Topological completions of the field of rational numbers which consist of Liouville numbers and rational numbers

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Abstract

Some subfields of the field of real numbers which consist exclusively of rational numbers and Liouville numbers are given. Each of these fields is a completion of the rational number field endowed with a field topology finer than the usual topology. © 1999 Elsevier science B.V. All rights reserved.

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1. Introduction

A topological ring \((R, \mathcal{T})\) is a ring \(R\) provided with a topology \(\mathcal{T}\) such that the algebraic operations \((x, y) \mapsto x \pm y\) and \((x, y) \mapsto xy\) are continuous. A topological field \((K, \mathcal{T})\) is a field \(K\) equipped with a ring topology \(\mathcal{T}\) such that the inversion \(x \mapsto x^{-1}\) is also continuous. A topological ring \(\hat{K}\) is the completion of the topological field \(K\) if \(\hat{K}\) is complete and \(K\) is a dense subfield of \(\hat{K}\). If a commutative topological field satisfies the first axiom of countability, then the completion \(\hat{K}\) is the quotient ring of the ring of Cauchy sequences in \(K\) by the ideal of sequences whose limit is zero. A topological field is called completable or full if its completion is a field. The field of rational numbers with the usual and \(p\)-adic topologies is completable.

We recall that a subset \(S\) of a commutative topological ring \(R\) is bounded if given any neighborhood \(V\) of zero, there exists a neighborhood \(U\) of zero such that \(SU \subseteq V\). If \(R\) is a nondiscretely topologized field, this is equivalent to saying that given any neighborhood \(V\) of zero, there exists a nonzero element \(x \in R\) such that \(Sx \subseteq V\) (see

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A ring topology on $R$ is locally bounded if there is a bounded neighborhood of zero.

In 1973 Shanks and Warner proved the following result:

**Lemma 1.** If $K$ is a Hausdorff, complete, locally bounded field of characteristic zero, then either $\mathbb{Q}$ is discrete or the closure of $\mathbb{Q}$ in $K$ is topologically isomorphic to $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime $p$.

**Proof.** See [11, 12] or [17, p. 393].

Therefore, the field of real numbers and the $p$-adic fields are the only locally bounded completions of the rational numbers field $\mathbb{Q}$. On the other hand, in 1968, Mutylin [9] constructed completable locally unbounded topologizations of $\mathbb{Q}$. He used an inductive procedure introduced in 1964 by Hinrichs [3]. In [7] I also introduced similar examples of completable locally unbounded field topologies on $\mathbb{Q}$. Some of these completions of $\mathbb{Q}$ are proper subfields of a $p$-adic field $\mathbb{Q}_p$.

In this article we present some subfields of the field of real numbers which are completions of $\mathbb{Q}$ with respect to field topologies finer than the usual one. Certainly, $\mathbb{Q}$ with these field topologies is not locally bounded. Another feature is that the remainder of these completions consists solely of Liouville numbers.

Using nonstandard analysis, the results in this article could be presented in a simpler way and proofs could be shorter. For an introduction to nonstandard analysis see [4, 16]. Likewise, for a knowledge about topological fields, the books [13, 17, 18] are recommended.

In this article, we shall only consider field topologies which satisfy the first axiom of countability, i.e., which have a countable basis of neighborhoods of zero. We recall that for a sequence $\{U_n\}_{n \in \mathbb{N}}$ of subsets of a commutative ring $R$ to be a fundamental system of neighborhoods of zero for a Hausdorff ring topology $\mathcal{T}$ on $R$, it suffices that the following properties hold.

\[
\begin{align*}
\text{For all } n & \quad 0 \in U_n, \quad U_n = -U_n, \quad U_{n+1} \subseteq U_n, \\
\text{For all } n \text{ there exists } k \text{ such that } & \quad U_k + U_k \subseteq U_n, \\
\text{For all } n \text{ there exists } k \text{ such that } & \quad U_k U_k \subseteq U_n, \\
\text{For all } n \text{ and } x \in R \text{ there exists } k \text{ such that } & \quad xU_k \subseteq U_n, \\
\bigcap_{n \in \mathbb{N}} U_n & = \{0\}.
\end{align*}
\]

If, in addition, $R$ is a field, then $\mathcal{T}$ is a field topology if $\{U_n\}_{n \in \mathbb{N}}$ also satisfies the following condition.

\[
\text{For all } n \text{ there exists } k \text{ such that } (1 + U_k)^{-1} \subseteq 1 + U_n.
\]

See [5; 8; 13, p. 4] or [18, p. 3], for instance.
2. Definition of the subfield $\mathbb{L}$

In this section, we shall define a subfield $\mathbb{L}$ of the field of real numbers, whose elements are either rational numbers or Liouville numbers. In the sequel, $a_n, c_n, e_n$ will denote rational integers and $b_n, d_n, f_n$ will denote nonzero natural numbers. Besides, $(p_n)_{n \in \mathbb{N}}$ will stand for a sequence of natural numbers that satisfies the following conditions:

$$p_1 = 1, \quad p_2 \geq 2 \quad\text{and}\quad p_{n+1} \geq (p_n)^n \quad\text{for all } n \in \mathbb{N}.$$  \hfill (7)

It is easy to check that this sequence also fulfills the following conditions:

$$p_n \geq 2^{(n-1)!} \quad\text{for all } n \geq 2,$$

$$\lim_{n \to \infty} \frac{\log(p_n)}{\log(p_{n+1})} = 0.$$  \hfill (8)

We define a subset of the field of real numbers:

$$\mathbb{L} = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} : \lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \right\},$$  \hfill (9)

$$a_n \in \mathbb{Z}, \quad b_n \in \mathbb{N}$$

Let us notice that the condition

$$\lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = 0$$  \hfill (10)

is equivalent to the following property: for all $s \in \mathbb{N}$ there exists $n_s \in \mathbb{N}$ such that

$$(b_n)^s < p_{n+1} \quad\text{for all } n \geq n_s.$$  \hfill (11)

It is straightforward to verify that all the series in the set $\mathbb{L}$ are absolutely convergent in $\mathbb{R}$ with its usual topology. Our goal is to prove that $\mathbb{L}$, with the sum and product inherited from $\mathbb{R}$, is a subfield; we shall need some lemmas.

**Lemma 2.** Let $(p_n)_{n \in \mathbb{N}}$ be the sequence defined in (7), and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers that satisfies

$$\lim_{n \to \infty} \frac{x_n}{\log(p_n)} = 0,$$

then

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{\log(p_n)} = 0.$$  \hfill (12)

**Proof.** A straightforward result of mathematical analysis. \hfill $\square$

An immediate consequence is the following:
Lemma 3. If \( \sum_{n=1}^{\infty} a_n/(b_n p_n) \in L \), then
\[
\lim_{n \to \infty} \frac{\log \left( \prod_{i=1}^{n} b_i p_i \right)}{\log(p_{n+1})} = 0.
\]

The following result will be used throughout this article.

Lemma 4. If the sequence \((p_n)_{n \in \mathbb{N}}\) satisfies the conditions (7), then
(a) \( \prod_{n=1}^{m} (p_n)^{2^{m-n}} < (p_{m+1})^{3/m} \).
(b) \( \prod_{n=1}^{m} p_n \leq (p_{m+1})^{1/(m-2)} \) for \( m \geq 3 \).

Proof. An elementary use of the principle of mathematical induction. \( \Box \)

We consider the sum and the product that \( L \) inherits from the field of real numbers.

Theorem 5. The subset \( L \) is a subfield of the field of real numbers.

Proof. It is easy to check that \( L \) is closed for the sum. Let us show that the product is also an inner operation. We consider two elements
\[
\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n}, \quad \beta = \sum_{n=1}^{\infty} \frac{c_n}{d_n p_n} \in L.
\]

We carry out their product by grouping terms in the following manner:
\[
\alpha \beta = \sum_{m=1}^{\infty} \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m p_m} + \sum_{m=2}^{\infty} \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right) \frac{a_m}{b_m p_m}.
\]

We denote these two summands by \( \delta \) and \( \gamma \), respectively. Let us verify that \( \delta \) belongs to \( L \). Since
\[
\sum_{n=1}^{\infty} \frac{a_n}{b_n p_n}
\]
is a convergent series in \( \mathbb{R} \) and the sequence \((c_m/d_m)_{m \in \mathbb{N}}\) converges to zero, then the coefficient of \( 1/p_m \) in \( \delta \) satisfies that
\[
\lim_{m \to \infty} \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m} = 0.
\]

Grouping this coefficient into a single fraction we get
\[
\left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m} = \left( \sum_{n=1}^{m} a_n/(b_n p_n) \right) \left( \prod_{n=1}^{m} b_n p_n \right) c_m \left( \prod_{n=1}^{m} b_n p_n \right)^{-d_m}.
\]

Applying Lemma 3, we conclude that the denominator satisfies that
\[
\lim_{m \to \infty} \frac{\log \left( \prod_{n=1}^{m} b_n p_n \right) d_m}{\log(p_{n+1})} = 0.
\]

