(2)$ with identity $\Phi$, satisfying for every $u, v \in V(C)$,

$$F(u \oplus v) \subseteq u \oplus F(v) \cup F(u) \oplus v, \tag{12}$$

$$u \oplus F'(v) \cup F'(u) \oplus v \subseteq F(u \oplus v) \cup F^{-1}(u \oplus v), \tag{13}$$

$$F(\Phi) = \varnothing.$$

Corollary 2 and Note (ii) above show that the contrajunctive compound $\mathfrak{C}(R)$ is an example of an abstract $\mathfrak{C}$-graph.

For studying abstract $\mathfrak{C}$-graphs, we start by stating a result on the extension of homomorphisms in vector spaces. We denote the vector space of all $n$-tuples over $K$ by $K^n = K^n_n$. For $t \leqslant n$, let

$$K^n_t = \{(\alpha_1, \ldots, \alpha_n) \colon \alpha_i \in K \ (1 \leqslant i \leqslant n), \ \alpha_j = 0 \ (t+1 \leqslant j \leqslant n)\}.$$

Then $K^n_t \cong K^t_t = K^t$ (where $\cong$ denotes isomorphism) under the mapping $\nu \colon K^n_t \to K^t_t$ defined by $\nu(\alpha) = \beta$, where $\alpha = (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0)$ ($n - t$ trailing 0's), $\beta = (\alpha_1, \ldots, \alpha_t)$. Since in the application $K = GF(2)$, we also use the notation $GF(2)^n$, $GF(2)^t$ for $K^n$, $K^t$ in the sequel.

LEMMA 3. *Let $V$ be an $n$-dimensional vector space over a field $K$, $U$ an $(m+t)$-dimensional subspace, and $T$ a homomorphism from $U$ onto $K^t$. Then there exists a (not necessarily unique) homomorphism $\tilde{T}$ from $V$ onto $K^{n-m}$, such that $v \in U$ if and only if $\nu(\tilde{T}(v)) = T(v)$.*

*Proof.* There exists an $(n-m-t)$-dimensional subspace $W$ of $V$, such that $V$ is the (internal) direct sum: $V = U \oplus W$. Every $v \in V$ can be written uniquely in the form $v = u \oplus w$, where $u \in U$, $w \in W$. Let $B$ be any isomorphism from $W$ onto $K^{n-m-t}$, $\tilde{T}$ the (external) direct sum of $T$ and $B$, i.e., $\tilde{T}(v) = (T(u), B(w))$. Clearly $\tilde{T}$ is a homomorphism from $V$ onto $K^{n-m}$. Also $v \in U$, if and only if $\nu(\tilde{T}(v)) = \nu(T(v), 0) = T(v)$. ∎

The following theorem gives the main properties of the GSG-function $G$ on an abstract $\mathfrak{C}$-graph. In the sequel, the values of $G$ are taken to be binary vectors but with the most significant bit at the right end.

THEOREM 5. *Let $C$ be an abstract $\mathfrak{C}$-graph. $|V(C)| = n$, $u, v \in V(C)$ and $G$ the GSG-function on $C$. Then:*

(i) $G(u) < \infty \Rightarrow G(u \oplus v) = G(u) \oplus G(v)$.

(ii) *$V_0(C)$ and $V^f(C)$ (defined in Section 2) are linear subspaces of $V(C)$. Moreover, $G$ is a homomorphism from $V^f(C)$ onto $GF(2)^t$ for some $t \geqslant 0$ with kernel $V_0(C)$, and quotient space $\{V_i(C) \colon 0 \leqslant i < 2^t\} = V^f(C)/V_0(C)$, $\dim(V^f(C)) = m + t$, where $m = \dim(V_0(C))$.*

(iii) *There exists a homomorphism $H$ from $V(C)$ onto $GF(2)^{n-m}$, with kernel $V_0(C)$, such that $G(u) < \infty$ if and only if $v(H(u)) = G(u)$, i.e., the $n-m-t$ rightmost bits of $H(u)$ are zero. Hence $G(u) = \infty$ if and only if $H(u) \notin GF(2)^n_t$. Moreover, $H(u) = H(v) \Rightarrow G(u) = G(v)$. In particular, $X \in V(C)/V_0(C)$, $u, v \in X \Rightarrow G(u) = G(v)$.*

*Proof.* (i) Define a mapping $\lambda$: $V(C) \times V(C) \to V(C)$ by $\lambda(u, v) = u \oplus v$. Then (12), (13) imply (9), (10), and the result follows directly from Lemma 2.

(ii) By (i), $u, v \in V^f(C) \Rightarrow u \oplus v \in V^f(C)$. Also $\Phi \in V^f(C)$. Hence $V^f(C)$ is a linear subspace of $V(C)$. Moreover, (i) implies that $G$ is a homomorphism from $V^f(C)$ into $GF(2)^t$ for sme $0 \leqslant t \leqslant n$, since $G(1 \cdot u) = G(u)$, $G(0 \cdot u) = G(\Phi) = 0$. If $G(u) = j$, then for every $0 \leqslant i < j$, there exists $v \in F(u)$ such that $G(u) = i$. This implies that $G$ is onto $GF(2)^t$. The kernel of $G$ is $V_0(C)$ by definition; hence $V_0(C)$ is a linear subspace of $V^f(C)$ and consequenly of $V(C)$. Now $m$ is the dimension of the kernel of $G$, and $t$ is the dimension of the range of $G$ over $V^f(C)$. Hence $\dim(V^f(C)) = m + t$. Since $V_0(C)$ is, in particular, a subgroup of $V(C)$, its cosets have the form $V_i(C) = w \oplus V_0(C) \in V^f(C)/V_0(C)$ for some $w \in V_i(C)$ and every $0 \leqslant i < 2^t$; and for every $w \in V^f(C)$ we have $w \oplus V_0(C) = V_i(C)$ for some $i = G(w)$.

(iii) There exists an $(n-m-t)$-dimensional subspace $W$ of $V$, such that $V$ is the direct sum $V = V^f(C) \oplus W$. Thus every $v \in V(C)$ can be written uniquely in the form $v = u \oplus w$, $u \in V^f(C)$, $w \in W$. Let $B$ be any isomorphism from $W$ onto $GF(2)^{n-m-t}$, and let $H(v) = (G(u), B(w))$. By Lemma 3, $H$ is a homomorphism from $V(C)$ onto $GF(2)^{n-m}$, the kernel of which is clearly $V_0(C)$. Moreover, $u \in V^f(C)$ if and only if $v(H(u)) = G(u)$. Let $v_1, v_2 \in V(C)$, $v_1 = u_1 \oplus w_1$, $v_2 = u_2 \oplus w_2$, $u_1, u_2 \in V^f(C)$, $w_1, w_2 \in W$, $H(v_1) = (G(u_1), B(w_1))$, $H(v_2) = (G(u_2), B(w_2))$. Suppose that $H(v_1) = H(v_2)$. Then $G(u_1) = G(u_2)$ and $w_1 = w_2$ (since $B$ is 1–1). By (i), $G(v_1) = G(u_1) \oplus G(w_1) = G(u_2) \oplus G(w_2) = G(v_2)$. ∎

EXAMPLE 3. Consider the graph $R$ of Fig. 5. The component $\mathfrak{C}^{(0)}(R)$ of $\mathfrak{C}(R)$ is given in Fig. 6. We use the same notation of vertices as in Example 2.

