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AN EXPLICIT FINITE ELEMENT METHOD FOR CONVECTION–DIFFUSION EQUATIONS USING RATIONAL BASIS FUNCTIONS

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Abstract—An explicit Galerkin method is formulated by using rational basis functions. The characteristics of the rational difference scheme are investigated with regard to consistency, stability and numerical convergence of the method. Numerical results are also presented.

INTRODUCTION

It is well known that classical Galerkin methods applied to convection–diffusion equations, where convection dominates the diffusion, generally yield non-physical oscillations when the cell Peclet number is large. The disturbances in the solution are rectified by using the technique of upwinding [1–3]. In the Petrov–Galerkin method test functions which are biased in the upstream direction are implemented. Therefore, the eigenvalues of the discretized scheme have reduced imaginary components, thereby increasing dissipative decay and reducing oscillations of the solution [1].

The one-dimensional rational basis functions developed by van Niekerk and van Niekerk [4, 5], are naturally biased in the upstream direction and therefore simulates the effect of upwinding. In this paper an explicit numerical scheme with rational basis functions are devised and the scheme is analysed to demonstrate its effectiveness. Higher order rational basis functions are also constructed and it is shown that in the numerical scheme the artificial diffusion coefficient becomes smaller with increasing order, thereby improving the consistency, stability and convergence of the scheme.

RATIONAL BASIS FUNCTION

The (S, T) rational approximant over the interval $[0, h]$ is defined by

$$R_{S,T}(x) = a + \frac{c + c_1 x + \cdots + c_S x^S}{1 + \frac{x}{h} + \cdots + \left(\frac{x}{h}\right)^T}, \quad x \in [0, h].$$

Particularly, consider the $(0, T)$ rational approximants, namely

$$\phi_1(x) = a_1 + \frac{b_1}{1 + \frac{x}{h} + \cdots + \left(\frac{x}{h}\right)^T}$$

and

$$\phi_0(x) = a_0 + \frac{b_0}{1 + \frac{x}{h} + \cdots + \left(\frac{x}{h}\right)^T},$$

where $x \in [0, h]$. In order to construct a rational basis function the constants a_0, b_0, a_1 and b_1 are determined from the interpolation constraints

$$\phi_1(0) = 0,$$

$$\phi_1(h) = 1,$$

$$\phi_0(0) = 1$$

and

$$\phi_0(h) = 0.$$

From these rational functions a $(0, T)$ rational basis function at an interior node x_i with local support on $[x_{i-1}, x_{i-1} + 2h]$ is defined by

$$\psi_i(x) = \begin{cases} \phi_1(x - x_{i-1}), & x \in [x_{i-1}, x_i], \\ \phi_0(x - x_i), & x \in [x_i, x_{i+1}]. \end{cases} \tag{1}$$

Thus, a $(0, 1)$ rational basis function at node x_i is given by definition (1), where

$$\phi_1(x) = \frac{2x/h}{1 + \dfrac{x}{h}}, \quad 0 \leqslant x \leqslant h,$$

and

$$\phi_0(x) = 1 - \phi_1(x), \quad 0 \leqslant x \leqslant h.$$

Similarly, for $(0, 2)$ and $(0, 3)$ rational basis functions

$$\phi_1(x) = \frac{3}{2} - \frac{3/2}{1 + \dfrac{x}{h} + \left(\dfrac{x}{h}\right)^2}$$

and

$$\phi_1(x) = \frac{4}{3} - \frac{4/3}{1 + \dfrac{x}{h} + \left(\dfrac{x}{h}\right)^2 + \left(\dfrac{x}{h}\right)^3},$$

respectively, where $x \in [0, h]$. Note that all cases satisfy the relation

$$\phi_0(x) + \phi_1(x) = 1, \quad x \in [0, h].$$

The $(0, T)$ rational basis functions ψ_i remain defined on the interval $[x_{i-1}, x_{i-1} + 2h]$. In this manner the danger of introducing real singularities is avoided. A further advantage of the method of construction is that all $(0, T)$ rational basis functions couple three adjoining nodes leaving the banded structure of the matrices in a Galerkin method intact.

