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Asymptotic-numerical method for buckling analysis of shell structures with large rotations

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Abstract

In this paper the buckling and post-buckling of elastic structures taking into account large rotations are investigated using an asymptotic-numerical method. The critical points are detected by two different ways: first by a bifurcation indicator, second by analysing the poles of a Padé approximant. The first step of the post-buckling branch is computed starting from the bifurcation point and using an extended system. The remaining bifurcating branch is followed by the same algorithm as for the fundamental path. Several examples are tested to show the effectiveness of the proposed method.

Keywords: Asymptotic-numerical method; Padé approximants; Bifurcation indicator; Buckling; Finite rotations; Finite elements

1. Introduction

In nonlinear structural mechanics, the buckling is a common phenomenon which requires algorithms able to detect the singular points and to follow the postcritical branches. Post-buckling analysis allows one to estimate the effect of imperfections and to predict the possibilities of jumping.

In the finite element framework, the response curve of imperfect structures is computed by using an incremental iterative algorithm. This can induce a significant computational cost and leads to numerical difficulties especially near singular points. In the presence of load or displacement limit points, a control parameter based on the arc length procedure is necessary [2,11,12].

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An alternative procedure to the classical iterative algorithms is the asymptotic-numerical method (ANM) [7]. It is based on the association of the perturbation techniques and the finite element method (FEM). The variables are developed in power series transforming the nonlinear problem into a sequence of linear ones solved by FEM [5]. To improve the validity range of the series representation, one uses Padé approximants which reduce at least by half the number of steps necessary to describe the whole solution branch [6,9].

In the present work, we propose to extend this method to the buckling of structures with large rotations. Two techniques are performed to detect the bifurcation point. The first one does not require any supplementary calculus, it consists in analysing the denominator of the Padé approximants which vanishes exactly at the bifurcation point. The second technique uses a bifurcation indicator well adapted to the ANM framework. The bifurcating branch is searched by asymptotic expansions starting from the bifurcation point. As the tangent operator is singular, an extended system is used. The possible bifurcation points on the bifurcating branch can be detected in the same manner as for the fundamental branch.

To illustrate the effectiveness of the method, we present some numerical applications.

2. Formulation of the problem taking into account large rotations

The shell formulation used in this paper is detailed in [4,17]. The unknowns of the problem are the mid-surface displacement and its director vector. The large rotations are taken into account without any rotation matrices. The current director vector is obtained by adding a simple vector to the one of the initial configuration. A complete three-dimensional constitutive law without condensation can be accounted by introducing a linear strain variable through the thickness. This extra variable is incorporated via the enhanced assumed strain (EAS) concept [13]. The principle consists in introducing an enhanced part of the strain $\tilde{\gamma}$ independent of the displacement u and orthogonal to the stress field S

$$\gamma - Bu = \tilde{\gamma}, \qquad \int_v ({}^tS:\tilde{\gamma})\,\mathrm{d}v = 0, \tag{1}$$

where $Bu = \gamma_\ell(u) + \gamma_{\mathrm{n}\ell}(u,u)$ is the compatible Green Lagrange strain which can be decomposed into a linear and a quadratic part.

The stationary condition of the Hu–Washizu functional leads to an equation in the following form:

$$R(U,\lambda) = L(U) + Q(U,U) - \lambda F = 0, \tag{2}$$

where $U = \left(u, \tilde{\gamma}, S\right)$ is a mixed unknown vector, $L(.)$ a linear operator, $Q(.,.)$ a quadratic one, F the external load vector and R the residual vector.