Thus \( \delta \in L \); analogously \( \gamma \in L \). Consequently, \( \alpha \beta = \delta + \gamma \in L \).
Let $x = \sum_{n=1}^{\infty} a_n/(b_n p_n) \in L \setminus \mathbb{Q}$, we are going to construct its inverse $\beta = \sum_{n=1}^{\infty} c_n/(d_n p_n)$ by defining inductively the coefficients $c_n$ and $d_n$. We assume that $a_1 \neq 0$ and $\sum_{n=1}^{m} a_n/(b_n p_n) \neq 0$ for all $m \in \mathbb{N}$. We set $d_1 = |a_1|$ and $c_1 = b_1 a_1/|a_1|$. Now, suppose we have defined $c_n$ and $d_n$ for $n = 1, \ldots, m-1$; we shall determine $c_m$ and $d_m$. For $\beta$ to be the inverse of $x$, it suffices that the product $x\beta$ has the coefficient of $1/p_m$ equal to zero for all $m \geq 2$:

$$\left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m} + \frac{a_m}{b_m} \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right) = 0.$$ 

Hence, we define

$$\frac{c_m}{d_m} = -\frac{a_m \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right)}{b_m \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right)}.$$ 

(11)

Since the element $x \neq 0$ is given by an absolutely convergent series in $\mathbb{R}$, there exists $r \in \mathbb{R}$, $r > 0$, such that

$$r < \left| \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right| \quad \text{for all } m \in \mathbb{N}. \quad (12)$$

As $(a_n/b_n)_{n \in \mathbb{N}} \rightarrow 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n/b_n|/r < \frac{1}{2}$ for all $n \geq n_0$. Therefore, for all $m \geq n_0$, the inequality

$$\left| \frac{c_m}{d_m} \right| \leq \frac{1}{2} \left| \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right|$$

holds. It follows that $(c_m/d_m)_{m \in \mathbb{N}}$ is a bounded sequence, which implies that $(\sum_{n=1}^{m} \frac{c_n}{(d_n p_n)})_{m \in \mathbb{N}}$ is also a bounded sequence. This fact, together with (11) and (12), implies that

$$\lim_{m \to \infty} \frac{c_m}{d_m} = 0.$$ 

Now, we shall show that the denominators $d_m$ satisfy the condition (10). Given $s \in \mathbb{N}$, we are going to see that there exists a natural number $m_s$ such that $(d_m)^s \leq p_{m+1}$ for all $m \geq m_s$. We call $r = 3s$. We express the fraction (11) in the following way

$$\frac{c_m}{d_m} = -\frac{a_m \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right)}{b_m \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right)} \left( \prod_{n=1}^{m-1} b_n d_n p_n \right) p_m.$$ 

There exists a real constant $M > 0$ such that $|\sum_{n=1}^{m} a_n/(b_n p_n)| < M$ for all $m \in \mathbb{N}$. Consequently, we get the bound

$$d_m \leq b_m M \left( \prod_{n=1}^{m} b_n d_n p_n \right) p_m.$$ 

(13)
Considering Lemma 3, we obtain that

$$\lim_{m \to \infty} \frac{\log (M \prod_{n=1}^{m} b_n p_n)}{\log (p_{m+1})} = 0.$$ 

So there exists \( l \in \mathbb{N}, l \geq 3 \) such that

$$\left( M \prod_{n=1}^{m} b_n p_n \right)^{r} < p_{m+1} \quad \text{for all } m \geq l.$$ 

Hence, from (13) we deduce that

$$ (d_m)^r \leq \left( \prod_{n=1}^{m-1} d_n \right)^r p_{m+1} \quad \text{for all } m \geq l. \quad (14)$$

We set \( d = \left( \prod_{n=1}^{l-1} d_n \right)^r \). Using the inequality (14), we inductively get the bounds

$$ (d_l)^r \leq d p_{l+1}, $$

$$ (d_{l+1})^r \leq d^2 p_{l+1} p_{l+2}, $$

... and

$$ (d_{l+k})^r \leq d^{2^k} \left( \prod_{n=1}^{k} (p_{l+n})^{2^{k-n}} \right) p_{l+k+1}. $$

We now apply Lemma 4(a) to the last expression obtaining that it is bounded by

$$ d^{2^k} (p_{l+k+1})^{3(l+k)} p_{l+k+1} \leq d^{2^l} (p_{l+k+1})^2. $$

From these two inequalities one concludes that

$$ (d_m)^r \leq d^{2^m} (p_{m+1})^2 \quad \text{for all } m \geq l. $$

There exists \( m_0 \in \mathbb{N} \) such that \( d^{2^m} \leq p_m \) for all \( m \geq m_0 \). Therefore

$$ (d_m)^r \leq (p_{m+1})^3 \quad \text{for all } m > \max\{l, m_0\}, $$

and the condition (10) is proven. Hence \( \beta = \alpha^{-1} \in L. \) \( \square \)

Our next goal is to prove that all non–rational elements of the field \( L \) are Liouville numbers. We recall the corresponding definition.

**Definition 6.** A real number \( \alpha \) is a Liouville number if, to each natural number \( k \), there corresponds a rational number \( c_k/d_k \), with \( d_k > 1 \), such that

$$ 0 < \left| \alpha - \frac{c_k}{d_k} \right| < \left( \frac{1}{d_k} \right)^k. $$

**Theorem 7.** Every element in \( L \setminus \mathbb{Q} \) is a Liouville number.
Proof. Let \( x = \sum_{n=1}^{\infty} a_n/(b_n p_n) \in L \setminus \mathbb{Q} \), where infinitely many coefficients \( a_n \) are nonzero. There exists a natural number \( m_0 \) such that \( |a_n/b_n| \leq 1 \) for all \( n \geq m_0 \). Hence, if \( m \geq m_0 \) we obtain

\[
|x - \sum_{n=1}^{m-1} \frac{a_n}{b_n p_n}| = \left| \sum_{n=m}^{\infty} \frac{a_n}{b_n p_n} \right| \leq \sum_{n=m}^{\infty} \frac{1}{p_n} \leq \frac{2}{p_m}.
\]

Let \( d_m \) be the denominator of \( \sum_{n=1}^{m-1} a_n/(b_n p_n) \). This denominator satisfies the inequality \( d_m \leq \left| \prod_{n=1}^{m-1} b_n p_n \right| \). We suppose that \( d_m > 1 \) for \( m > m_0 \). We consider any natural number \( k \). As a result of Lemma 3, there exists a natural number \( m_k \geq m_0 \) such that, for all \( m \geq m_k \), the inequality

\[
\left| \prod_{n=1}^{m-1} \frac{b_n p_n}{p_n} \right|^k < \frac{p_m}{2}
\]

holds. Thus, for all \( m \geq m_k \), we have

\[
|x - \sum_{n=1}^{m-1} \frac{a_n}{b_n p_n}| \leq \frac{2}{p_m} < \frac{1}{\left| \prod_{n=1}^{m-1} b_n p_n \right|^k} \leq \frac{1}{(d_m)^k}.
\]

It follows that \( x \) is a Liouville number. \( \square \)

We recall the first practical criterion whereby transcendental numbers were constructed, formulated in 1844 by Liouville [6].

**Theorem 8.** If \( x \) is an algebraic real number of degree \( m > 1 \), then there exists a constant \( c = c(x) > 0 \) such that the inequality

\[
|\frac{x - p}{q}| > \frac{c}{q^m}
\]

holds for every pair of integers \( p, q \) with \( q \geq 1 \).

**Proof.** We refer, for instance, to [1, p. 1; 2, p. 125; 14, p. 23; 15, pp. 69-70]. \( \square \)

It is an easy consequence that every Liouville number is transcendental and the subsequent result:

**Corollary 9.** The field of rational numbers \( \mathbb{Q} \) is algebraically closed in \( L \).

At this point, it is useful to define a subset \( S \) of the field of real numbers which contains the subfield \( L \). We consider the set of series

\[
\sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} \quad \text{with} \quad a_n \in \mathbb{Z}, \quad b_n \in \mathbb{N}
\]
that satisfy \( \lim_{n \to \infty} \log(b_n)/\log(p_{n+1}) = 0 \), and \( |a_n/b_n| < \sqrt{p_n} \) for all \( n \), except possibly finitely many. Each of these series is absolutely convergent in \( \mathbb{R} \) endowed with its usual topology. Let \( S \) be the set of real numbers which are the limit values of these series. The elements of \( S \) have an almost unique representation with respect to these series.

**Lemma 10.** Let \( \alpha \) be an element of \( S \) which is represented by two series,

\[
\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} - \sum_{n=1}^{\infty} \frac{c_n}{d_n p_n}.
\]

Then, there exists an index \( n_1 \in \mathbb{N} \) such that \( a_n/b_n = c_n/d_n \) for all \( n \geq n_1 \).