Computing the GSG-function $G$ on $\mathfrak{C}^{(0)}(R)$ gives

$$V_0 = \{0000, 0101\}, \qquad V_1 = \{0110, 0011\} = 0110 \oplus V_0,$$

$$V_2 = \{1100, 1001\} = 1100 \oplus V_0, \qquad V_3 = \{1010, 1111\} = 1010 \oplus V_0.$$

This shows that $t = 2$, $m = 1$. Note that $\beta^f = \{0101, 0110, 1100\}$ is a basis of $V^f$. Adjoin the vertex 1000 to complete $\beta^f$ to a basis $\beta = \beta^f \cup \{1000\}$ for $V(\mathfrak{C}(R))$, which can be written as the direct sum: $V(\mathfrak{C}(R)) = V^f(\mathfrak{C}(R)) \oplus W$,

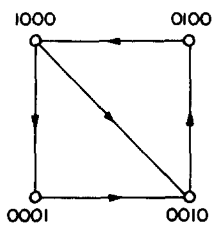


FIGURE 5

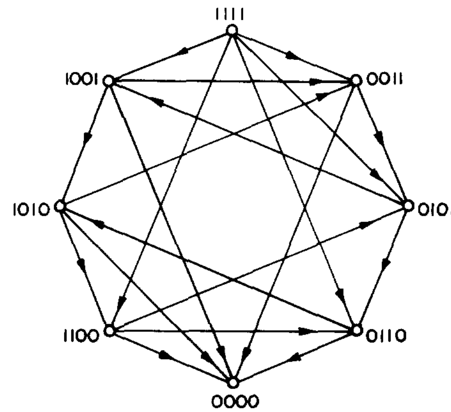


FIGURE 6

where $W$ is the linear subspace with basis $\{1000\}$. Define $B$: $W \to GF(2)^1$ by $B(1000) = 1$. Each $\bar{u} \in V(\mathfrak{C}(R))$ can be written uniquely in the form $\bar{u} = \bar{u}_1 \oplus \bar{u}_2$, where $\bar{u}_1 \in V^f(\mathfrak{C}(R))$, $\bar{u}_2 \in W$, and from this $H(\bar{u}) = (G(\bar{u}_1), B(\bar{u}_2))$ can be computed. This is exhibited in Table I for the four vertices of $R$. From this $H$ and hence $G$ can be computed for every $\bar{u} \in V(\mathfrak{C}(R))$. Thus $H(1010) = H(0010 \oplus 1000) = H(0010) \oplus H(1000) = 111 \oplus 001 = 110$. Since the rightmost position $(n - m - t = 1)$ of $H$ is zero, we have by Theorem 5(iii), $\nu(H(\bar{u})) = \nu(110) = 11 = G(\bar{u})$ in binary, that is, $G(\bar{u}) = 3$ in decimal. However, $H(1011) = H(1000) \oplus H(0010) \oplus H(0001) = 001 \oplus 111 \oplus 011 = 101$. Since the rightmost position is 1, we have $G(1011) = \infty$.

EXAMPLE 4. We exhibit an abstract $\mathfrak{C}$-graph which is not the contrajunctive compound of any digraph. Let $C$ be the game-graph of a Nim-heap with $m = 2^n$ vertices numbered $0, 1,..., 2^n - 1$. (The case $n = 2$ is given in Fig. 7.) Then $V(C)$ is a vector space under addition $\oplus$ over $GF(2)$, and $F(\Phi) = \varnothing$ for $\Phi = 0$.

Since $k \in F(u \oplus v) \Leftrightarrow k < u \oplus v$, the basic property of nim-sum implies (12); and (13) holds since $F(u \oplus v) \cup F^{-1}(u \oplus v)$ includes all vertices except

TABLE I

| $\bar{u}$ | $\bar{u}_1$ | $\bar{u}_2$ | $H(\bar{u})$ |
|---|---|---|---|
| 1000 | 0000 | 1000 | 001 |
| 0100 | 1100 | 1000 | 011 |
| 0010 | 1010 | 1000 | 111 |
| 0001 | 1001 | 1000 | 011 |

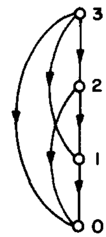


FIGURE 7

$u \oplus v$. Moreover, vertex $2^n - 1$ has $2^n - 1$ followers, whereas a vertex in any contrajunctive compound with $2^n$ vertices has at most $O(n^2)$ followers. This is the case because the followers of $\bar{u}$ in any contrajunctive compound are a subset of the set $\{\bar{u} \oplus \bar{z}_k \oplus \bar{z}_l\colon 1 \leqslant k \leqslant l \leqslant n\}$ (Definition 5), which has $\binom{n}{2} + 1$ members. Hence for $n$ sufficiently large, $C$ is an abstract $\mathfrak{C}$-graph which is neither the contrajunctive compound $\mathfrak{C}(R)$ nor $\mathfrak{C}^{(0)}(R)$ of any digraph.

*Notation for abstract $\mathfrak{C}$-graphs.* The linear span of a set $S$ of vectors over a given field is denoted by $\mathfrak{L}(S)$. If $C$ is an abstract $\mathfrak{C}$-graph, let

$$S(C) = \{u \in V(C)\colon F(u) = \varnothing\} \qquad \text{(the set of sinks of } C\text{)},$$

$$E_1(C) = \{x \oplus y\colon x \in V(C),\, y \in F(x)\} - \{\Phi\},$$

$$E_2(C) = \{x \oplus y\colon x \in V(C),\, y \in F(F(x))\} - \{\Phi\},$$

$$A_0(C) = (E_2(C) \cap V_0(C)) \cup S(C),$$

$$q = 2^t - 1, \qquad \text{where } 2^t - 1 = \max_{u \in V^f}(G(u)) \text{ (existence of such } t \text{ follows from Theorem 5(ii))},$$

$$W_j(C) = \bigcup_{i=1}^{j} V_i(C) \qquad \text{(that is, } W_j(C) = \text{set of all positions whose GSG-value} \leqslant j\text{), for } j = 2^k - 1 \ (0 \leqslant k \leqslant t).$$

We shall now show that $G$ on any vertex of any abstract $\mathfrak{C}$-graph $C$ can be computed from $S(C)$ and the $G$-values on $E_1$ and $E_2$.

THEOREM 6. *Let $C$ be any abstract $\mathfrak{C}$-graph. Then*

(i) $V_0(\mathfrak{C}) = \mathfrak{L}(A_0(C))$.

(ii) $\{G(u)\colon u \in V^f(C)\} = \{G(u)\colon u \in ((E_1(C) \cap V^f(C)) \cup \{\Phi\})\}$.

(iii) $V_i(C) = w_i \oplus V_0(C)$, *where* $w_i \in ((E_1(C) \cap V^f(C)) \cup \{\Phi\})$, $G(w_i) = i$ $(0 \leqslant i < 2^t)$.

(iv) $W_j$ *is a linear subspace of* $V^f$ *and* $W_j(C) = \mathfrak{L}(A_0(C) \cup (E_1(C) \cap$

$W_j(C))$), $j=2^k-1$, $0\leqslant k\leqslant t$. In particular, $V^f(C)=W_q(C)=\mathfrak{L}(A_0(C)\cup(E_1(C)\cap V^f(C)))$.

(v) $G(u)=\infty(L)$, $L\neq\varnothing\Rightarrow u\in z\oplus V_0(C)$ *for some* $z\in E_1(C)\cup E_2(C)$, *and* $G(z)=\infty(L)$.