3. Computation of the solution branches by ANM

The basic idea of the ANM consists in searching the solution branches of the nonlinear problem (2) under an asymptotic expansion form in terms of a control parameter "a". This expansion is developed in the neighbourhood of a known solution (U_0, λ_0) and the series is truncated at order n:

$$U(a) - U_0 = \sum_{i=1}^{n} a^i U_i, \qquad \lambda(a) - \lambda_0 = \sum_{i=1}^{n} a^i \lambda_i. \tag{3}$$

By substituting this expansion into Eq. (2) and equating the coefficients of the same power of the parameter "a", one transforms the nonlinear problem (2) into a sequence of linear ones solved by the finite element method. The validity range of the power series can be improved by Padé approximants [6]

$$U(a) - U_0 = \sum_{i=1}^{n-1} f_i(a)\ a^i U_i, \qquad \lambda(a) - \lambda_0 = \sum_{i=1}^{n-1} f_i(a)\ a^i \lambda_i, \tag{4}$$

where $f_i(a)$ are rational fractions admitting the same denominator. To obtain the whole solution branch, the path following technique has been recently presented [9]. The validity range of the solution of (4) is defined by the maximal value 'a_{mp}' of the control parameter "a". Requiring that the relative difference between the displacements at two consecutive orders must be smaller than a given parameter δ leads to

$$\delta = \frac{\|u_n(a_{mp}) - u_{n-1}(a_{mp})\|}{\|u_n(a_{mp}) - u_0\|}. \tag{5}$$

In this manner, one performs a step-by-step procedure allowing to describe the whole solution branches. Note that the high-order predictor can be associated with a high-order corrector which leads to a reliable and efficient algorithm [10].

4. Detection of bifurcation points by Padé approximants

A first simple method to detect bifurcation points can be established by an a posteriori analysis of the rational representation (4). It has been early recognized [6] that a bifurcation point corresponds to a root of the denominator of the fraction $f_n(a)$. Of course the rational representation (4) has many other poles. Nevertheless, it is rather easy to recognize if a given pole characterizes a bifurcation point. First, one limits himself to the smallest real pole. Second the quality of the approximated solution (4) remains very good beyond the pole. Third, the tangent matrix must be almost singular at this pole.

In other words, the rational representation can provide on the one hand the response curve before and after the bifurcation and on the other hand the exact location of the bifurcation point. Note that it is not easy to get the same results from the representation by series (3). Indeed, the convergence radius of the series coincides generally with the smallest real or complex pole [9]. Nevertheless, a more or less reliable method to detect bifurcation points from series has been discussed in [3,14], that relies on the technique presented in the next part.

5. Detection of bifurcation points by an indicator

Let f be a fictitious perturbating force applied to the structure at a point of the solution branch. $\Delta\mu$ is the unknown intensity of this force and ΔU its associated response. By superposing the fictitious perturbation and the applied load and neglecting the second order terms, one obtains the following tangent problem:

$$L_t(\Delta U) = \Delta\mu f, \tag{6}$$

where $L_t(.) = L(.) + 2Q(U, .)$ is the tangent operator at the equilibrium point. To obtain a unique solution of (6), one needs an additional condition [3]

$$\langle\langle \Delta U - \Delta U_0, \Delta U_0 \rangle\rangle = 0, \tag{7}$$

where $\langle\langle , \rangle\rangle$ denotes a scalar product. Here we choose the one associated with the tangent operator L_t^0 considered at a starting point. ΔU_0 is chosen as the solution of the following problem:

$$L_t^0(\Delta U_0) = f. \tag{8}$$

Eqs. (6) and (7) constitute a linear system with respect to ΔU and $\Delta\mu$. It is solved by ANM in the same manner as for the equilibrium path: computation of the unknown by power series improved by Padé approximants. So the obtained approximation of $\Delta\mu(a)$ is highly accurate inside the validity range $[0, a_{mp}]$. The bifurcation and the limit points correspond exactly to the values of the load for which the operator L_t is singular, i.e. the roots of the indicator $\Delta\mu$

$$\Delta\mu(a) = 0. \tag{9}$$

6. Computation of the post-buckling branch by the ANM

Assume that the critical point is computed with the procedures presented in Sections 4 and 5. We propose to follow the bifurcating branches starting from the singular point. Because of the tangent stiffness matrix is singular, the first step of the bifurcating branch is computed in a specific way. First, the tangent directions from the simple bifurcation point are obtained by the classical bifurcation analysis which leads to the well-known quadratic bifurcation equation [8,15,16]. Second, the linear problems resulting from the asymptotic expansion are solved via an extended system [16].

The next steps of the bifurcating branch are computed in the same manner as for the fundamental branch. In any case, the solution is searched using power series improved by Padé approximants and the step length is defined by Eq. (5).