**Proof.** We set \( e_n = a_n d_n - c_n b_n \) and \( f_n = b_n d_n \); thus

\[
0 = \sum_{n=1}^{\infty} \left( \frac{a_n}{b_n} \cdot \frac{c_n}{d_n} \right) \frac{1}{p_n} = \sum_{n=1}^{\infty} e_n \frac{1}{f_n p_n}.
\]

There exists a natural number \( n_0 \) such that \( |e_n/f_n| < 2\sqrt{p_n} \) for all \( n \geq n_0 \). Furthermore \( \lim_{n \to \infty} \log(f_n)/\log(p_{n+1}) = 0 \). Applying Lemma 3, we conclude that

\[
\lim_{n \to \infty} \frac{\log(\prod_{k=1}^{n} f_k p_k)}{\log(p_{n+1})} = 0.
\]

Hence, there exists a natural number \( n_1 \geq \max\{n_0, 6\} \) such that

\[
\left( \prod_{k=1}^{n} f_k p_k \right)^4 < p_{n+1} \quad \text{for all} \ n \geq n_1.
\]

We are going to show that \( \sum_{n=1}^{m} e_n/(f_n p_n) = 0 \) for all \( m \geq n_1 \); from which we conclude that \( e_n = 0 \) for all \( n > n_1 \) and the lemma is proven. We argue by way of contradiction. If there is \( m \geq n_1 \) such that \( \sum_{n=1}^{m} e_n/(f_n p_n) \neq 0 \), then we have

\[
\left| \sum_{n=1}^{m} \frac{e_n}{f_n p_n} \right| > \frac{1}{\prod_{n=1}^{m} f_n p_n} > \frac{1}{(p_{m+1})^{1/4}}.
\]

On the other hand

\[
\left| \sum_{n=m+1}^{\infty} \frac{e_n}{f_n p_n} \right| \leq \sum_{n=m+1}^{\infty} \frac{2}{\sqrt{p_n}} < 2 \frac{2}{\sqrt{p_{m+1}}} < \frac{1}{(p_{m+1})^{1/4}},
\]

and this is the contradiction we sought. \( \square \)

An immediate consequence of the previous lemma is that each number in \( L \) has an almost unique representation.
Corollary 11. Let $\alpha$ be an element of $L$ represented by two series

$$\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} = \sum_{n=1}^{\infty} \frac{c_n}{d_n p_n},$$

which fulfill the conditions (8). Then, there exists a natural number $n_0$ such that $a_n/b_n = c_n/d_n$ for all $n \geq n_0$.

Corollary 12. Let $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{(b_n p_n)}$ be an element of $L$ whose series has infinitely many coefficients $a_n \neq 0$, then $\alpha \notin \mathbb{Q}$.

Moreover, we have constructed a large set of real numbers which are not in $L$.

Corollary 13. Let $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{(b_n p_n)} \in S$ be such that $\lim_{n \to \infty} a_n/b_n \neq 0$. Then $\alpha \not\in L$.

For example, the real number $\sum_{n=1}^{\infty} \frac{1}{p_n} \notin L$.

3. A field topology on the field $L$

Our intention in this section is to define a field topology on the field $L$. For this purpose we give a fundamental system of neighborhoods at zero. For each natural number $m \geq 2$, we define the following subset:

$$W_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n p_n} \in L : (b_n)^m < p_{n+1}, \quad \left| \frac{a_n}{b_n} \right| < \frac{1}{2^m} \text{ for } n \geq m \right\}. \quad (15)$$

It is easy to verify that

$$W_{m+1} \subseteq W_m \quad \text{for all } m \geq 2,$$

$$W_m = -W_m \quad \text{for all } m \geq 2,$$

$$W_{2m} \pm W_{2m} \subseteq W_m \quad \text{for all } m \geq 2.$$

Therefore, the family $\mathcal{B} := \{W_m\}_{m\geq2}$ is a fundamental system of neighborhoods of zero for a group topology on $(L, +)$. We denote this topology by $\mathcal{T}$. The following result will be useful later for the study of the topology $\mathcal{T}$.

Lemma 14. For all $m \geq 6$, each element $\alpha \in W_m$ has a unique representation $\alpha = \sum_{n=m}^{\infty} \frac{a_n}{(b_n p_n)}$, with the coefficients $a_n$ and $b_n$ satisfying conditions (15).

Proof. It suffices to show the statement only for $W_6$. We reason by contradiction, suppose that an element $\alpha \in W_6$ can be written in two different ways:

$$\alpha = \sum_{n=6}^{\infty} \frac{a_n}{b_n p_n} = \sum_{n=6}^{\infty} \frac{c_n}{d_n p_n} \in W_6.$$
We set \( e_n = a_n d_n - c_n b_n \) and \( f_n = c_n d_n \), so that

\[
0 = \sum_{n=6}^{\infty} e_n \frac{1}{f_n p_n} \in W_6 - W_6 \subseteq W_3.
\]

Let \( l \geq 6 \) be the least index such that \( e_l \neq 0 \); we get

\[
-\frac{e_l}{f_l p_l} = \sum_{n=l+1}^{\infty} e_n \frac{1}{f_n p_n} \in W_3.
\]

In order to see that this is not possible, we bound both sides of the equality. On the one hand,

\[
\left| \frac{e_l}{f_l p_l} \right| \geq \frac{1}{f_l p_l} \geq \frac{1}{(p_{l+1})^{1/3} p_l} \geq \frac{1}{(p_{l+1})^{1/3+1/l}} \geq \frac{1}{(p_{l+1})^{1/3+1/6}} = \frac{1}{\sqrt{p_{l+1}}};
\]

on the other,

\[
\left| \sum_{n=l+1}^{\infty} e_n \frac{1}{f_n p_n} \right| \leq \frac{1}{2^{3/2}} \sum_{n=l+1}^{\infty} \frac{1}{p_n} \leq \frac{2}{2^{3/2}} \frac{2}{p_{l+1}} = \frac{1}{4 p_{l+1}}.
\]

It is easy to verify that

\[
\frac{1}{\sqrt{p_{l+1}}} > \frac{1}{4 p_{l+1}}
\]

for all \( l \geq 6 \), and we get a contradiction. \( \square \)

For two indices \( k > m \geq 6 \), the representation of an element of \( W_k \) coincides with its representation as an element of \( W_m \). In the sequel, given an element \( \sum_{n=m}^{\infty} a_n/(b_n p_n) \in W_n \), we will always assume that the conditions (15) are satisfied.

Now, we are going to show that \( \mathcal{B} \) is a neighborhood basis for a field topology on \( L \).

**Theorem 15.** The family \( \mathcal{B} = \{W_m\}_{m \geq 2} \) is a fundamental system of neighborhoods of zero for a field topology \( \mathcal{T} \) on \( L \).

**Proof.** We have already shown that \( (L,+,\mathcal{T}) \) is a topological group. It remains to verify properties (3)–(6). An immediate consequence from Lemma 14 is that

\[
\bigcap_{m \geq 2} W_m = \{0\},
\]

so \( \mathcal{T} \) is a Hausdorff group topology, i.e., property (5) is proven. Let us show property (3); for each \( W_l \in \mathcal{B} \) the inclusion \( W_{8l} W_{8l} \subseteq W_l \) holds. Set \( k = 8l \). We take two elements in \( W_k \),

\[
\alpha = \sum_{n=k}^{\infty} \frac{a_n}{b_n p_n}, \quad \beta = \sum_{n=k}^{\infty} \frac{c_n}{d_n p_n},
\]
and check that \( \alpha \beta \in W_I \). We carry out the product by grouping terms in the same way as we did in the proof of Theorem 5:

\[
\alpha \beta = \sum_{m=k}^{\infty} \left( \frac{1}{n} \sum_{n=k}^{\infty} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m p_m} + \sum_{m=k+1}^{\infty} \left( \frac{1}{n} \sum_{n=k}^{\infty} \frac{c_n}{d_n p_n} \right) \frac{a_m}{b_m p_m}.
\]

We denote these two summands by \( \delta \) and \( \gamma \), respectively. It suffices to check that \( \delta, \gamma \in W_{2I} \), since then

\[
\alpha \beta = \delta + \gamma \in W_{2I} + W_{2I} \subseteq W_I.
\]

We express the coefficient of each \( 1/p_m \) in \( \delta \) as a single fraction

\[
\delta = \sum_{m=k}^{\infty} \left( \frac{1}{n} \sum_{n=k}^{\infty} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m p_m} = \sum_{m=k}^{\infty} \left( \frac{1}{n} \sum_{n=k}^{\infty} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m p_m} \frac{1}{p_m}.
\]

and check the required conditions (15). First, we bound the denominator

\[
\left( \prod_{n=k}^{m} b_n p_n \right) d_m \leq \left( \prod_{n=k}^{m} (p_{n+1})^{1/k} p_n \right) (p_{m+1})^{1/k} < \left( \prod_{n=k}^{m} (p_n)^{1+1/k} \right) (p_{m+1})^{2/k};
\]

applying Lemma 4(b), we see that the last expression is less than or equal to

\[
(p_{m+1})^{(1+1/k)(m-2)} (p_{m+1})^{2/k} \leq (p_{m+1})^{4/k} = (p_{m+1})^{(2I)}.
\]

Now, we bound the whole coefficient of \( 1/p_m \):

\[
\left| \sum_{n=k}^{m} \frac{a_n}{b_n p_n} \right| \frac{c_m}{d_m} \leq \left( \sum_{n=k}^{m} \frac{1}{2^k p_n} \right) \frac{1}{2^k} = \left( \sum_{n=k}^{m} \frac{1}{p_n} \right) \frac{1}{2^{2k}} \leq \frac{1}{2^{2k}} \leq \frac{1}{2^{2I}}.
\]

We have proved that \( \delta \in W_{2I} \). Analogously \( \gamma \in W_{2I} \).