*Proof.* (i) Clearly $\mathfrak{L}(A_0(C))\subseteq V_0(C)$. If the result is false, let $u$ with $c(u)$ minimal satisfy $u\in V_0(C)$, $u\notin\mathfrak{L}(A_0(C))$. In particular, $u\notin S(C)$. Hence there exist $v\in F(u)$ and $w\in F(v)\cap V_0(C)$ with $c(w)<c(u)$. Therefore $w\in\mathfrak{L}(A_0(C))$. Let $z=u\oplus w$. Then $z\in E_2(C)\cap V_0(C)\subseteq A_0(C)$. Since $\mathfrak{L}(A_0(C))$ is a subspace, it follows that $u=z\oplus w\in\mathfrak{L}(A_0(C))$, a contradiction.

(ii) Denote the left-hand set by $S_l$ and the right-hand set by $S_r$. Clearly $S_r\subseteq S_l$. Let $j\in S_l$. If $j=0$, then $j=G(\Phi)\in S_r$. If $j>0$, pick $u\in V_j(C)$. There exists $v\in F(u)$ such that $G(v)=0$. Let $w=u\oplus v$. Then $w\in E_1(C)\cap V^f(C)$ and $G(w)=j$.

(iii) By Theorem 5(ii), $V_i(C)=u_i\oplus V_0(C)$ for any $u_i\in V_i(C)$. By (ii) there exists $w_i\in((E_1(C)\cap V^f(C))\cup\{\Phi\})$ such that $G(w_i)=i$.

(iv) Let $u,v\in W_j$. Then $G(u\oplus v)=G(u)\oplus G(v)\leqslant j=2^k-1$, and also $\Phi\in W_j$. Hence $W_j$ is a linear subspace of $V^f$. Clearly $A_0\cup(E_1\cap W_j)\subseteq W_j$; hence $\mathfrak{L}(A_0\cup(E_1\cap W_j))\subseteq W_j$. Let $u\in W_j$. If $u\in V_0$, then by (i), $u\in\mathfrak{L}(A_0)\subseteq\mathfrak{L}(A_0\cup(E_1\cap W_j))$. Otherwise let $v\in F(u)\cap V_0$, $w=u\oplus v$. Then $w\in E_1\cap W_j$, and $u=w\oplus v\in\mathfrak{L}((E_1\cap W_j)\cup A_0)$ by (i).

(v) Let $j\in L$, $v\in F(u)$ with $G(v)=j$. If $j=0$, let $z=u\oplus v$. Then $z\in E_1(C)$, $G(z)=G(u)\oplus G(v)=\infty(L)$, and $u=z\oplus v\in z\oplus V_0(C)$. If $j>0$, let $w\in F(v)\cap V_0(C)$, $z=u\oplus w$. Then $z\in E_2(C)$, $G(z)=\infty(L)$, and $u=z\oplus w\in z\oplus V_0(C)$. ∎

## 5. The Contrajunctive Compound

Recall that $\mathfrak{C}(R)$ is an abstract $\mathfrak{C}$-graph, and that by Theorem 6(iv), $A_0(\mathfrak{C}(R))\cup(E_1(\mathfrak{C}(R)\cap V^f(\mathfrak{C}(R))))$ is a generating set of $V^f(\mathfrak{C}(R))$. This leads to one of the two main results, namely, that there exists a basis of $V^f\mathfrak{C}(R)$ which is contained in a subgraph comprising $O(n^4)$ vertices and $O(n^5)$ edges. Moreover, the Restriction Principle below implies that $G$ can be computed polynomially on this subgraph. From this a basis of $V^f(\mathfrak{C}(R))$ can be computed, to be extended to a basis of $(V(\mathfrak{C}(R)))$. Then the homomorphism $H$ can be computed by standard linear-algebra elimination techniques over $GF(2)$.

Definition 7. Let $R$ be any digraph. A subset $W\subseteq V(R)$ is called a *restriction* of $V(R)$ if $F(W)\subseteq W$.

LEMMA 4 (Restriction Principle). *Let $W$ be a restriction of $V(R)$, $R'$ the vertex subgraph of $R$ whose vertex set is $W$. Then the GSG-function as computed on $R'$ alone (without considering $R$) is identical with the GSG-function on $R$, restricted to $W$.*

*Proof.* Let $G'$ be the GSG-function on $R$ restricted to $W = V(R')$. Since $u \in V(R') \Rightarrow F_{R'}(u) = F_R(u)$, Definition 1 (Section 2) shows that $G'$ is a GSG-function on $R'$ ∎

*Notation for contrajunctive compounds.* To simplify the notation, we shall write $\bar{R}$ for $\mathfrak{C}(R)$ throughout, where $R$ is any digraph with $|V(R)| = n$ and $V(R) = (z_1, z_2, ..., z_n)$. Let

$$Y^i(\bar{R}) = \left\{\bar{u} \in V(\bar{R}) \colon \sum_{k=1}^{n} u^k = i\right\},$$

that is, the set of all $\bar{u}$ with $i$ 1-bits (see notation in Section 4, prior to Definition 5),

$$Yf^i(\bar{R}) = Y^i(\bar{R}) \cap V^f(\bar{R}),$$

$$Q_j^i(\bar{R}) = Y^i(\bar{R}) \cap V_j(\bar{R}),$$

$$W_j^i(\bar{R}) = Y^i(\bar{R}) \cap W_j(\bar{R}),$$

$$B(\bar{R}) = Y^1(\bar{R}) \cap S(\bar{R}),$$

$$Q_0(\bar{R}) = Q_0^2(\bar{R}) \cup Q_0^4(\bar{R}) \cup B(\bar{R}),$$

and

$$V^{[m]}(\bar{R}) = \left\{\bar{u} \in V(\bar{R}) \colon \sum_{k=1}^{n} u^k \leqslant m\right\},$$

that is, the set of all $\bar{u}$ with at most $m$ 1-bits. If $\bar{u} \in V(\bar{R})$, we denote its transpose by $\bar{u}'$. For $\bar{u} \in V(\bar{R})$, let $\bar{u}_f \in V(\bar{R})$ be defined by $u_f^k = 1$ if and only if $u^k = 1$ and $z_k \in V^f(R)$ $(1 \leqslant k \leqslant n)$. Also $\bar{u}_\infty = \bar{u} \oplus \bar{u}_f$.

DEFINITION 8. The *standard basis* of $V(\bar{R})$ is the set of unit vectors $(\bar{z}_1, ..., \bar{z}_n)$, where $z_j^i = 1$ if $i = j$; 0 otherwise $(1 \leqslant j \leqslant n)$.

THEOREM 7. *Let $R$ be any digraph, $|V(R)| = n$, $\bar{R} = \mathfrak{C}(R)$ the contrajunctive compound of $R$. Then*

(i) $E_1(\bar{R}) \subseteq Y^2(\bar{R})$, $E_2(\bar{R}) \subseteq Y^2(\bar{R}) \cup Y^4(\bar{R})$, $S(\bar{R}) = \mathfrak{L}(B(\bar{R}))$.

(ii) $W_j(\bar{R}) = \mathfrak{L}(Q_0(\bar{R}) \cup W_j^2(\bar{R}))$, $j = 2^k - 1$, $0 \leqslant k \leqslant t$. *In particular,* $V_0(\bar{R}) = \mathfrak{L}(Q_0(\bar{R}))$, $V^f(\bar{R}) = \mathfrak{L}(Q_0(\bar{R}) \cup Yf^2(\bar{R}))$.

(iii) $\{G(\bar{u}) \colon \bar{u} \in V^f(\bar{R})\} = \{G(\bar{u}) \colon \bar{u} \in Yf^2(\bar{R}) \cup \{\Phi\}\}$. *Furthermore,* $t \leqslant 1 + \log_2 n$.