A $(0, T)$ rational approximation of the function u over the interval $[x_i, x_{i+1}]$ can now be respresented by

$$u_T(x) = u(x_i)\psi_i(x) + u(x_{i+1})\psi_{i+1}(x),$$

where $x \in [x_i, x_{i+1}]$.

DISCRETIZING THE PROBLEM

Consider the convection–diffusion equation

$$u_t = \epsilon u_{xx} - \delta u_x, \quad \epsilon > 0, \quad \delta > 0, \quad x \in (0, 1), \quad t \in (0, T], \tag{2}$$

with initial condition

$$u(x, 0) = \begin{cases} 10x - 2, & 0.2 \leqslant x \leqslant 0.3, \\ -10x + 4, & 0.3 \leqslant x \leqslant 0.4, \\ 0, & \text{elsewhere}, \end{cases}$$

and boundary conditions

$$u(0, t) = 0, \quad t \geqslant 0,$$
$$u(1, t) = 0, \quad t \geqslant 0.$$

Divide the interval [0, 1] in N subintervals of length h and the time-interval [0, T] in M subintervals of length k. Introduce the rational basis functions

$$\psi_i(x), \quad i = 1, \ldots, N-1,$$

which are independent of time at the nodes. Galerkin's method seeks an approximate solution to condition (2) in the form

$$u(x, t) = \sum_{i=1}^{N-1} u_i(t)\psi_i(x), \tag{3}$$

which satisfies the following system:

$$(u_t + \delta u_x - \epsilon u_{xx}, \psi_j) = 0 \quad j = 1, \ldots, N-1, \tag{4}$$

where

$$(u, v) = \int_0^1 uv \, \mathrm{d}x.$$

Substitution of forms (3) into (4) leads to

$$\begin{aligned}
&(\psi_{j-1}, \psi_j)\dot{u}_{j-1} + (\psi_j, \psi_j)\dot{u}_j + (\psi_{j+1}, \psi_j)\dot{u}_{j+1} \\
&\quad + \delta(\psi'_{j-1}, \psi_j)u_{j-1} + \delta(\psi'_j, \psi_j)u_j + \delta(\psi'_{j+1}, \psi_j)u_{j+1} \\
&\quad + \epsilon(\psi'_{j-1}, \psi'_j)u_{j-1} + \epsilon(\psi'_j, \psi'_j)u_j + \epsilon(\psi'_{j+1}, \psi'_j)u_{j+1} = 0, \quad \text{for} \quad j = 1, \ldots N-1.
\end{aligned}$$

This system of first order differential equations is discretized forward in time, thus

$$\begin{aligned}
&(\psi_{j-1}, \psi_j)u_{j-1}^{n+1} + (\psi_j, \psi_j)u_j^{n+1} + (\psi_{j+1}, \psi_j)u_{j+1}^{n+1} \\
&\quad = \{(\psi_{j-1}, \psi_j) - \delta k(\psi'_{j-1}, \psi_j) - \epsilon k(\psi'_{j-1}, \psi'_j)\}u_{j-1}^n + \{(\psi_j, \psi_j) - \delta k(\psi'_j, \psi_j) - \epsilon k(\psi'_j, \psi'_j)\}u_j^n \\
&\quad + \{(\psi_{j+1}, \psi_j) - \delta k(\psi'_{j+1}, \psi_j) - \epsilon k(\psi'_{j+1}, \psi'_j)\}u_{j+1}^n,
\end{aligned} \tag{5}$$

where

$$u_i^n \equiv u(ih, nk) \quad \text{and} \quad j = 1, \ldots, N-1.$$

The ith equation for the (0, 1) rational basis function is typically

$$\begin{aligned}
&(6h \ln 2 - 4h)u_{i-1}^{n+1} + (9h - 12h \ln 2)u_i^{n+1} + (6h \ln 2 - 4h)u_{i+1}^{n+1} \\
&\quad = \left(6h \ln 2 - 4h + \frac{7\epsilon k}{6h} + \frac{\delta k}{2}\right)u_{i-1}^n + \left(9h - 12h \ln 2 - \frac{7\epsilon k}{3h}\right)u_i^n \\
&\quad + \left(6h \ln 2 - 4h + \frac{7\epsilon k}{6h} - \frac{\delta k}{2}\right)u_{i+1}^n.
\end{aligned} \tag{6}$$