We now show property (4). It is easy that for all \( q \in Q \) and all \( W_m \in B \), there exists \( W_I \in B \) such that \( q W_I \subseteq W_m \). For an element \( x \in L \) and a neighborhood \( W_m \in B \), let us see that there exists \( W_I \) such that \( xW_I \subseteq W_m \). Let \( W_k \in B \) satisfying \( W_k W_k \subseteq W_{2m} \). We split \( x \) into the sum \( x = q + x_k \), where \( q \in Q \) and \( x_k \in W_k \). There exists \( W_l \in B \), with \( l \geq k \), such that \( q W_l \subseteq W_{2m} \). We conclude that

\[
xW_l \subseteq q W_l + x_k W_l \subseteq W_{2m} + W_k W_k \subseteq W_{2m} + W_{2m} \subseteq W_m.
\]

It remains to show property (6), i.e., continuity of the inversion. We are going to prove that

\[
(1 + W_{3k})^{-1} \subseteq 1 + W_k \text{ for all } k \geq 5.
\]
For each element \( \alpha = \sum_{n=3k}^{\infty} a_n/(b_n p_n) \in W_{3k} \), there will be an element \( \beta = \sum_{n=3k}^{\infty} c_n/(d_n p_n) \in W_k \) that fulfills \( (1 + \alpha)(1 + \beta) = 1 \). Therefore, \( (1 + \alpha)^{-1} = 1 + \beta \in 1 + W_k \).

We define the coefficients \( c_n \) and \( d_n \) inductively. In the product

\[
(1 + \alpha)(1 + \beta) = \left(1 + \sum_{n=3k}^{\infty} \frac{a_n}{b_n p_n}\right) \left(1 + \sum_{n=3k}^{\infty} \frac{c_n}{d_n p_n}\right) = 1,
\]

the coefficient of each \( 1/p_n \) must be zero for \( n \geq 3k \). First, we have

\[
\left(\frac{a_{3k}}{b_{3k}} + \frac{c_{3k}}{d_{3k}} + \frac{a_{3k} c_{3k}}{b_{3k} d_{3k}}\right) \frac{1}{p_{3k}} = 0,
\]

from which we obtain

\[
\frac{c_{3k}}{d_{3k}} = \frac{-a_{3k}}{b_{3k} \left(1 + \frac{a_{3k}}{b_{3k} p_{3k}}\right)} = \frac{-a_{3k} p_{3k}}{b_{3k} p_{3k} + a_{3k}}.
\]

Let us see that \( c_{3k}/d_{3k} \) satisfy the conditions (15) for \( \beta \in W_k \). Since \( 3k \geq 15 \) it is clear that

\[
\left|1 + \frac{a_{3k}}{b_{3k} p_{3k}}\right| \geq \frac{1}{2},
\]

then

\[
\left|\frac{c_{3k}}{d_{3k}}\right| \leq 2 \left|\frac{a_{3k}}{b_{3k}}\right| \leq \frac{2}{2^{3k}} < \frac{1}{2^k}.
\]

Moreover

\[
d_{3k} = \left|b_{3k} p_{3k} + a_{3k}\right| \leq 2b_{3k} p_{3k} \leq 2(p_{3k+1})^{2/(3k)} < (p_{3k+1})^{1/k}.
\]

Now, we assume that \( c_n \) and \( d_n \) are defined for \( n = 3k, \ldots, m - 1 \) satisfying the conditions (15). We are going to define \( c_m \) and \( d_m \) satisfying also the same conditions. The coefficient of \( 1/p_m \) in the product \( (1 + \alpha)(1 + \beta) \) is

\[
\left(1 + \sum_{n=3k}^{m} \frac{a_n}{b_n p_n}\right) \frac{c_m}{d_m} + \frac{a_m}{b_m} \left(1 + \sum_{n=3k}^{m-1} \frac{c_n}{d_n p_n}\right) = 0.
\]

Hence

\[
\frac{c_m}{d_m} = \frac{-a_m \left(1 + \sum_{n=3k}^{m-1} c_n/(d_n p_n)\right)}{b_m \left(1 + \sum_{n=3k}^{m-1} a_n/(b_n p_n)\right)} \leq \frac{-a_m \left(\prod_{n=3k}^{m-1} b_n d_n p_n\right) p_m}{b_m \left(\prod_{n=3k}^{m} a_n/(b_n p_n)\right) p_m}.
\]
Taking into account that \( m \geq n \geq 3k \geq 15 \), and that \(|a_n/b_n| \leq 1/2^{15}\), we see that

\[
\left| \sum_{n=3k}^{m} a_n b_n / b_n p_n \right| \leq \frac{1}{2^{14}},
\]

\[
1 + \frac{1}{2^{14}} \geq 1 + \left| \sum_{n=3k}^{m} a_n / b_n p_n \right| \geq 1 - \frac{1}{2^{14}}.
\]

By induction, \(|c_n/d_n| \leq 1/2^5\) for \( n = 3k, \ldots, m - 1 \). Whence,

\[
1 + \sum_{n=3k}^{m-1} c_n / d_n p_n \leq 1 + \frac{1}{2^4},
\]

from which we obtain the bound

\[
\left| \frac{c_m}{d_m} \right| \leq \left| \frac{a_n}{b_n} \right| \frac{1 + 1/2^4}{1 - 1/2^{14}} \leq \frac{1 + 1/2^4}{2^{3k}} \leq \frac{1}{2^k}.
\]

Now, we shall check the bound for the denominator \( d_m \):

\[
d_m \leq b_m \left( 1 + \frac{1}{2^{14}} \right) \left( \prod_{n=3k}^{m-1} b_n d_n p_n \right) p_m.
\]

Since \( b_n \leq (p_{n+1})^{1/(3k)} \) and, by induction, \( d_n \leq (p_{n+1})^{1/k} \), we gather that

\[
d_m \leq (p_{m+1})^{1/(3k)} \left( 1 + \frac{1}{2^{14}} \right) \left( \prod_{n=3k}^{m-1} (p_{n+1})^{1/(3k)} p_n \right) p_m
\]

\[
< \left( 1 + \frac{1}{2^{14}} \right) (p_{m+1})^{1/(3k)} \prod_{n=3k}^{m} (p_n)^{1/(3k + 4/(3k))}.
\]

Applying Lemma 4(b), we get

\[
d_m < \left( 1 + \frac{1}{2^{14}} \right) (p_{m+1})^{1/(3k)} (p_{m+1})^{(1+4/(3k))/(m-2)}.
\]

The exponent of \( p_{m+1} \) is bounded as follows:

\[
\frac{1}{3k} + \left( 1 + \frac{4}{3k} \right) \frac{1}{m-2} < \frac{5}{6k},
\]

from which we conclude that

\[
d_m < (p_{m+1})^{1/k}.
\]

The proof is now complete. \( \square \)
4. The field $L$ is a completion of $Q$

Our goal in this section is to prove that the topological field $(L, \mathcal{F})$ is a completion of the field of rational numbers endowed with $\mathcal{F} \cap Q$, which is the restriction of $\mathcal{F}$ to $Q$. The topology $\mathcal{F} \cap Q$ is also a field topology on $Q$. We shall use the same symbol $\mathcal{F}$ for this subspace topology $\mathcal{F} \cap Q$. A fundamental system of neighborhoods of zero for this topology is $\mathcal{B}_Q = \{ V_m \}_{m \geq 2}$, where $V_m = W_m \cap Q$. Taking into account Lemma 14, we conclude that

\[ (b_n)_{m < n} < p_n + 1, \quad \left| \frac{a_n}{b_n} \right| < \frac{1}{2^m} \quad \text{for all } n \geq m; \quad l \geq m \]

for all $m \geq 6$.

The topological field $(Q, \mathcal{F})$ is not complete; we are going to show that $(L, \mathcal{F})$ is the completion of $(Q, \mathcal{F})$. This completion is constructed in the usual manner, i.e., as the quotient ring of the ring of Cauchy sequences of $(Q, \mathcal{F})$ by the ideal of sequences whose limit is zero. We say that two Cauchy sequences in $(Q, \mathcal{F})$ are equivalent if they represent the same element in this quotient ring. In order to prove the subsequent theorem we need two technical results.

Lemma 16. Let $(x_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $(Q, \mathcal{F})$. There exist a sequence of rational integers $(a_n)$ and a sequence of natural numbers $(b_n)$ such that the sequence $(a_m)_{m \in \mathbb{N}}$, defined by

\[ a_m = \sum_{n=1}^{m} \frac{a_n}{b_n p_n}, \]

is a Cauchy sequence equivalent to $(x_m)_{m \in \mathbb{N}}$. In addition, $\sum_{n=1}^{m} a_n/(b_n p_n) \in V_{13}$ for all $m \geq 13$, i.e., the coefficients satisfy $(b_n)^{13} < p_{n+1}$ and $|a_n/b_n| \leq 1/2^{13}$ for all $n \geq 13$.