(iv) $G(\bar{u}) = \infty(L)$, $L \neq \varnothing \Rightarrow \bar{u} \in \bar{z} \oplus V_0(\bar{R})$ *for some* $\bar{z} \in Y^2(\bar{R}) \cup Y^4(\bar{R})$ *with* $G(\bar{z}) = \infty(L)$.

(v) $G(\bar{u}) = G(\bar{u}_\infty) \oplus \sum'_{u_f^k=1} G(z_k)$ [5, Theorem III].

*Proof.* (i) If $\bar{v} = \bar{u} \oplus \bar{z}_k \oplus \bar{z}_l$, $\bar{w} = \bar{v} \oplus \bar{z}_p \oplus \bar{z}_q$, then $\bar{u} \oplus \bar{v} = \bar{x}$, where $\bar{x} = \bar{z}_k \oplus \bar{z}_l$. If $\bar{x} \neq \Phi$, then $\sum_{j=1}^n x^j = 2$; $\bar{y} = \bar{u} \oplus \bar{w} = \bar{z}_k \oplus \bar{z}_l \oplus \bar{z}_p \oplus \bar{z}_q$. Hence if $\bar{y} \neq \Phi$, then $\sum_{j=1}^n y^j = 2$ or 4. The last part follows from the definition of a sink.

(ii) By Theorem 6(iv), $W_j(\bar{R}) = \mathfrak{L}((E_2(\bar{R}) \cap V_0(\bar{R})) \cup S(\bar{R}) \cup (E_1(\bar{R}) \cap W_j(\bar{R})))$. By (i), $E_2(\bar{R}) \cap V_0(\bar{R}) \subseteq Q_0^2(\bar{R}) \cup Q_0^4(\bar{R})$, $E_1(\bar{R}) \cap W_j(\bar{R}) \subseteq Y^2(\bar{R}) \cap W_j(\bar{R}) = W_j^2(\bar{R})$. Hence $W_j(\bar{R}) \subseteq \mathfrak{L}(Q_0(\bar{R}) \cup W_j^2(\bar{R}))$, where we used $\mathfrak{L}(S_1 \cup \mathfrak{L}(S_2)) = \mathfrak{L}(S_1 \cup S_2)$ for sets $S_1$ and $S_2$. The opposite inclusion is obvious. The two special cases follow from $W_0^2(\bar{R}) = Y^2(\bar{R}) \cap W_0(\bar{R}) = Y^2(\bar{R}) \cap V_0(\bar{R}) = Q_0^2(\bar{R})$, and $E_1(\bar{R}) \cap W_q(\bar{R}) = E_1(\bar{R}) \cap V^f(\bar{R}) \subseteq Y^2(\bar{R}) \cap V^f(\bar{R}) = Yf^2(\bar{R})$.

(iii) By (i) and Theorem 6(ii), $\{G(\bar{u})\colon \bar{u} \in V^f(\bar{R})\} \subseteq \{G(\bar{u})\colon \bar{u} \in Yf^2(\bar{R}) \cup \{\Phi\}\}$, and the opposite inclusion is obvious. The out-degree of any $\bar{u} \in Yf^2(\bar{R})$ is clearly at most $2n-1$. Hence $2^t - 1 \leqslant 2n - 1 \Rightarrow t \leqslant 1 + \log_2 n$.

(iv) Follows from (i) and from Theorem 6(v).

(v) The set $Z_1(\bar{R}) = (\bar{z}_1, \ldots, \bar{z}_n)$ of unit vectors is a restriction of $V(\bar{R})$ and is isomorphic to $V(R)$. Hence $G(\bar{z}_i) = G(z_i)$ $(1 \leqslant i \leqslant n)$. By Theorem 5(i), $G(\bar{u}) = G(\bar{u}_\infty) \oplus G(u_f)$. Now $\bar{u}_f = \sum'_{u_f^k=1} \bar{z}_k$. By repeated use of Theorem 5(i), $G(\bar{u}_f) = \sum'_{u_f^k=1} G(\bar{z}_k) = \sum'_{u_f^k=1} G(z_k)$. ∎

*Notation.* If $A$ is a matrix with $m$ columns, its $i$th column is denoted by $A_i$, and $A = \|A_1, \ldots, A_m\|$.

DEFINITION 9. An $n \times m$ matrix $D$ of rank $k$ is said to be in *row-echelon* form if:

(i) There are $k$ columns numbered $1 \leqslant i_1 < \cdots < i_k \leqslant m$ which are the unit vectors $\bar{z}'_1, \ldots, \bar{z}'_k$, respectively (Definition 8).

(ii) If $1 \leqslant j < i_1$, then $D_j$ is the zero vector.

(iii) If $i_l < j < i_{l+1}$, then the last $n - l$ elements of $D_1$ are zero $(1 \leqslant l < k)$. If $j > i_k$, the last $n - k$ elements of $D_j$ are zero.

For details see, e.g., Noble [15].

THEOREM 8. *The following can be computed in* $O(n^6)$ *steps*:
*A basis* $\beta_0 = \{\bar{b}_1, \ldots, \bar{b}_m\}$ *of* $V_0(\bar{R})$, *where* $b_i \in Q_0(\bar{R})$ $(1 \leqslant i \leqslant m)$.

*Vectors* $\bar{v}_i \in Yf^2(\bar{R})$ *such that* $G(\bar{v}_i) = 2^{i-1}$ $(1 \leqslant i \leqslant t)$. *Furthermore,* $\beta^f = \{\bar{v}_1,..., \bar{v}_t\} \cup \beta_0$ *is a basis of* $V^f(\bar{R})$.

*Vectors* $\bar{w}_1,..., \bar{w}_{n-m-t}$ *such that* $\beta = \{\bar{w}_1,..., \bar{w}_{n-m-t}\} \cup \beta^f$ *is a basis of* $V(\bar{R})$.

*An* $n \times n$ *matrix* $\Gamma$ *over* $GF(2)$ *such that for every* $\bar{u} \in V(\bar{R})$, $\Gamma \cdot \bar{u}' = (\varepsilon_1,..., \varepsilon_n)'$ *is the coordinate vector of* $\bar{u}$ *in the basis* $\beta$, *i.e.,* $\bar{u} = \sum_{i=1}^{\prime m} \varepsilon_i \bar{b}_i \oplus \sum_{i=1}^{\prime t} \varepsilon_{m+i} \bar{v}_i \oplus \sum_{i=1}^{\prime n-m-t} \varepsilon_{m+t+i} \bar{w}_i$. *Furthermore,* $H(\bar{u}) = (\varepsilon_{m+1},..., \varepsilon_n)$ *is a homomorphism satisfying Theorem* 5(iii).

*A function* $\delta_0^4$, *which is the* $\delta_0$*-function for the vertex subgraph* $R'$ *of* $\bar{R}$ *with vertex set* $(V^{[4]}(\bar{R}) \cap V^{(0)}(\bar{R})) \cup B(\bar{R})$, *and functions* $\delta_j^2$ *which are the* $\delta_j$*-functions for the vertex subgraph* $R''$ *of* $R$ *with vertex set* $V^{[2]}(\bar{R}) \cap V^{(0)}(\bar{R})$ $(0 \leqslant j < 2^t)$. ($V^{(0)}$ *was defined in Section* 4 *and the* $\delta$-*functions in Section* 2.)

The proof of Theorem 8 is effected by the following algorithm.

*Algorithm* B *for the GSG-Function on* $V(\mathfrak{C}(R))$

*Input*: A digraph $R$ with $|V(R)| = n$.