In matrix notation the difference scheme may be written in the form

$$A(u^{n+1} - n^n) + kBu^n + kCu^n = 0,$$

where $u = (u_1, \ldots, u_{N-1})^{\mathrm{T}}$ and A, B and C are tridiagonal matrices. This provides an explicit scheme which gives nodal values at time level $(n + 1)k$.

CONSISTENCY

In order to develop the consistency of the rational approximation, consider the discrete equation (5). The following relations hold for any $(0, T)$ rational basis function, i.e.

$$\begin{aligned}
&(\psi_{j-1}, \psi_j) = (\psi_{j+1}, \psi_j), \quad (\psi_{j-1}, \psi_j) + (\psi_j, \psi_j) + (\psi_{j+1}, \psi_j) = h, \quad (\psi'_{j-1}, \psi_j) + (\psi'_{j+1}, \psi_j) = 0, \\
&(\psi'_j, \psi_j) = 0, \quad (\psi'_{j-1}, \psi'_j) = (\psi'_{j+1}, \psi'_j), \quad (\psi'_{j-1}, \psi'_j) + (\psi'_j, \psi'_j) + (\psi'_{j+1}, \psi'_j) = 0.
\end{aligned} \tag{7}$$

Define the local truncation error $T_{j,n}$ at (jh, nk) by

$$T_{j,n} = \frac{1}{hk}[(\psi_{j-1}, \psi_j)u_{j-1}^{n+1} + (\psi_j, \psi_j)u_j^{n+1} + (\psi_{j+1}, \psi_j)u_{j+1}^{n+1}$$
$$- \{(\psi_{j-1}, \psi_j) - \epsilon k(\psi'_{j-1}, \psi'_j) - \delta k(\psi'_{j-1}, \psi_j)\}u_{j-1}^n$$
$$- \{(\psi_j, \psi_j) - \epsilon k(\psi'_j, \psi'_j) - \delta k(\psi'_j, \psi_j)\}u_j^n$$
$$- \{(\psi_{j+1}, \psi_j) - \epsilon k(\psi'_{j+1}, \psi'_j) - \delta k(\psi'_{j+1}, \psi_j)\}u_{j+1}^n]. \tag{8}$$

By using the Taylor's expansion and relations (7) it follows that

$$T_{j,n} = u_t + \epsilon h(\psi'_{j-1}, \psi'_j)u_{xx} - 2\delta u_x(\psi'_{j-1}, \psi_j) + 0(h^2, k). \tag{9}$$

Since, the (0, 1) rational basis function satisfies

$$(\psi'_{j-1}, \psi'_j) = -\frac{7}{6h}$$

and

$$(\psi'_{j-1}, \psi_j) = -\tfrac{1}{2},$$

it follows from equation (9) that

$$\lim_{h,k\to 0} T_{j,n} = u_t - \tfrac{7}{6}\epsilon u_{xx} + \delta u_x. \tag{10}$$

This equation differs from the original equation (2) due to the added diffusion coefficient $\epsilon/6$, which represents a numerical or artificial diffusion.

The (0, 2) and (0, 3) rational basis functions yield

$$\lim_{h,k\to 0} T_{j,n} = u_t - 1.1046\epsilon u_{xx} + \delta u_x$$

and

$$\lim_{h,k\to 0} T_{j,n} = u_t - 1.0764\epsilon u_{xx} + \delta u_x,$$

respectively.