Proof. Let $(y_m)_{m \in \mathbb{N}}$ be a subsequence of $(x_m)_{m \in \mathbb{N}}$ that satisfies $y_{m_1} - y_{m_2} \in V_{m+1}$ for all $m_1, m_2 \geq m$. Therefore, $y_m - y_{12} \in V_{13}$ for all $m \geq 13$, and so

\[ y_m = y_{12} + \sum_{n=13}^{m} \frac{a_{n}}{b_{n} p_{n}}, \]

where $(b_{m})^{13} < p_{n+1}$, $|a_{n}/b_{m}| < 1/2^{13}$. We suppose that $\mu_m > m$, adding null terms in (17) if it is necessary. We consider an arbitrary element $y_k$ with $k > \mu_m$. We also have

\[ y_k - y_{12} = \sum_{n=13}^{k} \frac{a_k}{b_{k} p_{n}} \in V_{13}. \]
Since \( \gamma_k - \gamma_m \in V_{m+1} \), we can express it as
\[
\gamma_k - \gamma_m = \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n p_n} \in V_{m+1}.
\]
The coefficients \( u_n, v_n \) satisfy the conditions (16), in addition, we assume that \( \kappa > k \).
Therefore, we have the equalities
\[
\gamma_k = \gamma_m + \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n p_n} = \gamma_{12} + \sum_{n=13}^{m} \frac{a_{mn}}{b_{mn} p_n} + \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n p_n} = \gamma_{12} + \sum_{n=13}^{m} \frac{a_{mn}}{b_{mn} p_n} + \sum_{n=m+1}^{\mu_m} \left( \frac{a_{mn}}{b_{mn}} + \frac{u_n}{v_n} \right) \frac{1}{p_n} + \sum_{n=\mu_m+1}^{\kappa} \frac{u_n}{v_n p_n}.
\]
We note that
\[
\sum_{n=13}^{\mu_m} \frac{a_{mn}}{b_{mn} p_n} + \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n p_n} \in V_{13} + V_{m+1} \subseteq V_6,
\]
and recall (18). In view of the unique representation of the elements in \( V_6 \) (see Lemma 14), we deduce that \( \kappa = \mu_k \) and also the following equalities:
\[
\frac{a_{mn}}{b_{mn}} = \frac{a_{kn}}{b_{kn}} \quad \text{for } n \in \{13, \ldots, m\}, \quad (19)
\]
\[
\frac{a_{mn}}{b_{mn} p_n} + \frac{u_n}{v_n} = \frac{a_{kn}}{b_{kn}} \quad \text{for } n \in \{m + 1, \ldots, \mu_m\}, \quad (20)
\]
\[
\frac{u_n}{v_n} = \frac{a_{kn}}{b_{kn}} \quad \text{for } n \in \{\mu_m + 1, \ldots, \kappa\}. \quad (21)
\]
The equalities (19) also hold for each \( k > m \).
We define \( a_1/b_1 = \gamma_{12} \), \( a_n/b_n = 0 \) for \( n = 2, \ldots, 12 \), and \( a_n/b_n = a_{mn}/b_{mn} \) for \( 13 \leq n \leq m \). Consequently, we define \( \omega_m = \gamma_{12} \) for \( m \leq 12 \), and for each \( m > 12 \),
\[
\omega_m = \gamma_{12} + \sum_{n=13}^{m} \frac{a_n}{b_n p_n} = \gamma_{12} + \sum_{n=13}^{m} \frac{a_{mn}}{b_{mn} p_n}.
\]
In order to prove that \( (\omega_m)_{m \in \mathbb{N}} \) is a Cauchy sequence equivalent to \( (\gamma_m)_{m \in \mathbb{N}} \), we show that, for \( m \geq 13 \), the difference \( \gamma_k - \omega_k \in V_{m+1} \) for all \( k > \mu_m \) [\( \mu_m \) defined in (17)].
We have
\[
\omega_k = \gamma_{12} + \sum_{n=13}^{k} \frac{a_{kn}}{b_{kn} p_n}.
\]
Taking into account the equalities (19)–(21), we conclude that
\[
\gamma_k - \omega_k = \sum_{n=\mu_m+1}^{\kappa} \frac{u_n}{v_n p_n} \in V_{m+1} \quad \text{(for all } k > \mu_m)\).
\]
The lemma is proven. \( \square \)
Proposition 17. Let \((x_m)_{m \in \mathbb{N}}\) be a Cauchy sequence of \((\mathbb{Q}, \mathcal{F})\). There exist a sequence of rational integers \((a_n)\) and a sequence of natural numbers \((b_n)\) which satisfy
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = 0,
\]
such that the sequence \((\omega_m)_{m \in \mathbb{N}}\), defined by
\[
\omega_m = \sum_{n=1}^{m} \frac{a_n}{b_n p_n},
\]
is a Cauchy sequence equivalent to \((x_m)_{m \in \mathbb{N}}\).

Proof. We consider the Cauchy sequence \((\omega_m)_{m \in \mathbb{N}}\) defined in the previous lemma and show that it meets the requirements. For each neighborhood of zero \(V_k \in \mathcal{B}_\mathbb{Q}\) with \(k \geq 13\), there exists a natural number \(m_k \geq 13\) such that
\[
\omega_{m_2} - \omega_{m_1} = \sum_{m=m_1+1}^{m_2} \frac{a_n}{b_n p_n} \in V_k
\]
for all \(m_1, m_2 \geq m_k\). In addition, \((b_n)^{13} < p_{n+1}\) and \(|a_n/b_n| < 1/2^{13}\) for all \(n \geq m_k\). Considering the uniqueness of the representation of the elements of \(V_k\) for \(k \geq 6\) (see Lemma 14), we obtain that
\[
(b_n)^k < p_{n+1} \quad \text{and} \quad \left| \frac{a_n}{b_n} \right| < \frac{1}{2^k} \quad \text{for all } n > m_k.
\]
The lemma is proven. \(\square\)

Looking at the definition of \(L\) in (8), we conclude immediately the following result.

Theorem 18. The topological field \((L, \mathcal{F})\) is the completion of the topological field \((\mathbb{Q}, \mathcal{F})\).

Let us now show a concrete example. Let \(p \in \mathbb{N} \setminus \{0, 1\}\) and define the sequence \((p_n)_{n \in \mathbb{N}}\) by \(p_1 = 1\) and \(p_n = p^{n!}\) for all \(n \geq 2\). We obtain the field
\[
L = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n p^{n!}} : a_n \in \mathbb{Z}, b_n \in \mathbb{N}, \lim_{n \to \infty} \frac{\log(b_n)}{(n+1)!} = \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \right\}.
\]

5. Relationship of the sequence \((p_n)_{n \in \mathbb{N}}\) and the field \(L\)

In this section, we study how the field \(L\) depends on the sequence \((p_n)_{n \in \mathbb{N}}\) satisfying properties (7). We present some cases in which a field \(L_1\) is a proper subfield of another \(L_2\); for instance, when we take a subsequence of a given sequence \((p_n)_{n \in \mathbb{N}}\).
The field $L$ defined in (8) can also be expressed in the following equivalent way:

$$L = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d_n} : a_n \in \mathbb{Z}, d_n \in \mathbb{N}, \lim_{n \to \infty} \frac{\log(d_n)}{\log(p_{n+1})} = \lim_{n \to \infty} \frac{a_n}{d_n} p_n = 0 \right\}. \tag{22}$$

An easy consequence of this second description of $L$ is the following result:

**Lemma 19.** If $\left( p_n \right)_{n \in \mathbb{N}}$ and $\left( q_n \right)_{n \in \mathbb{N}}$ are two sequences of natural numbers which satisfy conditions (7), and such that $\left\{ p_n/q_n \right\}_{n \in \mathbb{N}}$ and $\left\{ q_n/p_n \right\}_{n \in \mathbb{N}}$ are bounded subsets of $\mathbb{Q}$, then the same field $L$ is obtained from both sequences.

In view of the definition (22) of the field $L$, we define a new basis of neighborhoods of zero $\mathcal{B}_d = \{ U_m \}_{m \geq 2}$ in $L$, where

$$U_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{d_n} \in L : (d_n)^m < p_{n+1}, \left| \frac{a_n}{d_n} \right| < \frac{1}{2^m p_n} \right\}$$

for all $n \geq m$.

We compare this basis $\mathcal{B}_d$ with the basis $\mathcal{B} = \{ W_m \}_{m \geq 2}$ defined in Section 3.

**Lemma 20.** The two basis $\mathcal{B}$ and $\mathcal{B}_d$ define the same field topology $\mathcal{T}$ on $L$.

**Proof.** It is easy to check that $U_m \subseteq W_m$ and $W_{2m} \subseteq U_m$ for all $m \geq 2$. \hfill $\square$

With this new basis $\mathcal{B}_d$, it is straightforward to prove the following result:

**Lemma 21.** If $\left( p_n \right)_{n \in \mathbb{N}}$ and $\left( q_n \right)_{n \in \mathbb{N}}$ are two sequences of natural numbers which satisfy conditions (7), and such that $\left\{ p_n/q_n \right\}_{n \in \mathbb{N}}$ and $\left\{ q_n/p_n \right\}_{n \in \mathbb{N}}$ are bounded subsets of $\mathbb{Q}$, then the same field topology $\mathcal{T}$ on the field $L$ is obtained from both sequences.

In some cases, our field $L$ is a subfield of another field $L'$ corresponding to a different sequence $\left( p'_n \right)_{n \in \mathbb{N}}$; but it is not a topological subfield of $(L', \mathcal{T}')$. The next results are examples of this situation.

**Proposition 22.** Let $\left( p_n \right)_{n \in \mathbb{N}}$ and $\left( q_n \right)_{n \in \mathbb{N}}$ be two sequences of natural numbers which satisfy the conditions (7), from which we define, respectively, the topological fields $(L_p, \mathcal{T}_p)$ and $(L_q, \mathcal{T}_q)$. If the subsets $\left\{ q_n/p_n \right\}_{n \in \mathbb{N}}$ and $\left\{ \log(p_n)/\log(q_n) \right\}_{n \geq 2}$ of $\mathbb{R}$ are bounded, then $L_p$ is a subfield of $L_q$, and the topology $\mathcal{T}_p$ is finer than $\mathcal{T}_q$ restricted to $L_p$.

**Proof.** First, we prove the inclusion $L_p \subseteq L_q$. Let $x = \sum_{n=1}^{\infty} a_n/(b_n p_n) \in L_p$. We rewrite this element as follows:

$$x = \sum_{n=1}^{\infty} \frac{a_n q_n}{b_n p_n q_n}.$$
We check that \( \alpha \) fulfills the conditions (8) required to be an element of \( L_q \). Since 
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = 0, \]
we have that 
\[ \lim_{n \to \infty} \frac{(a_n q_n)}{(b_n p_n)} = 0. \]
We look at the other limit:
\[ \lim_{n \to \infty} \frac{\log(b_n p_n)}{\log(q_{n+1})} = \lim_{n \to \infty} \frac{\log(p_{n+1})}{\log(q_{n+1})} \cdot \left( \frac{\log(b_n)}{\log(p_{n+1})} + \frac{\log(p_n)}{\log(p_{n+1})} \right). \]

In the left-hand expression we have the following: inside the brackets two sequences whose limit is zero, outside the brackets a bounded sequence. Consequently, the whole limit is zero.

Second, we show that the topology \( \mathcal{F}_p \) is finer than \( \mathcal{F}_q \cap L_p \). Let \( \{U^p_m\}_{m \geq 2} \) and \( \{U^q_m\}_{m \geq 2} \) two fundamental system of neighborhoods of zero for the topologies \( \mathcal{F}_p \) and \( \mathcal{F}_q \), respectively. The neighborhoods are
\[
U^p_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{d_n} \in L_p : (d_n)^m < p_{n+1}, \quad \left| \frac{a_n}{d_n} \right| < \frac{1}{2^m p_n} \text{ for all } n \geq m \right\},
\]
\[
U^q_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{d_n} \in L_q : (d_n)^m < q_{n+1}, \quad \left| \frac{a_n}{d_n} \right| < \frac{1}{2^m q_n} \text{ for all } n \geq m \right\}.
\]

There exist two real numbers \( r > 0 \) and \( s > 0 \) such that
\[
\frac{q_n}{p_n} < r, \quad \frac{\log p_n}{\log q_n} < s, \quad p_n < (q_n)^s, \quad \text{for all } n \in \mathbb{N}.
\]

Let \( k \) be a natural number that fulfills \( r < 2^k \) and \( s < k \). Let us show that \( U^p_{km} \subseteq U^q_{m} \cap L_p \). Let
\[
\alpha = \sum_{n=km}^{\infty} \frac{a_n}{d_n} \in U^p_{km} \subseteq L_p.
\]
The terms in the series defining \( \alpha \) satisfy the inequalities
\[
(d_n)^m < p_{n+1} \quad \text{and} \quad \left| \frac{a_n}{d_n} \right| < \frac{1}{2^m p_n},
\]
from which we conclude that
\[
(d_n)^m < (p_{n+1})^{1/k} < (q_{n+1})^{1/k} < q_{n+1}, \quad \left| \frac{a_n}{d_n} \right| < \frac{1}{2^m p_n} < \frac{r}{2^m q_n} < \frac{1}{2^m q_n}.
\]
Therefore \( \alpha \in U^q_m \cap L_p \). \( \square \)

If \( \{ p_n/q_n \} \) is not a bounded subset of \( \mathbb{R} \), then \( L_p \) is a proper subfield of \( L_q \), and the topology \( \mathcal{F}_p \) is strictly finer than \( \mathcal{F}_q \cap L_p \).
We show an example of a pair of sequences that satisfy the hypotheses of Proposition 22.

\[ p_1 = q_1 = 1, \quad p_n = 3^n, \quad q_n = 2^n \quad \text{for all } n \geq 2. \]

**Proposition 23.** Let \((p_n)_{n \in \mathbb{N}}\) be a sequence of natural numbers which satisfies the conditions (7). Let \((p_{n_k})_{k \in \mathbb{N}}\) be a subsequence which satisfies \(p_{n_k} = p_1 = 1\) and \(n_k \geq 3k\) for \(k\) big enough. The respective topological fields are \((L_1, \mathcal{T}_1)\) and \((L_2, \mathcal{T}_2)\). Then \(L_1 \subseteq L_2\) and \(\mathcal{T}_1\) is finer than the subspace topology \(\mathcal{T}_2 \cap L_1\).

**Proof.** First, we show that \(L_1 \subseteq L_2\). Let \(x = \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} \in L_1\). We rewrite this element grouping the terms in the series as follows:

\[
x = \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} = \sum_{k=1}^{\infty} \left( \sum_{n=n_k}^{n_k+1-1} \frac{a_n}{b_n p_n} \right) = \sum_{k=1}^{\infty} \left( \sum_{n=n_k}^{n_k+1-1} \frac{a_n}{b_n p_n} \right) \cdot \frac{1}{p_{n_k}} := \sum_{k=1}^{\infty} \frac{c_k}{d_k} \frac{1}{p_{n_k}}.
\]

We see that the coefficients \(c_k/d_k\) satisfy the conditions (8) required for \(x \in L_2\). We consider the denominator

\[
d_k = \prod_{n=n_k}^{n_k+1-1} b_n p_n.
\]

Applying Lemma 3, we conclude that

\[
\lim_{k \to \infty} \frac{\log(d_k)}{\log(p_{n_k+1})} = 0.
\]

We check that \(\lim_{k \to \infty} c_k/d_k = 0\). If \(n_{k+1} > n_k + 1\), we split this coefficient into two summands

\[
c_k = a_n \frac{1}{b_n} \sum_{n=n_k+1}^{n_{k+1}-1} \frac{a_n}{b_n p_n} = \frac{a_n}{b_n} \sum_{n=n_k+1}^{n_{k+1}-1} \frac{p_n}{b_n p_n} = \frac{1}{d_k} \sum_{n=n_k+1}^{n_{k+1}-1} \frac{1}{p_n}.
\]

By hypothesis, we have \(\lim_{k \to \infty} a_n/b_n = 0\). There exists a natural number \(k_0\) such that \(|a_n/b_n| < 1\) for all \(n \geq n_0\). Hence, for all \(k \geq k_0\), the absolute value of the second summand in (23) is bounded by the expression

\[
\sum_{n=n_k+1}^{n_{k+1}-1} \frac{p_n}{p_n} < \sum_{n=n_k+1}^{\infty} \frac{1}{\sqrt{p_n}}.
\]

This expression converges to zero for \(k \to \infty\). The inclusion \(L_1 \subseteq L_2\) is proven.

Second, we show that \(\mathcal{T}_1\) is finer than \(\mathcal{T}_2 \cap L_1\). A neighborhood basis of zero for the topology \(\mathcal{T}_2\) is \(\{W^2_n\}_{n \geq 2}\), where

\[
W^2_n = \left\{ \sum_{k=t}^{\infty} \frac{a_n}{p_n} \frac{1}{p_n} : \left| \frac{a_n}{b_n} \right| < \frac{1}{2^t}, \ (b_n)' < p_{n_k+1} \text{ for all } k \geq t \right\}.
\]
And a neighborhood basis of zero for the topology \( T_1 \) is \( \{ W_m \}_{m \geq 2} \); the neighborhoods are taken according to (15).