*Output*: $\beta_0$, $\beta^f$, $\beta$, $\Gamma$, $H(\bar{z}_i)$, where the $\bar{z}_i$ are unit vectors $(1 \leqslant i \leqslant n)$, $\delta_0^4(\bar{u})$ for all $\bar{u} \in V(R') \cap N$, $\delta_j^2(\bar{u})$ for all $\bar{u} \in V(R'') \cap N$ $(0 \leqslant j < 2^t)$.

*Note*: Since $H$ is a homomorphism satisfying Theorem 5(iii), $H(\bar{z}_i)$ $(1 \leqslant i \leqslant n)$ completely determines $G$ on $V(\bar{R})$.

*Procedure*: (i) Construct the vertex subgraph $R'$. This involves generating all subsets of $V(R)$ with two and four elements, together with $\Phi$ and $Y^1 \cap S(\bar{R})$. (This subgraph has $O(n^4)$ vertices and $O(n^5)$ edges.)

(ii) Apply the first iteration of Algorithm A (Section 2) to $R'$ ($N, P, T$ labeling). Store $Q_0(\bar{R}) = Q_0^2(\bar{R}) \cup Q_0^4(\bar{R}) \cup B(\bar{R})$ and $\{\delta_0^4(\bar{u})$: $\bar{u} \in A_0(R')\}$, sorted in monotonic order of the numerical value of the vertices $\bar{u}$ ($O(n^5)$ steps).

(iii) Apply Algorithm A to the vertex subgraph $R''$. This yields $G$ for all elements of $Yf^2(\bar{R})$, and from this the following is found and stored: the $\bar{v}_i$, $t$ and $\delta_j^2$ $(0 \leqslant j < 2^t)$ ($O(n^5)$ steps).

(iv) Construct a matrix $M = \|\bar{a}_1',..., \bar{a}_p', \bar{v}_1',..., \bar{v}_t', \bar{z}_1',..., \bar{z}_n'\|$, where $\{\bar{a}_1',..., \bar{a}_p'\} = Q_0(\bar{R})$, $\bar{v}_i \in Yf^2(\bar{R})$ with $G(\bar{v}_i) = 2^{i-1}$ $(1 \leqslant i \leqslant t)$, and $\bar{z}_1',..., \bar{z}_n'$ are the unit vectors. ($O(n^5)$ steps, since $M$ is of order $n \times O(n^4)$.)

(v) By elementary row transformations over $GF(2)$, transform $M$ into a matrix $D$ in row-echelon form ($O(n^6)$ steps).

(vi) Let $1 \leqslant i_1 < \cdots < i_n \leqslant p + t + n$ be the indices of $D$ mentioned in Definition 9. Store the basis $\beta_0 = \{M_{i_1},..., M_{i_m}\}$ and $m$, where $m$ is the largest integer such that $i_m \leqslant p$. Also store the bases $\beta^f = \{M_{i_{m+1}},..., M_{i_{m+t}}\} \cup \beta_0$ and $\beta = \{M'_{i_{m+t+1}},..., M'_{i_n}\} \cup \beta^f$, and the matrix $\Gamma = \|D_{p+t+1},..., D_{p+t+n}\|$. Then $H(\bar{z}_i) = \{\varepsilon_{m+1}^i,..., \varepsilon_n^i)$, where $\Gamma \cdot \bar{z}_i' = (\varepsilon_1^i,..., \varepsilon_n^i)'$ $(1 \leqslant i \leqslant n)$.

*Validity proof of Algorithm* B. By the definition of $Q_0$, $Q_0(\bar{R}) = V(R') \cap V_0(\bar{R})$. Thus by the Restriction Principle (Lemma 4), the application of the first iteration of Algorithm A to $R'$ yields $Q_0(\bar{R})$ and $\{\delta_0^4(\bar{u}) \colon \bar{u} \in A_0(R')\}$. Also $V(R'')$ is a restriction of $V(\bar{R})$. Therefore the application of Algorithm A to $R''$ yields $Yf^2(\bar{R})$, hence the $\bar{v}_i$, $t$ and the $\delta_j^2$ $(0 \leqslant j < 2^t)$, establishing the validity of (ii) and (iii).

The matrix $M = \|M_1,..., M_{p+t+n}\|$ constructed in step (iv) has clearly rank $n$. For the matrix $D$ obtained in (v), let $1 \leqslant i_1 < \cdots < i_n \leqslant p+t+n$ be as in Definition 9. Let $1 \leqslant j \leqslant p+t+n$, $r$ the largest integer such that $i_r \leqslant j$, $M^{(j)} = \|M_1,..., M_j\|$, $D^{(j)} = \|D_1,..., D_j\|$ and $E = E_s \cdots E_1$, where $E_1,..., E_s$ are the elementary row transformation matrices used in transforming $M$ to $D$. Then $D^{(j)} = EM^{(j)}$. Since $E$ is non-singular, $D^{(j)}$ and $M^{(j)}$ have the same rank. Definition 9 implies that $D_{i_1},..., D_{i_r}$ is a basis of the subspace of $V(\bar{R})$ spanned by the columns of $D^{(j)}$. Since $D_{i_l} = EM_{i_l}$ $(1 \leqslant l \leqslant r)$, $M_{i_1},..., M_{i_r}$ is a basis for the subspace spanned by $M_1,..., M_j$. It follows that $\beta_0 = \{M_{i_1},..., M_{i_m}\}$ is a basis of $V_0(\bar{R})$ if $m$ is the largest integer such that $i_m \leqslant p$, since $Q_0(\bar{R}) = \{\bar{a}'_1,..., \bar{a}'_p\}$ and $Q_0(\bar{R})$ is a generating set of $V_0(\bar{R})$ (Theorem 7(ii)). By Theorem 5(i),

$$\sum_{i=1}^{m}{}' \alpha_i \bar{b}^i \oplus \sum_{i=1}^{t}{}' \beta_i \bar{v}_i = \Phi \Rightarrow \sum_{i=1}^{t}{}' \beta_i G(\bar{v}_i) = 0 \Rightarrow \sum_{i=1}^{t}{}' \beta_i 2^{i-1} = 0$$
$$\Rightarrow \beta_1 = \cdots = \beta_t = 0 \Rightarrow \alpha_1 = \cdots = \alpha_m = 0.$$

Thus since $\dim V^f(\bar{R}) = m+t$ and since $\bar{b}_1,..., \bar{b}_m$, $\bar{v}_1,..., \bar{v}_t$ are linearly independent over $GF(2)$,

$$\begin{aligned}\beta^f &= \beta_0 \cup \{M_{i_{m+1}},..., M_{i_{m+t}}\} = \beta_0 \cup \{M_{i_{p+1}},..., M_{i_{p+t}}\} \\ &= \{\bar{b}_1,..., \bar{b}_m, \bar{v}_1,..., \bar{v}_t\}\end{aligned}$$

is a basis of $V^f(\bar{R})$.

The same argument used in showing that $M_{i_1},..., M_{i_m}$ is a basis of $V_0(\bar{R})$ shows that $\beta = \beta^f \cup \{M_{i_{m+t+1}},..., M_{i_n}\}$ is a basis of $V(\bar{R})$. Thus the columns of $\Delta = \|M_{i_1},..., M_{i_n}\|$ are the coordinates of the elements of $\beta$ in the standard basis. Then clearly $\Delta^{-1}$ effects the change of basis from the standard basis to the basis $\beta$, namely, $\Delta^{-1}\bar{u}' = (\varepsilon_1,..., \varepsilon_n)'$, where $(\varepsilon_1,..., \varepsilon_n)$ is the coordinate vector of $\bar{u}$ in the basis $\beta$.