7. Numerical applications

The proposed method has been applied to several examples. Here, we limit ourselves to two tests which present bifurcation points and nonlinear prebuckling. The first problem concerns a deep circular arch and the second one refers to a thin cylindrical roof.

Two main parameters are important for an ANM step: the truncation order n and the parameter δ. It has been shown in previous papers that the optimum order of truncation is generally in the range 10–20. δ defines the step length. A large value of δ allows one to reduce the number of steps but does not ensure a good residual at the end of the calculus. However, a small value of δ increases the number of steps to describe the whole solution branch but ensures a good residual along the steps. For the two tests, we choose $n = 15$ and $\delta = 10^{-4}$.

7.1. Buckling analysis of a deep arch under point load

We consider a deep circular arch, simply supported on the two edges and submitted to a vertical load at the center as described in Fig. 1. The arch presents a nonlinear symmetric prebuckling and a first bifurcation point with an antisymmetric buckling mode (symmetric stable bifurcation). The structure is discretised with 20 quadratic shell elements: 8 nodes per element and 618 degrees of freedom.

Fig. 2a shows the evolution of the indicator versus the load parameter for two truncation orders 10 and 15.

The first bifurcation point corresponding to $\lambda_1 = 3.353$ has been detected both by the pole of Padé approximants and by the bifurcation indicator. This critical point is exactly the same as that found in [16]. To reach the bifurcation point, one needs only one step and then one decomposition of the tangent stiffness matrix. The first buckling mode is then obtained (Fig. 3b).

Note that the validity range of the Padé approximants is not limited by the pole (i.e., bifurcation point), the solution remains well approximated beyond the bifurcation point. In Fig. 2b are presented the response curves of the arch, the star indicates the end of the first step ($\lambda = 8.42$) corresponding to a relative residual less than 10^{-3}.

Fig. 3a represents a deformed state of the arch corresponding to a fundamental equilibrium load.

Solving the classical bifurcation equation, one gets all the tangents at the bifurcation point. The first bifurcating branch is followed automatically. The bifurcation indicator is evaluated through this branch, it vanishes at the limit point L2 and at the second bifurcation point B2 ($\lambda_2 = -0.278$). The second bifurcating branch is then computed. Beyond the limit point L2, the first bifurcating branch becomes unstable and the arch jumps to the point S.

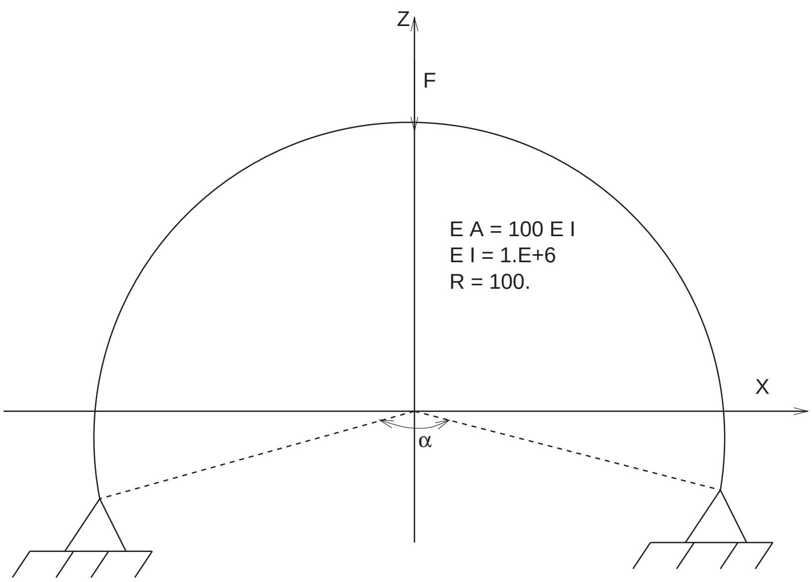
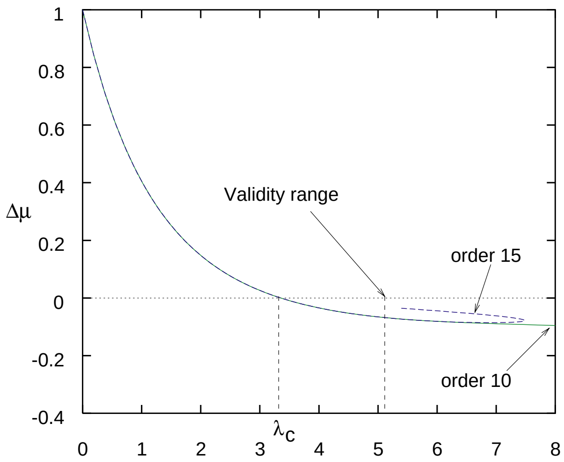
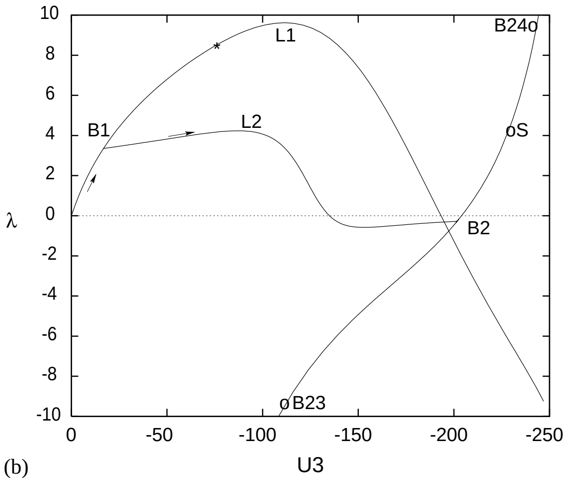