Observe that with higher order rational basis functions the artificial diffusion coefficient decreases and the discrete equation almost coincides with equation (2). It is important to note that the artificial diffusion coefficient is independent of h and k and is negligible for small values of ϵ. From this argument it is evident that the numerical scheme is naturally dissipative and will tend to damp out numerical oscillations arising from the convection term.

STABILITY

The stability of discrete equation (5) is examined by the standard Fourier stability analysis [6]. Substitution of

$$u_j^n = \xi^n e^{\hat{\imath} j\gamma h}$$

(where $\hat{\imath}^2 = -1$ and $\gamma \geqslant 0$), into equation (5) and the use of relations (7), yield

$$\xi = [h - \{2(\psi_{j-1}, \psi_j) + \epsilon k(\psi'_j, \psi'_j)\}(1 - \cos\gamma h)$$
$$- 2\hat{\imath}\delta k(\psi'_{j+1}, \psi_j)\sin\gamma h]/[h - 2(\psi_{j-1}, \psi_j)(1 - \cos\gamma h)].$$

A necessary condition for stability is that the amplification factor ξ satisfies $|\xi|^2 \leqslant 1$. This condition is equivalent to

$$k \leqslant \frac{2\epsilon(\psi'_j, \psi'_j)[h - 2(\psi_{j-1}, \psi_j)(1 - \cos\gamma h)]}{\{\epsilon(\psi'_j, \psi'_j)\}^2(1 - \cos\gamma h) + \delta^2(1 + \cos\gamma h)}. \tag{11}$$

An extreme value is attained at

$$\delta^2 = \frac{h\epsilon^2(\psi_j', \psi_j')^2}{h - 4(\psi_{j-1}, \psi_j)},$$

which renders the stability condition

$$k \leqslant \frac{h - 4(\psi_{j-1}, \psi_j)}{\epsilon(\psi_j', \psi_j')}.$$

Thus, in particular the (0, 1), (0, 2) and (0, 3) rational basis functions provide

$$k \leqslant 0.156\frac{h^2}{\epsilon},$$

$$k \leqslant 0.168\frac{h^2}{\epsilon}$$

and

$$k \leqslant 0.174\frac{h^2}{\epsilon},$$

respectively. From the theoretical investigation it is clear that the stability condition weakens with increasing order of the rational basis functions.

NUMERICAL RESULTS

In this section, some numerical results for the explicit difference scheme are presented. The scheme has been solved for different values of h, namely $h = 0.033$, 0.02, 0.0167 and 0.0125. The parameters are $\epsilon = 0.01$ and $\delta = 1$ with timestep $k = 0.001$. The results are compared by means of the relative L_2-norm, i.e.

$$\| E \|_2 = \frac{\| v - u \|_2}{\| v \|_2},$$

where u is the approximant and v the analytical solution which is given in van Niekerk [4]. The results at time $T = 0.6$ for (0, 1), (0, 2) and (0, 3) rational basis functions are shown in Table 1.

From Table 1 it is clear that higher order methods, except for the first column, improve the results and that the accuracy increases with smaller h. In Figs 1–3 the numerical and analytical solutions at $T = 0.6$ with $h = 0.0125$, $k = 0.001$ and different rational basis functions are compared. The graphs clearly demonstrate the superiority of the higher order rational basis functions. The better performances of the higher order basis functions correspond with the consistency analysis.

The numerical scheme is also implemented at different timesteps to verify the stability condition. In Table 2 the discrete L_2-norm,

$$\| E \|_2^2 = h \sum_{i=1}^{N} (u_i - v_i)^2,$$

at time $T = 1.0$ is tabulated for different rational basis functions. A dash in the table indicates that the stability condition is being violated.