Moreover, $E \cdot \|\bar{z}'_1,..., \bar{z}'_n\| = EI = E = \|D_{p+t+1},..., D_{p+t+n}\| = \Gamma$, and $E \cdot \|M_{i_1},..., M_{i_n}\| = \|D_{i_1},..., D_{i_n}\| = \|\bar{z}'_1,..., \bar{z}'_n\| = I$. Hence $\Gamma = \Delta^{-1}$, establishing the validity of (vi).

To complete the proof of Theorem 8 we note that $H$ is a homomorphism from $V(\bar{R})$ onto $GF(2)^{n-m}$ with kernel $V_0(\bar{R})$. Also $\bar{u} \in V^f(\bar{R}) \Leftrightarrow \varepsilon_i = 0$ $(m+t+1 \leqslant i \leqslant n \Leftrightarrow v(H(\bar{u})) = (\varepsilon_{m+1},..., \varepsilon_{m+t}) = G(\bar{u})$ in its binary vector form (with its most significant bit on the right, both here and below), since $G(\bar{u}) = \sum_{i=1}^{t'} \varepsilon_{m+i} 2^{i-1}$ by Theorem 5(i).

Let $\bar{u}, \bar{v} \in V(\bar{R})$, $\Gamma \cdot \bar{u}' = (\varepsilon_1, \ldots, \varepsilon_n)'$, $\Gamma \cdot \bar{v}' = (\varphi_1, \ldots, \varphi_n)'$. Then $H(\bar{u}) = H(\bar{v}) \Leftrightarrow \varepsilon_i = \varphi_i$ $(1 \leqslant i \leqslant n-m)$. Let $\sum_{i=m+t+1}^{n} \varepsilon_i \bar{w}_i = \bar{w}_\varepsilon$, $\sum_{i=m+t+1}^{n} \varphi_i \bar{w}_i = \bar{w}_\varphi$. By Theorem 5(i), $G(\bar{u}) = G(\bar{w}_\varepsilon) \oplus \sum_{i=1}^{t'} \varepsilon_{m+i} G(\bar{v}_i) = G(\bar{w}_\varphi) \oplus \sum_{i=1}^{t'} \varphi_{m+i} G(\bar{v}_i) = G(\bar{v})$. ∎

EXAMPLE 5. Let $R$ be the graph given in Fig. 8. It is convenient to use the decimal equivalents of the corresponding 5-tuples over $GF(2)$. That is, the binary $(\varepsilon_1, \ldots, \varepsilon_5)$ is replaced by $\sum_{i=1}^{5} \varepsilon_i 2^{i-1}$. Thus (10110) is replaced by 13, (01001) by 18, etc.

Applying steps (i)–(iii) of Algorithm B gives

$$Q_0^2(\bar{R}) = \{5, 10\}, \qquad Q_0^4(\bar{R}) = \{15\}, \qquad B(\bar{R}) = \{\Phi\},$$
$$Yf^2(\bar{R}) = Q_0^2(\bar{R}) \cup Q_1^2(\bar{R}), \qquad Q_1^2(\bar{R}) = \{3, 6, 9, 12\}.$$

From this a matrix $M$ is constructed, and elimination leads to the following matrix $D$ in row-echelon form:

$$M = \begin{pmatrix} 0&1&0&1&1&1&0&0&0&0 \\ 0&0&1&1&1&0&1&0&0&0 \\ 0&1&0&1&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 0&1&0&1&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&1&1&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} = D.$$

From $D$ and $M$ we get $\beta_0 = \{5, 10\}$, $\beta^f = \{5, 10, 3\}$, $\beta = \{5, 10, 3, 1, 16\}$, $m = 2$, $t = 1$ and

$$\Gamma = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&1&0&1&0 \\ 1&1&1&1&0 \\ 0&0&0&0&1 \end{pmatrix}.$$

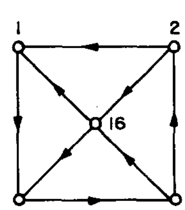


FIGURE 8

From this we get $\Gamma \cdot \bar{z}_1' = \Gamma \cdot 1 = \Gamma \cdot (10000)' = (00010)' = 8$. To get $H(1)$ from $\Gamma \cdot 1$ delete $\varepsilon_1, \ldots, \varepsilon_m$ (Theorem 8), in the present case, $\varepsilon_1$ $(=0)$ and $\varepsilon_2$ $(=0)$. Thus $H(1) = (010) = 2$. Similarly, $H(2) = 3$, $H(4) = 2$, $H(8) = 3$, $H(16) = 4$. Note that the values of $H$ on the unit vectors are obtained by deleting the first $m$ $(=2)$ rows of $\Gamma$. From these $H$-values we can get all $G$-values. For example, $H(3) = H(1 \oplus 2) = H(1) \oplus H(2) = 2 \oplus 3 = 1 = (100)$ in binary. Now $n - m - t = 2$. Since the two rightmost bits of the binary representation of $H(3)$ are 0, Theorem 5(iii) implies $G(3) = H(3) = 1$. Similarly, $H(20) = H(4 \oplus 16) = H(4) \oplus H(16) = 2 \oplus 4 = 6 = (011)$ in binary. Since the two rightmost bits are not 0, we have $G(20) = \infty$.

If we reverse the direction of edge (2, 1) in Fig. 8, but leave all the others unchanged, we get $Q_0(\bar{R})$ and $Q_1^2(\bar{R})$ as above, but $t = 2$, $Q_2^2 = \{18, 23, 24, 29\}$, $Q_3^2 = \{17, 20, 27, 30\}$. Hence

$$M = \begin{pmatrix} 0&1&0&1&1&0&1&0&0&0&0 \\ 0&0&1&1&1&1&0&1&0&0&0 \\ 0&1&0&1&0&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&0&0&1 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 0&1&0&1&0&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&1&0&1&1 \\ 0&0&0&0&0&1&0&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix} = D,$$

$\beta_0 = \{5, 10\}$, $\beta^f = \{5, 10, 3, 18\}$, $\beta = \{5, 10, 3, 18, 1\}$,

$$\Gamma = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&1&0&1&1 \\ 0&0&0&0&1 \\ 1&1&1&1&1 \end{pmatrix},$$

$H(16) = 7$, $H(8) = 5$, $H(4) = 4$, $H(2) = 5$, $H(1) = 4$. Now $n - m - t = 1$, and therefore $H(3) = 1 = G(3)$ as before (since $\varepsilon_5 = 0$). Also $H(20) = H(4) \oplus H(16) = 4 \oplus 7 = 3 = (110)$ in binary. Since $\varepsilon_5 = 0$, we have now $G(20) = H(20) = 3$.

## 6. A Winning Strategy for the Contrajunctive Compound

The main purpose of this section is to give a method for winning the annihilation game, when starting from an $N$-position, in $O(n^5)$ moves and $O(n^6)$ computation steps.

*Forcing a draw.* By using $\Gamma$, the $N$, $P$, $T$ membership of any $\bar{u} \in V(\bar{R})$ can be decided in $O(n^2)$ steps. If $\bar{u} \in T$, then $\bar{v} \in F(\bar{u}) \cap T$ can be found polynomially by scanning $F(\bar{u})$ $(|F(\bar{u})| < n^2)$. Similarly, if $\bar{u} \in N$, then $\bar{v} \in F(\bar{u}) \cap P$ can be found polynomially. This, however, does not guarantee a win, because of the possibility of cycling and never reaching a sink. Nevertheless, the strategy of moving from $\bar{u} \in N$ to any $\bar{v} \in F(\bar{u}) \cap P$ guarantees a non-losing outcome.