For each index \( k \) it holds that

\[
\sum_{n=n_k + 1}^{\infty} \frac{p_n}{p_n} < 1. \tag{24}
\]

There exists \( l \in \mathbb{N} \) such that \( n_k \geq 3k \) for all \( k \geq l \). For each index \( n_r \) with \( r \geq l \), let \( t \) be the least natural number that \( 3t \geq n_r \), which implies that \( t \geq r \). We are going to show that

\[
W^1_{3t} \subseteq W^2_{n_r} \cap L_1.
\]

For any element \( x = \sum_{n=3t}^{\infty} a_n/(b_n p_n) \in W^1_{3t} \), we group the terms of the series as follows:

\[
x = \sum_{k=r}^{\infty} \left( \sum_{n=n_k}^{n_{k+1}-1} \frac{a_n}{b_n p_n} \right) \frac{1}{p_n} = \sum_{k=r}^{\infty} \frac{c_k}{d_k} \frac{1}{p_n}.
\]

For those indices \( n \) smaller than \( 3t \), we define \( a_n = 0 \), \( b_n = 1 \). We verify that the coefficients \( c_k, d_k \) satisfy the conditions required for \( x \in W^2_{n_r} \). We bound the denominator:

\[
d_k \leq \prod_{n=1}^{n_{k+1}-1} b_n p_n \leq \left( \prod_{n=1}^{n_{k+1} - 2} (p_n)^{1+1/(3t)} \right) (p_{n_{k+1}})^{1/(3t)}.
\]

Applying Lemma 4, we bound this last expression by

\[
(p_{n_{k+1}})^{(1+1/(3t))((n_{k+1}-2)+1/(3t))} < (p_{n_{k+1}})^{1/t}.
\]

Therefore, \( (d_k)^{1/t} < p_{n_{k+1}} \) for all \( k \geq r \). To bound the fraction \( c_k/d_k \), we split it in the same way as we did in (23). Taking into account (24) and the fact that \( |a_n/b_n| < 1/2^{3t} \) for all \( n \geq 3t \), we get the bound

\[
\left| \frac{a_n}{b_n} \right| \leq \frac{1}{2^{3t}} + \frac{1}{2^{3t}} < \frac{1}{2},
\]

We have proved that \( x \in W^2_{n_r} \). \( \square \)

6. Other similar subfields of \( \mathbb{R} \)

In the sequel, we introduce other subfields of the field of real numbers obtained with some slight changes in the definition (8) of the field \( L \). These fields have the same properties of the field \( L \) constructed above: they are completions of the field of rational numbers \( \mathbb{Q} \) with respect to field topologies finer than the usual one, and they consist solely of rational numbers and Liouville numbers. We shall skip most of the proofs, which are similar to the corresponding ones in the previous sections.
In what follows, \((f_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence of natural numbers such that
\[
\lim_{n \to \infty} \frac{\log(f_n)}{\log(p_{n+1})} = 0.
\]

We denote a sequence of these by \(f\), and the family of all these sequences by \(\mathcal{F}\). We select one of these sequences and define the following subset of the field of real numbers:
\[
L_f = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} : a_n \in \mathbb{Z}, b_n \in \mathbb{N}, \lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = \lim_{n \to \infty} \frac{a_n}{b_n} f_n = 0 \right\}.
\] (25)

**Theorem 24.** The subset \(L_f\), with the sum and product inherited from \(\mathbb{R}\), is a subfield which consists exclusively of rational numbers and Liouville numbers.

For each sequence \(f = (f_n)_{n \in \mathbb{N}}\), the strict inclusion \(L_f \subset L\) holds.

There is a field topology \(\mathcal{T}_f\) on the field \(L_f\). A fundamental system of neighborhoods of zero is \(\mathcal{B}_f = \{S_m\}_{m \geq 2}\), where
\[
S_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n p_n} \in L_f : (b_n)^m < p_{n+1}, \left| \frac{a_n}{b_n} \right| < \frac{1}{f_n 2^m} \text{ for all } n \geq m \right\}.
\]

Each topological field \((L_f, \mathcal{T}_f)\) is the completion of the field of rational numbers \(\mathbb{Q}\) endowed with the subspace topology \(\mathcal{T}_f \cap \mathbb{Q}\).

**Theorem 25.** For a fixed sequence \((p_n)_{n \in \mathbb{N}}\) satisfying the conditions (7), the intersection of all the fields in the family \(\{L_f : f \in \mathcal{F}\}\) is the field of rational numbers.

**Proof.** Since every field \(L_f\) is a subfield of \(L\), it suffices to show that, for each \(\alpha \in L \setminus \mathbb{Q}\), there exists a field \(L_f\) such that \(\alpha \notin L_f\). Let \(\alpha = \sum_{n=1}^{\infty} a_n/(b_n p_n) \in L \setminus \mathbb{Q}\). We define \(f_n = n + \max\{(b_k)^2 : k = 1, \ldots, n\}\). The sequence \(f := (f_n)_{n \in \mathbb{N}}\) is strictly increasing and satisfies
\[
\lim_{n \to \infty} \frac{\log(f_n)}{\log(p_{n+1})} = 0.
\]
Thus \(f \in \mathcal{F}\). Besides, it is clear that \(\lim_{n \to \infty}(a_n/b_n) f_n \neq 0\). Therefore, \(\alpha \notin L_f\). \(\Box\)

Our next goal is to define another family of fields similar to the previous one. We need a strictly increasing sequence of natural numbers \((g_n)_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} \log(g_n)/\log(p_n) = 0\). We denote such a sequence by \(g\), and the family of all of them by \(\mathcal{G}\). For a fixed sequence \(g\), we define the following subset of \(\mathbb{R}\):
\[
L_g = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} : a_n \in \mathbb{Z}, b_n \in \mathbb{N}, \lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = \lim_{n \to \infty} \frac{a_n}{b_n g_n} = 0 \right\}.
\] (26)
This subset \( L_g \), with the sum and product inherited from \( \mathbb{R} \), is also a subfield which contains solely rational numbers and Liouville numbers. The strict inclusion \( L \subset L_g \) holds for each sequence \( (g_n)_{n \in \mathbb{N}} \).

The field \( L_g \) has a field topology \( \mathcal{T}_g \). A neighborhood basis of zero for this topology is \( \mathcal{B}_0 = \{ V_m \}_{m \geq 2} \), where

\[
V_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n p_n} \in L_g : (b_n)^m < p_{n+1}, \left| \frac{a_n}{b_n} \right| < \frac{g_n}{2^m} \text{ for all } n \geq m \right\}.
\]

Each topological field \( (L_g, \mathcal{T}_g) \) is also the completion of the rational numbers field \( \mathbb{Q} \) with respect to the subspace topology \( \mathcal{T}_g \cap \mathbb{Q} \).

7. The subfield \( \mathcal{L} \subset \mathbb{R} \)

We introduce another subfield of \( \mathbb{R} \). This time, we make in the definition (8) a change with more consequences. We use again a sequence of natural numbers \( (p_n)_{n \in \mathbb{N}} \) which satisfies properties (7). We set \( \log(0) = -\infty \) and define the following subset of \( \mathbb{R} \):

\[
\mathcal{L} = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n p_n} : a_n \in \mathbb{Z}, b_n \in \mathbb{N}, \lim_{n \to \infty} \frac{\log(b_n)}{\log(p_{n+1})} = 0, \limsup_{n \to \infty} \frac{\log |a_n|}{\log(p_n)} \leq 0 \right\}.
\]  

(27)

Notice that the condition

\[
\limsup_{n \to \infty} \frac{\log |a_n|}{\log(p_n)} \leq 0
\]

(28)

is equivalent to the following one: for all \( s \in \mathbb{N} \), there exists \( n_s \in \mathbb{N} \) such that

\[
\left| \frac{a_n}{b_n} \right|^s < p_n \text{ for all } n \geq n_s.
\]

It is easy to check that each series that represent an element in \( \mathcal{L} \) is absolutely convergent in \( \mathbb{R} \) with its usual topology.

**Theorem 26.** The subset \( \mathcal{L} \), with the sum and product inherited from \( \mathbb{R} \), is a subfield of \( \mathbb{R} \).

**Proof.** The fact that the sum and product are inner operations in \( \mathcal{L} \) is shown in a similar way as we did in the proof of Theorem 5. Nevertheless, the proof that \( \mathcal{L} \) contains the inverse of all its nonzero elements is somehow different.

Let \( \alpha = \sum_{n=1}^{\infty} \frac{a_n}{(b_n p_n)} \in \mathcal{L} \setminus \mathbb{Q} \); we construct its inverse \( \beta = \sum_{n=1}^{\infty} \frac{c_n}{(d_n p_n)} \) by defining inductively the coefficients \( c_n \) and \( d_n \). We assume that \( a_1 \neq 0 \) and \( \sum_{n=1}^{m} \frac{a_n}{(b_n p_n)} \neq 0 \) for all \( m \in \mathbb{N} \). We define \( d_1 = |a_1| \) and \( c_1 = b_1 a_1 / |a_1| \). Assuming we have computed \( c_n \) and \( d_n \) for \( n = 1, \ldots, m-1 \), let us determine \( c_m \) and \( d_m \). The product \( \alpha \beta \)
can be expressed as follows:

\[ \alpha \beta = \frac{a_1}{b_1} \frac{c_1}{d_1} + \sum_{m=2}^{\infty} \left( \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m} + \frac{a_m}{b_m} \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right) \right) \frac{1}{p_m}. \]

If the coefficient of \( \frac{1}{p_m} \) in \( \alpha \beta \) is zero for all \( m \geq 2 \), then \( \alpha \beta \) is the inverse of \( \alpha \). Hence, we require

\[ \left( \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right) \frac{c_m}{d_m} + \frac{a_m}{b_m} \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right) = 0 \]

for all \( m \geq 2 \), and we define

\[ \frac{c_m}{d_m} = \frac{-a_m \left( \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right)}{b_m \left( \sum_{n=1}^{m-1} \frac{a_n}{d_n p_n} \right)}. \tag{29} \]

We show that the coefficients \( \frac{c_m}{d_m} \) fulfill the condition (28). There exists \( r \in \mathbb{R}, r > 0 \), such that

\[ r < \left| \sum_{n=1}^{m} \frac{a_n}{b_n p_n} \right| \quad \text{for all } m \in \mathbb{N}. \tag{30} \]

There exists \( m_0 \geq 5 \) such that

\[ \frac{|a_m|}{b_m} < (p_m)^{1/8}, \quad \frac{1}{r} < (p_m)^{1/8}, \tag{31} \]

for all \( m \geq m_0 \). Consequently, taking into account (29)–(31), we get that

\[ \left| \frac{c_m}{d_m} \right| < (p_m)^{1/4} \left| \sum_{n=1}^{m-1} \frac{c_n}{d_n p_n} \right| \quad \text{for all } m \geq m_0. \tag{32} \]

We denote

\[ R = \sum_{n=1}^{m_0-1} \left| \frac{c_n}{d_n p_n} \right|. \]

Applying (32), we get inductively the following inequalities:

\[ \left| \frac{c_{m_0}}{d_{m_0}} \right| < (p_{m_0})^{1/4} R, \]

\[ \left| \frac{c_{m_0+1}}{d_{m_0+1}} \right| < (p_{m_0+1})^{1/4} \left( R + \frac{(p_{m_0})^{1/4} R}{p_{m_0}} \right) < (p_{m_0+1})^{1/4} 2R, \]

\[ \ldots \]

\[ \left| \frac{c_{m_0+k}}{d_{m_0+k}} \right| < (p_{m_0+k})^{1/4} \left( R + \frac{(p_{m_0})^{1/4} R}{p_{m_0}} + \frac{(p_{m_0+1})^{1/4} 2R}{p_{m_0+1}} + \ldots \right) \]

\[ < (p_{m_0+k})^{1/4} (k + 1) R. \tag{33} \]
There exists \( k_0 \in \mathbb{N} \) such that \((k+1)R < (p_{m_0+k})^{1/4}\) for all \( k \geq k_0 \). Therefore, from the inequality (33) we conclude that

\[
\left| \frac{c_n}{d_n} \right| < (p_n)^{1/2} \quad \text{for all } n > m_0 + k_0. \]

Thus, the series \( \sum_{n=1}^{\infty} c_n/(d_n p_n) \) is absolutely convergent in \( \mathbb{R} \) with its usual topology. There exists \( P \in \mathbb{N} \) such that

\[
\left| \frac{c_n}{d_n} \right| < P \quad \text{for all } m \in \mathbb{N}. \]

Using again (29) and (30), we gather that

\[
\left| \frac{c_n}{d_n} \right| \leq \left| \frac{a_n}{b_n} \right| \frac{P}{r} \quad \text{for all } n \in \mathbb{N}. \]

From this inequality we deduce that

\[
\limsup \frac{\log |c_n/d_n|}{\log(p_n)} \leq 0. \]

One can prove that the denominators \( d_m \) satisfy properties (9) or (10) in the same way as we did in the proof of Theorem 5. Thus, \( \beta = x^{-1} \in L \). \( \Box \)

The field \( L \) also satisfies that its elements are solely rational numbers and Liouville numbers. For a fixed sequence \( (p_n)_{n \in \mathbb{N}} \), this field \( L \) is the union of all the fields \( L_n \) defined in the previous section, i.e.,

\[ L = \bigcup_{n \in \mathbb{N}} L_n. \]

The field \( L \) has a field topology \( \widetilde{\mathcal{F}} \). A basis of neighborhoods of zero for this topology is \( \{U_m\}_{m \geq 2} \), where

\[
U_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n p_n} \in L : \left| \frac{a_n}{b_n} \right|^m < p_n, \ (b_n)^m < p_{n+1} \text{ for all } n \geq m \right\}. \quad (34)
\]

The topological field \( (L, \widetilde{\mathcal{F}}) \) is also the completion of \( \mathbb{Q} \) with respect to the subspace topology \( \widetilde{\mathcal{F}} \cap \mathbb{Q} \).

The field \( L \) can be also expressed in the following form:

\[
L = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d_n} : a_n \in \mathbb{Z}, \ d_n \in \mathbb{N}, \ \lim_{n \to \infty} \frac{\log(d_n)}{\log(p_{n+1})} = 0, \ \limsup_{n \to \infty} \frac{\log |a_n/d_n|}{\log(p_n)} \leq -1 \right\}. \quad (35)
\]

An immediate consequence of this last definition of \( L \) is the following result:
Lemma 27. Let \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) be two sequences of natural numbers which satisfy the conditions (7). If \(\lim_{n \to \infty} \log(p_n)/\log(q_n) = 1\), then we get the same field \(\mathcal{L}\) from both sequences.

8. The subfield \(\mathcal{L} \subset \mathbb{R}\)

Finally, we exhibit another subfield of \(\mathbb{R}\) with similar properties. We fix a sequence of natural numbers \((p_n)_{n \in \mathbb{N}}\) which fulfills the conditions (7). We define the following subfield of \(\mathbb{R}\):

\[
\mathcal{L} = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d_n} : a_n \in \mathbb{Z}, \quad \frac{\log(d_n)}{\log(p_{n+1})} = 0, \quad \lim_{n \to \infty} \frac{\log|a_n/d_n|}{\log(p_n)} = -\infty \right\}.
\]

Now, the field topology defined in \(\mathcal{L}\) has the basis of neighborhoods of zero \(\{S_m\}_{m \geq 2}\), where

\[
S_m = \left\{ \sum_{n \geq m} \frac{a_n}{d_n} \in \mathcal{L} : (d_n)^m < p_{n+1}, \quad \left| \frac{a_n}{d_n} \right| < \left( \frac{1}{p_n} \right)^m \quad \text{for all } n \geq m \right\}.
\]

The topological field \(\mathcal{L}\) is also a completion of \(\mathbb{Q}\), and consists exclusively of rational numbers and Liouville numbers.

Notice that, in the definition (36) of the field \(\mathcal{L}\), the sequence \((d_n)_{n \in \mathbb{N}}\) is quite squeezed by the sequence \((p_n)_{n \in \mathbb{N}}\), since

\[
\lim_{n \to \infty} \frac{\log(d_n)}{\log(p_{n+1})} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\log(d_n)}{\log(p_n)} = +\infty.
\]

These limits are equivalent to the following condition: for every \(s \in \mathbb{N}\) there exists \(n_s \in \mathbb{N}\) such that \((d_n)^s < p_{n+1}\) and \(d_s > (p_n)^s\) for all \(n \geq n_s\).

An immediate consequence of the definition (36) of the field \(\mathcal{L}\) is the next result:

Lemma 28. Let \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) two sequences of natural numbers which satisfy the conditions (7). If

\[
\left\{ \frac{\log p_n}{\log q_n} \right\}_{n \in \mathbb{N}} \quad \text{and} \quad \left\{ \frac{\log q_n}{\log p_n} \right\}_{n \in \mathbb{N}}
\]

are bounded subsets of \(\mathbb{R}\), then we get the same field \(\mathcal{L}\) with both sequences.

We compare all the fields introduced throughout this article. We fix a sequence \((p_n)_{n \in \mathbb{N}}\) which satisfies the conditions (7). We have the following strict inclusions of fields:

\[
\mathcal{L} \subset L \subset L_\phi \subset \mathcal{L} \subset \mathbb{R}, \quad g \in \mathcal{G};
\]

\[
L_f, \quad f \in \mathcal{F}.
\]
We consider each field with the field topology that we have constructed for it, and \( \mathbb{R} \) with its usual topology. For two fields \( L_1 \) and \( L_2 \) in the above diagram with their respective topologies \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), if \( L_1 \subset L_2 \), then \( \mathcal{T}_1 \) is strictly finer than the subspace topology \( \mathcal{T}_2 \cap L_1 \).

We consider now a particular example. We define the sequence \( (p_n)_{n \in \mathbb{N}} \) to be \( p_1 = 1 \) and \( p_n = p^{n!} \) for all \( n \geq 2 \), where \( p \in \mathbb{N} \setminus \{0, 1\} \). Then, the definition (35) of the field \( \mathcal{L} \) is equivalent to the following one:

\[
\mathcal{L}_p = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d_n} : a_n \in \mathbb{Z}, \lim_{n \to \infty} \frac{\log(d_n)}{(n+1)!} = 0, \limsup_{n \to \infty} \frac{\log|a_n/d_n|}{n!} \leq -\log(p) \right\}.
\]

The intersection of these fields is the field \( \mathbb{R} \):

\[
\mathbb{R} = \bigcap_{p \in \mathbb{N} \setminus \{0, 1\}} \mathcal{L}_p
\]

\[
= \left\{ \sum_{n=1}^{\infty} \frac{a_n}{d_n} : a_n \in \mathbb{Z}, \lim_{n \to \infty} \frac{\log(d_n)}{(n+1)!} = 0, \lim_{n \to \infty} \frac{\log|a_n/d_n|}{n!} = -\infty \right\}.
\]

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References