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Note

Covering Weighted Graphs by Even Subgraphs

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A weighted graph is one in which each edge e is assigned a nonnegative number $w(e)$, called the weight of e. The weight of a subgraph is the sum of the weights of its edges. An even graph is a graph every vertex of which is of even degree. A cover of a graph G is a collection of its subgraphs which together cover each edge of G at least once. A cover is called an (l, m)-cover if each edge of G is covered either exactly l or exactly m times. We prove that every bridgeless graph has a (2, 4)-cover by four even subgraphs of total weight at most $(20/9)\, w(G)$. As a corollary, this result yields a weighted generalization of a result found by J. C. Bermond, B. