The Basis Number of a Graph

EDWARD F. SCHMEICHEL*

San José State University, San José, California 95192
Communicated by the Editors
Received February 13, 1978

MacLane proved that a graph is planar if and only if it has a 2-fold basis for its cycle space. We define the basis number of a graph G to be the least integer k such that G has a k-fold basis for its cycle space. We investigate the basis number of the complete graphs, complete bipartite graphs, and the n-cube.

1. Introduction

Throughout this paper, we consider only finite, undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated. For undefined terms, see [1].

Let G be a connected graph, and let $e_1, e_2, ..., e_q$ be an ordering of the edges in G. Then any subset of edges $S$ corresponds to a $(0, 1)$-vector $(a_1, a_2, ..., a_q)$ in the usual way, with $a_i = 1$ (resp., $a_i = 0$) if and only if $e_i \in S$ (resp., $e_i \notin S$). These vectors form a $q$-dimensional vector space over the field $\mathbb{Z}_2$.

The vectors corresponding to the cycles in G generate a subspace of $(\mathbb{Z}_2)^q$ called the cycle space of G, and denoted by $\mathcal{C}(G)$. (For brevity in the sequel, we will say that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\mathcal{C}(G)$.) It is well known that the dimension of $\mathcal{C}(G)$ is $q - p + 1$, where $p$ and $q$ denote respectively the number of vertices and edges in G. In fact, a basis of $\mathcal{C}(G)$ may be obtained as follows: Let $T$ be any spanning tree in G. If $e$ is an edge of $G - T$, then $T + e$ contains exactly one cycle, say $C_e$. It is readily shown that the $q - p + 1$ cycles $C_e$, $e \in G - T$, form a basis for $\mathcal{C}(G)$. The basis for $\mathcal{C}(G)$ obtained in this way is usually termed the fundamental basis corresponding to $T$.

Note that while each edge in $G - T$ occurs in exactly one cycle of the fundamental basis corresponding to $T$, an edge of $T$ may occur in relatively many cycles of this fundamental basis. This observation suggests the following.

* Supported in part by the Air Force Office of Scientific Research under Grant AFOSR-76-3017.
DEFINITION. A basis of $\mathcal{C}(G)$ is called $k$-fold if each edge of $G$ occurs in at most $k$ of the cycles in the basis. The basis number of $G$ (denoted by $b(G)$) is the smallest integer $k$ such that $\mathcal{C}(G)$ has a $k$-fold basis.

The first important use of these concepts occurred in 1937 when MacLane proved that a graph $G$ is planar if and only if $b(G) \leq 2$. For a proof, see [21].

The purpose of this paper is to investigate the basis number of certain classes of nonplanar graphs. We first consider the basis number of the complete graphs $K_n$. It is proved that $b(K_n) = 3$, for every integer $n \geq 5$. In contrast to the situation for complete graphs, it is shown that for any integer $r$, there exists a graph $G$ with $b(G) \geq r$. We then determine the basis number of almost all complete bipartite graphs and conclude by considering the basis number of the $n$-cube.

2. Main Results

THEOREM 1. For every integer $n \geq 5$, we have $b(K_n) = 3$.

Proof. Since $K_n$ is nonplanar for $n \geq 5$, it follows by the theorem of MacLane mentioned above that $b(K_n) \geq 3$ if $n \geq 5$. To prove the theorem, therefore, it suffices to show that $b(K_n) \leq 3$.

Note that a collection of cycles in $G$ which generates all the chordless cycles in $G$ in fact generates all of $\mathcal{C}(G)$, since every cycle can be expressed as a sum modulo 2 of chordless cycles. In particular, to generate $\mathcal{C}(K_n)$, it suffices to generate all the 3-cycles in $K_n$.

Denote the vertices of $K_n$ by the integers $Z_n = \{0, 1, 2, \ldots, n - 1\}$ modulo $n$. Let $B_n$ be the collection of 3-cycles in $K_n$ defined as follows: $B_n = \{abc | a + b + c \equiv 0, 1, \text{ or } 2 \pmod{n}\}$. We will show that every 3-cycle in $K_n$ (and hence all of $\mathcal{C}(K_n)$) can be generated from $B_n$. Since every edge of $K_n$ occurs in exactly three 3-cycles of $B_n$, it would follow that $B_n$ contains a 3-fold basis of $\mathcal{C}(K_n)$, and hence that $b(K_n) \leq 3$ as desired.

Let $abc$ be any 3-cycle in $K_n$ with say $a < b < c$. Let $a + b + c \equiv r \pmod{n}$, where $3 \leq r \leq n - 1$. Assuming that we generate any 3-cycle $a'b'c'$ from $B_n$, where $a' + b' + c' \equiv k \pmod{n}$ and $0 \leq k < r$, we will show inductively that we can also generate $abc$ from $B_n$.

Define two integers $x, y \in Z_n$ to be consecutive if $|x - y| = 1$ or $n - 1$. We now have essentially three possibilities for $abc$.

(1) $a, b, c$ are pairwise nonconsecutive. Then $a, b, c, a - 1, b - 1$, and $c - 1$ are all distinct. Moreover, note that

$$abc = ab(c - 1) + a(b - 1)c + (a - 1)bc + a(b - 1)(c - 1)$$

$$+ (a - 1)b(c - 1) + (a - 1)(b - 1)c$$

$$+ (a - 1)(b - 1)(c - 1) \pmod{2}.$$
Since each 3-cycle \(xyz\) on the right side satisfies \(x + y + z \equiv r - 1, r - 2,\) or \(r - 3 \pmod{n}\), it follows by assumption that each 3-cycle on the right side can be generated from \(B_n\). Hence also \(abc\) can be generated from \(B_n\).

(2) \(a, \ b\) are consecutive, but \(a, \ c\) and \(b, \ c\) are nonconsecutive. Then \(a, \ b, \ c, \ a - 1,\) and \(c - 1\) are all distinct. Moreover, we have
\[
abc = ab(c - 1) + (a - 1)ac + (a - 1)bc
+ (a - 1) a(c - 1) + (a - 1) b(c - 1) \pmod{2}.
\]
As in possibility (1), we conclude that we can generate \(abc\) from \(B_n\).

(3) \(a, \ b\) and \(b, \ c\) are consecutive, but \(a, \ c\) are nonconsecutive. Then \(a, \ b, \ c, \ a - 1\) are all distinct. We also have
\[
abc = (a - 1)ab + (a - 1)ac + (a - 1)bc \pmod{2}.
\]
As in possibility (1), we conclude that we can generate \(abc\) from \(B_n\).

This completes the proof of Theorem 1.

In contrast to the small basis number of the complete graphs, we now prove the existence of graphs having arbitrarily large basis number.

**Theorem 2.** For any positive integer \(r\), there exists a graph \(G\) with \(b(G) > r\).

**Proof.** Let \(G\) be a connected graph with girth \(r\) and average vertex degree \(\geq 2r\) (e.g., \(a(2r, r)\)-cage). We will prove that \(b(G) \geq r\).

Otherwise, suppose that \(C_1, C_2, \ldots, C_{q-p+1}\) are cycles in an \((r - 1)\)-fold basis for \(\mathcal{C}(G)\). Let \(|C_i|\) denote the length of cycle \(C_i\). We have at once that
\[
r(q - p + 1) \leq \sum_{i=1}^{q-p+1} |C_i| \leq (r - 1)q. \quad (1)
\]
(The left inequality in (1) comes from the fact that \(G\) has girth \(r\) so that \(|C_i| \geq r\), and the right inequality follows since the basis is \((r - 1)\)-fold.) From (1) we obtain at once that \(r \geq q/(p - 1)\).

On the other hand, note that \(q/(p - 1) > q/p \geq r\), since \(q/p\) is half the average degree in \(G\). This is a contradiction, and we conclude that \(b(G) \geq r\).

The proof of Theorem 2 is complete.

We now consider the basis number of the complete bipartite graphs \(K_{m,n}\), where \(m, n \geq 3\). Since all these graphs are nonplanar, we have \(b(K_{m,n}) \geq 3\) in all cases. We begin with the following result.

**Theorem 3.** For any \(n \geq 3\), \(b(K_{3,n}) = b(K_{4,n}) = 3\).

**Proof.** By the above paragraph, it suffices to exhibit a 3-fold basis for each of these graphs.
We begin with $K_{4,n}$. Let $\{a, b, c, d\}$ and $\{x_1, x_2, \ldots, x_n\}$ be the partition of the vertices of $K_{4,n}$ into independent sets. Consider the following collection of 4-cycles in $K_{4,n}$:

$$B(K_{4,n}) = \begin{align*}
&\{a \ x_i \ b \ x_{i+1} \ \text{for} \ i = 1, 2, \ldots, n - 1 \\
&c \ x_i \ d \ x_{i+1} \ \text{for} \ i = 1, 2, \ldots, n - 1 \\
a \ x_{2i-1} \ c \ x_{2i} \ \text{for} \ i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \\
b \ x_{2i} \ d \ x_{2i+1} \ \text{for} \ i = 1, 2, \ldots, \left\lfloor \frac{n - 1}{2} \right\rfloor.
\end{align*}$$

To show that $B(K_{4,n})$ generates all of $\mathcal{C}(K_{4,n})$, it suffices to show that $B(K_{4,n})$ generates all the chordless cycles of $K_{4,n}$ which would be the 4-cycles. Let $ax_i \beta x_k$ be any 4-cycle in $K_{4,n}$, with $a, \beta \in \{a, b, c, d\}$. Since

$$ax_i \beta x_k = \sum_{i \neq j} ax_i \beta x_{i+1} \quad (\text{mod } 2),$$

it suffices to generate any 4-cycle in $K_{4,n}$ of the form $ax_i \beta x_{i+1}$ which does not already belong to $B(K_{4,n})$.

We have first that

$$ax_{2i} \ c x_{2i+1} = bx_{2i} \ d x_{2i+1} + ax_{2i} \ b x_{2i+1} + cx_{2i} \ d x_{2i+1} \quad (\text{mod } 2).$$

Similarly, we can obtain $bx_{2i-1} \ d x_{2i}$ as a sum modulo 2 of three 4-cycles in $B(K_{4,n})$.

Next we have

$$ax_i \ d x_{i+1} = ax_i \ c x_{i+1} + cx_i \ d x_{i+1} \quad (\text{mod } 2).$$

Similarly, we can obtain $bx_i \ c x_{i+1}$ as a sum modulo 2 of two 4-cycles already generated.

It follows now that $B(K_{4,n})$ generates $\mathcal{C}(K_{4,n})$. Since $|B(K_{4,n})| = 3(n - 1) = \dim \mathcal{C}(K_{4,n})$, it follows that $B(K_{4,n})$ is indeed a basis of $\mathcal{C}(K_{4,n})$. It is a simple matter to verify that it is a 3-fold basis.

Next we consider $K_{3,n}$. Let $\{a, b, c\}$ and $\{x_1, x_2, \ldots, x_n\}$ be the partition of the vertices of $K_{3,n}$ into independent sets. Consider the following collection of $2(n - 1)$ 4-cycles in $K_{3,n}$:
The proof that \( B(K_{3,n}) \) is a 3-fold basis for \( \mathcal{G}(K_{3,n}) \) is analogous to the proof just given for \( K_{4,n} \), and is therefore omitted.

The proof of Theorem 3 is complete.

**THEOREM 4.** If \( m, n \geq 5 \), then \( b(K_{m,n}) \leq 4 \). Moreover, equality holds except possibly for the following: \( K_{5,5}, K_{5,6}, K_{5,7}, K_{5,8}, K_{6,6}, K_{6,7}, K_{6,8}, K_{6,10} \).

*Proof.* As noted in the proof of Theorem 3, to generate \( \mathcal{G}(K_{m,n}) \) it suffices to generate all the 4-cycles in \( K_{m,n} \).

Let \( \{x_1, x_2, \ldots, x_m\} \) and \( \{y_1, y_2, \ldots, y_n\} \) be the partition of the vertices of \( K_{m,n} \) into independent sets. Consider the \((m - 1)(n - 1)\) 4-cycles \( x_i y_j x_{i+1} y_{j+1} \) for \( 1 \leq i < m - 1, 1 \leq j < n - 1 \). If \( x_r y_s x_t y_u \) is any 4-cycle in \( K_{m,n} \), with \( r < t \) and \( s < u \), we have that

\[
x_r y_s x_t y_u = \sum_{j=3}^{u-1} x_r y_j x_t y_{j+1} = \sum_{j=3}^{u-1} \left( \sum_{i=r}^{t-1} x_i y_j x_{i+1} y_{j+1} \right) \quad \text{(mod 2)}.
\]

Thus these \((m - 1)(n - 1)\) 4-cycles generate all the 4-cycles in \( K_{m,n} \), and hence all of \( \mathcal{G}(K_{m,n}) \). Also, since \((m - 1)(n - 1) = mn - (m + n) + 1 = \dim \mathcal{G}(K_{m,n})\), they actually form a basis for \( \mathcal{G}(K_{m,n}) \). Finally, this basis is 4-fold, since any edge \( x_i y_j \) occurs in only the following cycles of this basis: \( x_{i-1} y_{j-1} x_i y_j, x_{i-1} y_j x_i y_{j+1}, x_i y_{j-1} x_{i+1} y_j, \) and \( x_i y_j x_{i+1} y_{j+1} \). So in general we have \( b(K_{m,n}) \leq 4 \).

On the other hand, suppose that \( K_{m,n} \) has a 3-fold basis. Then each vertex of degree \( m \) can occur in at most \( \lfloor 3m/2 \rfloor \) cycles in this 3-fold basis. It follows at once that the basis contains at most \( (n/2)\lfloor 3m/2 \rfloor \) cycles, and hence \( (n/2)\lfloor 3m/2 \rfloor \geq \dim \mathcal{G}(K_{m,n}) = (m - 1)(n - 1) \). Exchanging the roles of \( m \) and \( n \), we obtain the analogous inequality \( (m/2)\lfloor 3n/2 \rfloor \geq (m - 1)(n - 1) \). For \( m, n \geq 5 \), the only \( K_{m,n} \) for which both inequalities are satisfied are the graphs in the list of possible exceptions. Hence \( b(K_{m,n}) \geq 4 \) in all other cases.

The proof of Theorem 4 is complete.
(It seems likely that \( b(K_{m,n}) = 3 \) for the graphs listed as possible exceptions in Theorem 4.)

Consider next the basis number of the \( n \)-cube \( Q_n \). We begin by establishing an upper bound for \( b(Q_n) \).

**Theorem 5.** \( b(Q_n) \leq n - 1 \).

*Proof.* The result is easily verified for \( n = 2, 3, \) and \( 4 \), and thus we proceed by induction on \( n \).

We assume the vertices of \( Q_n \) correspond to the collection of all \((0, 1)\) \( n \)-tuples in the standard way. We will say that two vertices \( v = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( v' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n) \) in \( Q_n \) match if and only if \( \alpha_i = \alpha'_i \), for \( i = 1, 2, \ldots, n - 1 \), but \( \alpha_n \neq \alpha'_n \). Let \( X \) (resp., \( X' \)) denote the vertices of \( Q_n \) having \( \alpha_n = 1 \) (resp., \( \alpha_n = 0 \)). Then \( X \) and \( X' \) induce subgraphs \( Q_X \) and \( Q_{X'} \), respectively, which are isomorphic to \( Q_{n-1} \). Hence by our induction hypothesis, we can find an \((n - 2)\)-fold basis \( B_X \) (resp., \( B_{X'} \)) for \( \pi(Q_X) \) (resp., \( \pi(Q_{X'}) \)).

Let \( v_1, v_2, \ldots, v_{2n} \) be a hamiltonian path in \( Q_X \). Then the sequence of matching vertices \( v'_1, v'_2, \ldots, v'_{2n-1} \) form a hamiltonian path in \( Q_{X'} \). Moreover, the edges joining a vertex in \( X \) to a vertex in \( X' \) are precisely the edges \( v_i v'_i \).

Define a collection of cycles in \( Q_n \) (denoted by \( B \)) as follows: \( B \) will be \( B_X \cup B_{X'} \), together with the \((2^{n-1} - 1)\) 4-cycles \( C_i = v_i v_{i+1} v'_{i+1} v'_i \), for \( i = 1, 2, \ldots, 2^{n-1} - 1 \). Since \( \dim \pi(Q_n) = 2^{n-1}(n - 2) + 1 \), and \( B_X \) and \( B_{X'} \) are isomorphic to \( Q_{n-1} \), we have
\[
|B| = |B_X| + |B_{X'}| + (2^{n-1} - 1) = 2|B_X| + (2^{n-1} - 1) = 2(2^{n-2}n - 3) + 1 = 2^{n-1}(n - 2) + 1 = \dim \pi(Q_n).
\]

Thus to show that \( B \) is a basis of \( \pi(Q_n) \), it suffices to show that the cycles of \( B \) are independent.

Suppose therefore that some collection of cycles in \( B \), say \( S \subseteq B \), satisfies a nontrivial relation modulo 2 (that is, \( \sum_{C \in S} C = 0 \) (mod 2)). Since \( B_X \) and \( B_{X'} \) are themselves bases and no cycle in \( B_X \) has an edge in common with any cycle in \( B_{X'} \), it follows that \( S \) must include at least one cycle \( C_i \) in \( B - (B_X \cup B_{X'}) \). But then it follows easily that \( C_i \in S \). (Suppose \( C_i \in S \). Note that \( C_i \) contains the edge \( v_i v'_i \). The only other cycle in \( B \) containing the edge \( v_i v'_i \) is \( C_{i-1} \). Hence \( C_{i-1} \) must belong to \( S \) to cancel \( v_i v'_i \) modulo 2. Continuing, we see that \( S \) must contain \( C_1 \).) But the cycle \( C_1 \) contains the edge \( v_i v'_i \), which occurs in no other cycle of \( B \), and in particular in no other cycle of \( S \). This means that \( \sum_{C \in S} C \) could not be 0 modulo 2, a contradiction. Thus a nontrivial relation among the cycles of \( B \) is impossible, and so \( B \) is an independent collection and hence a basis of \( \pi(Q_n) \).

It is easy to see that \( B \) is an \((n - 1)\)-fold basis of \( \pi(Q_n) \), and the proof of Theorem 5 is complete.

Although the bound in Theorem 5 is precise for \( n = 2, 3, \) and \( 4 \), it appears that in general this bound is far from tight. It seems likely that \( b(Q_n) \leq 4 \) for
every integer \( n \). On the other hand, since \( Q_n \) has girth 4 and is regular of degree \( n \), it is easy to show (as in the proof of Theorem 2) that \( b(Q_n) \geq 4 \) for \( n \geq 8 \). Thus we put forth the following.

**Conjecture.** If \( n \geq 8 \), \( b(Q_n) = 4 \).

Finally, let \( \gamma(G) \) denote the genus of a graph \( G \). We conclude by giving an upper bound for \( b(G) \) in terms of \( \gamma(G) \).

**Theorem 6.** \( b(G) \leq 2\gamma(G) + 2 \).

**Proof.** Embed \( G \) on a sphere with \( \gamma(G) \) handles. By Euler's relation, \( p - q + f = 2 - 2\gamma \), and so \( \dim \mathcal{E}(G) = q - p + 1 = (f - 1) + 2\gamma \). Consider the cycles bounding the faces of the embedding with exactly one exception. Let \( F \) denote the resulting set of \( f - 1 \) cycles, and let \( \langle F \rangle \) denote the subspace of \( \mathcal{E}(G) \) generated by the cycles of \( F \). We have \( \dim(\mathcal{E}(G)/\langle F \rangle) = \dim \mathcal{E}(G) - \dim \langle F \rangle = 2\gamma \). Choose any collection of \( 2\gamma \) cycles in \( G \) representing \( 2\gamma \) independent cosets in \( \mathcal{E}(G)/\langle F \rangle \). It is easy to see these \( 2\gamma \) cycles together with \( F \) constitute a basis of \( \mathcal{E}(G) \) in which each edge of \( G \) occurs at most \( 2\gamma(G) + 2 \) times.

Although equality can hold in Theorem 6 when \( G \) is planar (i.e., \( \gamma(G) = 0 \)), it is not clear that equality ever holds when \( \gamma(G) > 0 \). In particular, if \( G \) is a toroidal graph (i.e., \( \gamma(G) = 1 \)), it appears that \( b(G) \leq 3 \), whereas the given bound would be \( b(G) \leq 4 \). Perhaps it is true in general that \( b(G) \leq \gamma(G) + 2 \).

*Note added in proof.* The above conjecture has been settled affirmatively by J. A. Banks.

**Acknowledgment**

The author would like to thank P. D. Seymour for suggesting many important improvements in the paper.

**References**