The numerical results correspond extremely well with the stability condition and also indicate that the higher order rational basis functions have a less rigid restriction on the timestep, which

Table 1. Relative L_2-norm for different rational basis functions

	h				
	0.033	0.025	0.020	0.0167	0.0125
(0, 1)	6.93E − 2	4.80E − 2	4.23E − 2	4.15E − 2	4.25E − 2
(0, 2)	7.80E − 2	4.11E − 2	2.80E − 2	2.28E − 2	2.07E − 2
(0, 3)	7.45E − 2	4.22E − 2	2.62E − 2	1.80E − 2	1.21E − 2

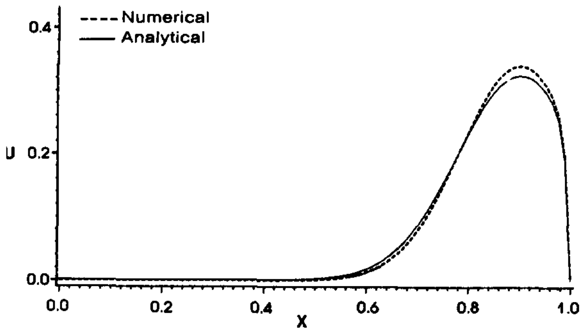

Fig. 1. (0, 1) Rational basis function.

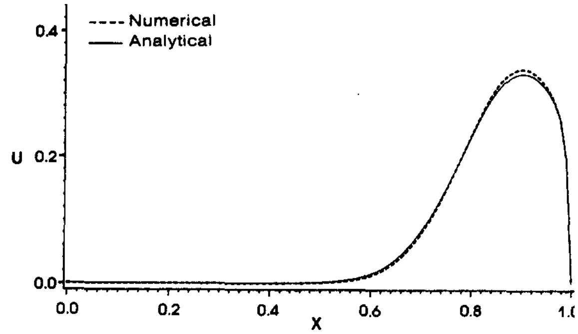

Fig. 2. (0, 2) Rational basis function.

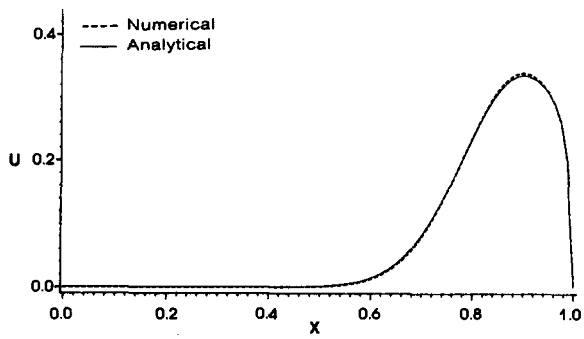

Fig. 3. (0, 3) Rational basis function.

Table 2. L_2-norm for different space and timesteps

h	k = 0.01	0.005	0.0025	0.001
	(0, 1) Rational Basis Function			
0.0333	1.80E − 3	0.91E − 3	1.74E − 3	2.37E − 3
0.0200	—	0.29E − 3	0.76E − 3	1.40E − 3
0.0167	—	—	0.65E − 3	1.28E − 3
0.0125	—	—	52.95E − 3	1.18E − 3
	(0, 2) Rational Basis Function			
0.0333	2.31E − 3	0.99E − 3	1.40E − 3	1.97E − 3
0.0200	—	0.29E − 3	0.31E − 3	0.88E − 3
0.0167	—	—	0.14E − 3	0.74E − 3
0.0125	—	—	0.05E − 3	0.63E − 3
	(0, 3) Rational Basis Function			
0.0333	2.53E − 3	1.13E − 3	1.31E − 3	1.81E − 3
0.0200	—	1.09E − 3	0.29E − 3	0.67E − 3
0.0167	—	1.15E − 3	0.21E − 3	0.57E − 3
0.0125	—	—	0.24E − 3	0.38E − 3

is in accordance with the theoretical analysis. The last column, $k = 0.001$, suggests that the error improves with $0(h^{1/2})$ when the order of the basis function increases. From this observation it is evident that higher order basis functions improve the numerical convergence of the scheme.

Finally, the numerical results validate the theoretical analysis that higher order basis functions yield a numerical scheme with improved consistency properties and higher accuracies. Moreover, one has the additional advantages of better stability and convergence with higher order rational basis functions, while the effect of upstream differencing is simulated in a natural way.

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