Fig. 1. Circular deep arch, simply supported and submitted to a vertical load at centre.



(a)



(b)

Fig. 2. Deep arch: (a) bifurcation indicator, (b) load—vertical displacement response of the loaded point, the star indicates the end of the first step.

Concerning the computation cost, the fundamental branch is obtained with 9 steps that is to say with only 9 decompositions of the tangent stiffness matrix and the first bifurcating branch is obtained with 10 steps.

7.2. *Buckling analysis of a thin cylindrical roof*

In this section we present the buckling of a cylindrical thin shell subjected to a point load at its centre. The geometry and boundary conditions are described in Fig. 4. In this study, only a half geometry is discretised with 18 quadratic shell elements leading to 438 degrees of freedom. This problem is studied in [1] to analyse the sensitivity of buckling load to geometrical shape

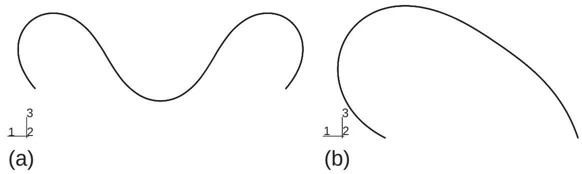


Fig. 3. Deep arch: (a) fundamental configuration and (b) first buckling mode.

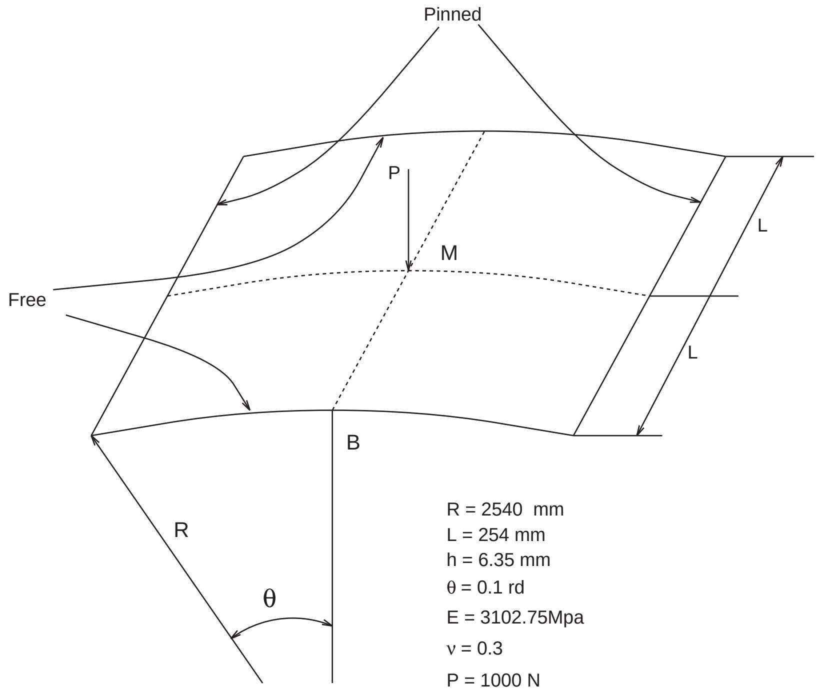


Fig. 4. Geometry description for the cylindrical roof.

imperfections. The prebuckling path is nonlinear and the first bifurcation is symmetric and unstable. The first buckling load is obtained with only one step $\lambda_1 = 0.520$. This value is detected by the two techniques presented in this work. In Fig. 5 the star indicates the end of the first step ($\lambda = 0.57$). Starting from the bifurcation point, one can follow the bifurcating branch and search possible secondary bifurcation points. The bifurcation indicator and the pole of Padé approximants indicate the second bifurcation point for $\lambda_2 = -0.224$.

Remark. In this problem, the first bifurcation point is located near the end of the first step. The indicator gives the bifurcation with a truncation order $n = 15$ but for the second technique it is necessary to choose a very high order $n \geqslant 35$. By reducing the step length via the parameter δ, the bifurcation point is detected at the second step both by the indicator and the pole of the Padé approximants with a truncation order $n = 15$. Note that the fundamental path is obtained with only 9 steps and the bifurcating path with 7 steps.

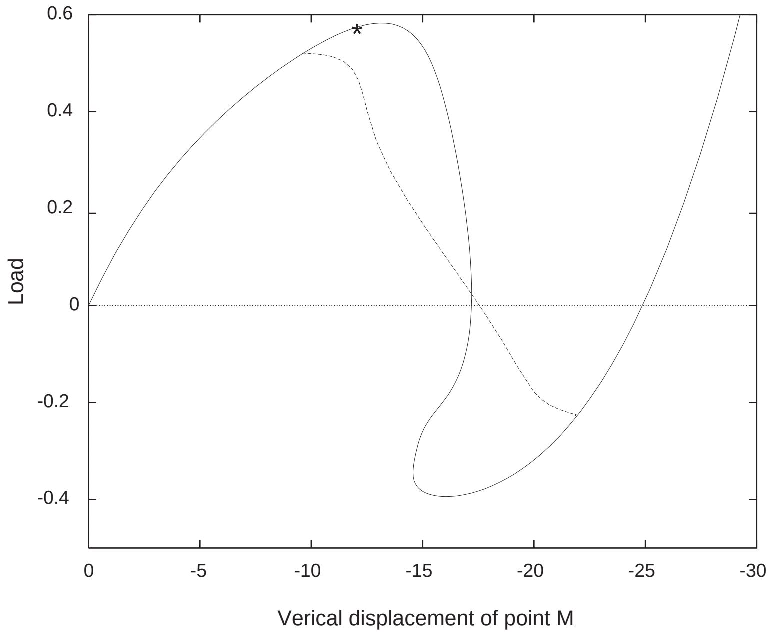


Fig. 5. Load/displacement response curve of the thin roof. The star indicates the end of the first step.

8. Conclusion

We have presented an asymptotic-numerical method for the computation of the critical points and the post-buckling of elastic structures taking into account large rotations. As the series representation is truncated at high order and improved by Padé approximants, one obtains an important reduction of the step number allowing to save a significant computation time. The critical points have been detected by two techniques: first by analysing the denominator of the Padé approximants and second by introducing a bifurcation indicator well adapted to the ANM framework. The tangent directions at the bifurcation points are computed by solving the bifurcation equation. Starting from the bifurcation point, the first step of the bifurcating branch is computed solving an extended system. The next steps are computed in the same manner as for the fundamental path. Secondary critical points can then be determined on the bifurcating branch.

The proposed method can be extended to the problems of multiple bifurcations and nonlinear vibrations of structures with large rotations.

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