*Forcing a win.* The problem of cycling is overcome by the method described below.

DEFINITION 10. Let $W(\bar{R}) \subseteq V(\bar{R})$, $\bar{u} \in V(\bar{R})$. A *representation* $\tilde{u} = (\bar{u}_1, \dots, \bar{u}_h)$ of $\bar{u}$ over $W$ is a subset $\{\bar{u}_1, \dots, \bar{u}_h\} \subseteq W$, $\bar{u}_i \neq \Phi$ $(1 \leqslant i \leqslant h)$, such that $\bar{u} = \mu(\tilde{u}) = \sum_{i=1}^{\prime h} \bar{u}_i$. We also call $\tilde{u} = (\bar{u}_1, \dots, \bar{u}_h)$ simply a *representation* of $\bar{u}$ when the subset $W$ is indicated by the context. The emty representation is denoted by $\tilde{\varnothing}$.

*Notes.* (i) For every set $S$, let $\pi(S)$ denote the set of all subsets of $S$. As usual, the symmetric different $\tilde{x} \cup \tilde{y} - \tilde{x} \cap \tilde{y}$ is denoted by $\tilde{x} \oplus \tilde{y}$.

(ii) If $S = \{\tilde{x}_1, \dots, \tilde{x}_m\}$, $\tilde{x}_i \in \pi(V(\bar{R}))$ $(1 \leqslant i \leqslant m)$, then, as customary, we define $\mu(S) = \bigcup_{i=1}^{m} \mu(\tilde{x}_i)$; hence $\mu(S_1 \cup S_2) = \mu(S_1) \cup \mu(S_2)$.

(iii) Every $\bar{u} \in V_0(\bar{R})$ has a representation over $Q_0(\bar{R})$, since $Q_0$ is a generating set of $V_0$.

For $\tilde{u} = (\bar{u}_1, \dots, \bar{u}_h) \in \pi(V(\bar{R}))$, define a follower function for representations by

$$\tilde{F}(\tilde{u}; \bar{u}_j, \bar{v}_j) = (\tilde{u} \oplus \{\bar{u}_j, \bar{v}_j\}) - \{\Phi\},$$

where $\bar{u}_j \in \tilde{u}$, $\bar{v}_j \in F_{\bar{R}}(\bar{u}_j)$ for any $1 \leqslant j \leqslant h$. Note that $\tilde{F}(\tilde{u}; \bar{u}_j, \bar{v}_j)$ is a representation of $\mu(\tilde{u}) \oplus \bar{u}_j \oplus \bar{v}_j = \sum_{i \neq j}^{\prime} \bar{u}_i \oplus \bar{v}_j$ over $V(\bar{R})$. We also define

$$\tilde{F}(\tilde{u}) = \bigcup_{j=1}^{h} \bigcup_{\bar{u}_j \in F(\bar{v}_j)} \tilde{F}(\tilde{u}; \bar{u}_j, \bar{v}_j), \qquad \tilde{F}'(\tilde{u}) = \tilde{F}(\tilde{u}) - \{\tilde{u}\}.$$

LEMMA 5. *For every* $\tilde{u} = (\bar{u}_1, \dots, \bar{u}_h) \in \pi(V(\bar{R}))$ *with* $\bar{u} = \mu(\tilde{u})$, *we have*:

(i) $F(\mu(\tilde{u})) \subseteq \mu(\tilde{F}(\tilde{u}))$.

(ii) $\mu(\tilde{F}'(\tilde{u})) \subseteq F(\mu(\tilde{u})) \cup F^{-1}(\mu(\tilde{u}))$.

(iii) *Let* $\tilde{n} = \tilde{F}(\tilde{u}; \bar{u}_j, \bar{v}_j)$, *where* $\bar{v}_j = F_{kl}(\bar{u}_j)$, $k \neq l$. *Then* $\mu(\tilde{u}) \in F(\mu(\tilde{v}))$ *if and only if either*: (a) $u^k = 0$, *or* (b) $u^l = 0$, $z_k \in F(z_l)$.

*Proof.* $\mu(\tilde{F}(\tilde{u})) = \mu(\bigcup_{j=1}^{h} \bigcup_{\bar{u}_j \in F(\bar{v}_j)} (\tilde{u} \oplus \{\bar{u}_j, \bar{v}_j\})) = \mu(\bigcup_{j=1}^{h} (\tilde{u} \oplus \{\bar{u}_j, F(\bar{u}_j)\})) = \bigcup_{j=1}^{h} (F(\bar{u}_j) \oplus \sum_{i \neq j}^{\prime} \bar{u}_i)$. Similarly, $\mu(\tilde{F}'(\tilde{u})) = \bigcup_{j=1}^{h} (F'(\bar{u}_j) \oplus \sum_{i \neq j}^{\prime} \bar{u}_i)$.

Now apply Lemma 1. ∎

Let $c$ be a counter function on $V(R') = (V^{[4]}(\bar{R}) \cap V^{(0)}(\bar{R})) \cup B(\bar{R})$. For $\tilde{u} = (\bar{u}_1, \dots, \bar{u}_h) \subseteq Q_0(\bar{R})$, define $\tilde{c}(\tilde{u}) = \sum_{i=1}^{h} c(\bar{u}_i)$. Note that there exists $c = O(n^4)$. Hence $\tilde{c} = O(n^5)$.

THEOREM 9. *A function $\Lambda_0$ can be computed in polynomial time, such that if $\tilde{u} = (\bar{u}_1, \ldots, \bar{u}_h) \subseteq Q_0(\bar{R})$, $\mu(\tilde{v}) \in F(\mu(\tilde{u}))$, then $\Lambda_0(\tilde{u}, \mu(\tilde{v})) = \tilde{w} \subseteq Q_0(\bar{R})$, $\mu(\tilde{w}) \in F(\mu(\tilde{v})) \cap Q_0(\bar{R})$, $\tilde{c}(\tilde{w}) < \tilde{c}(\tilde{u})$.*

*Note.* $\mu(\tilde{v}) \in F'(\mu(\tilde{u}))$ since $\mu(\tilde{u}) \in V_0(\bar{R})$.

*Proof.* Let $\mu(\tilde{v}^0) \in F(\mu(\tilde{u}))$. By Lemma 5(i) we may assume $\tilde{v}^0 = \tilde{F}(\tilde{u}; \bar{u}_1, \bar{v}_1)$, where $\bar{v}_1 = F_{kl}(\bar{u}_1)$. Since $G(\bar{v}_1) > 0$, we have $\bar{v}_1 \neq \bar{u}_i$ $(1 \leqslant i \leqslant h)$, and so $\tilde{v}^0 = (\bar{v}_1, \bar{u}_2, \ldots, \bar{u}_h)$ is the representation of $\bar{v}_1 \oplus \sum_{i=2}^{h} \bar{u}_i$ over $V(\bar{R})$. The computation of $\bar{v}_1$ requires $O(n)$ steps, and $\bar{v}_1 \in V(R')$. Let $\bar{w}_1 = \delta_0^4(\bar{v}_1)$, where $\delta_0^4(\bar{v}_1) \in F(\bar{v}_1)$ is located in the output of Algorithm B in $O(\log n)$ steps. Then $\bar{w}_1 \in Q_0(\bar{R}) \cap V(R')$, and by **B** of Definition 1, $c(\bar{w}_1) < c(\bar{u}_1)$; hence $\tilde{c}(\tilde{w}^0) < \tilde{c}(\tilde{u})$, where $\tilde{w}^0 = (\bar{w}_1, \bar{u}_2, \ldots, \bar{u}_h)$ if $\bar{w}_1 \neq \bar{u}_i$ $(2 \leqslant i \leqslant h)$, $\tilde{w}^0 = \{\bar{u}_2, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_h\}$ otherwise, and in any case $\tilde{w}^0 \subseteq Q_0(\bar{R}) \cap \tilde{F}'(\tilde{v}^0)$. By Lemma 5(ii), $\mu(\tilde{w}^0) \in F(\mu(\tilde{v}^0)) \cup F^{-1}(\mu(\tilde{v}^0))$.

If $\mu(\tilde{w}^0) \in F(\mu(\tilde{v}^0))$, we let $\Lambda(\tilde{u}, \mu(\tilde{v}^0)) = \tilde{w}^0$, which satisfies the desired requirements. If $\mu(\tilde{v}^0) \in F(\mu(\tilde{w}^0))$, we replace the ancestor $\mu(\tilde{u})$ of $\mu(\tilde{v}^0)$ by its ancestor $\mu(\tilde{w}^0)$ with representation $\tilde{w}^0$. This results in a new representation $\tilde{v}^1$ of $\mu(\tilde{v}^0) = \mu(\tilde{v}^1)$. Using $\delta_0^4$ as before, we get $\tilde{w}^1 \subseteq Q_0(\bar{R})$, $\tilde{c}(\tilde{w}^1) < \tilde{c}(\tilde{w}^0)$, with $u(\tilde{w}^1) \in F(\mu(\tilde{v}^1)) \cup F^{-1}(\mu(\tilde{v}^1))$.

This process thus leads to the formation of two sequences $\tilde{v}^0, \tilde{v}^1, \ldots$ and $\tilde{w}^0, \tilde{w}^1, \ldots$, where $\mu(\tilde{v}^0) = \mu(\tilde{v}^i)$ $(i = 1, 2, \ldots)$, $\tilde{w}^i \subseteq Q_0(R)$, $\mu(\tilde{w}^i) \in F(\mu(\tilde{v}^i)) \cup F^{-1}(\mu(\tilde{v}^i))$ $(i = 0, 1, \ldots)$. Since $\tilde{c}(\tilde{w}^0) > \tilde{c}(\tilde{w}^1) > \cdots$, these sequences must be finite. In fact, each sequence has at most $O(n^5)$ terms. Hence there exists $j = O(n^5)$ such that $\mu(\tilde{w}^j) \in F(\mu(\tilde{v}^j))$. We then define $\Lambda_0(\tilde{u}, \mu(\tilde{v}^0)) = \tilde{w}^j$, which satisfies the desired requirements.

Finally, it can be decided in $O(n)$ steps whether $\mu(\tilde{w}^i) \in F(\mu(\tilde{v}^i))$ or $\mu(\tilde{v}^i) \in F(\mu(\tilde{w}^i))$ by using Lemma 5(iii). ∎

THEOREM 10. *If player* A *moves from any N-position or player* B *moves from any P-position in an annihilation game, player* A *can win the game in $O(n^5)$ moves using an $O(n^6)$ step computation.*

*Proof.* We apply Algorithm B to $R$, and store the $H(\bar{z}_i)$ $(1 \leqslant i \leqslant n)$ (or compute them anew whenever needed). Without loss of generality we may assume that player B moves from $\bar{u}^0 \in P$, because if player A starts from some $\bar{v} \in N$, he can scan $F(\bar{v})$ $(|F(\bar{v})| \leqslant n^2)$ and use the $H(\bar{z}_i)$ to locate and move to $\bar{w} \in F(\bar{v}) \cap P$.

Player A now computes $\Gamma \cdot \bar{u}^{0\prime} = (\varepsilon_1, \ldots, \varepsilon_m, 0, \ldots, 0)$ $(O(n^2)$ steps), and so $\tilde{u}^0 = \{\bar{b}_i\colon\ \varepsilon_i = 1\ \ (1 \leqslant i \leqslant m)\} \subseteq Q_0(\bar{R})$, $\mu(\tilde{u}^0) = \bar{u}^0$. If $B$ moves to some $\bar{v}^0 \in F(\bar{u}^0)$, then A computes $\tilde{u}^1 = \Lambda_0(\tilde{u}^0, \bar{v}^0)$ in polynomial time and moves to $\bar{u}^1 = \mu(\tilde{u}^1) \in F(\bar{v}^0) \cap P$.

In general, if B moves from $\bar{u}^i = \mu(\tilde{u}^i) \in P$ with $\tilde{u}^i \subseteq Q_0(\bar{R})$ to $\bar{v}^i \in F(\bar{u}^i)$, player A computes $\tilde{u}^{i+1} = \Lambda_0(\tilde{u}^i, \bar{v}^i)$ and moves to $\bar{u}^{i+1} = \mu(\tilde{u}^{i+1}) \in$

$F(\bar{v}^i) \cap P$, where $\tilde{u}^{i+1} \subseteq Q_0(\bar{R})$. By Theorem 9, $\tilde{c}(\tilde{u}^0) > \tilde{c}(\tilde{u}^1) > \cdots$. Since $\tilde{c}(\tilde{u}^0) = O(n^5)$, player A can win in $O(n^5)$ moves. Since each iteration in the computation of $\Lambda_0$ requires $O(n)$ steps, the entire computation time invested by A comprises that of $O(n^6)$ computation steps. ∎

*Notes.* (i) The above is a winning strategy in the wide sense. We do not know whether a winning strategy in the narrow sense can always be computed in polynomial time for annihilation games.

(ii) For playing a single annihilation game it suffices to apply steps (i) and (ii) of Algorithm B. But for playing a disjunctive compound where at least one of the component games is an annihilation game, all the values of $G$ on $V(\bar{R})$ are necessary. For such a disjunctive compound, strategies $\Lambda_j$ $(j = 0, 1, \ldots)$ can be formulated in a way similar to $\Lambda_0$.

EXAMPLE 6. Consider the digraph of Fig. 8 with the starting position $\bar{u}^0 = 15 \in P$. Then $\Gamma \cdot \bar{u}^{0\prime} = (11000)'$, and so $\tilde{u}^0 = (\bar{u}_1, \bar{u}_2) \subseteq Q_0(\bar{R})$, $\mu(\tilde{u}^0) = \bar{u}^0$, where $\bar{u}_1 = 5$, $\bar{u}_2 = 10$. Suppose player B moves to $\bar{v}^0 = 6 = F_{18}(15)$. Then $\bar{v}_1 = F_{18}(\bar{u}_1) = 12$, $\tilde{v}^0 = (\bar{v}_1, \bar{u}_2)$, $\mu(\tilde{v}^0) = \bar{v}^0$. Now it may be assumed that $\delta_0^4(\bar{v}_1) = \Phi$, and so $\tilde{w}^0 = (\bar{u}_2)$, $\mu(\tilde{w}^0) = 10$. Since $10 \in F^{-1}(6)$, we replace the ancestor $\tilde{u}^0$ of $\tilde{v}^0$ by $\overline{w}^0$ with representation $\tilde{w}^0 = (\bar{u}_2)$. Then $\bar{v}_2 = F_{84}(\bar{u}_2) = 6$, and so $\tilde{v}^1 = (6)$. Now $\delta_0^4(\bar{v}_2) = \Phi = \bar{w}^1$ with representation $\tilde{w}^1 = \tilde{\varnothing} \in F(\mu(\tilde{v}^1))$. Thus $\Lambda_0(\tilde{u}^0, \mu(\tilde{v}^0)) = \tilde{w}^1$, and so A moves to $\bar{w}^1$ for realizing his win.

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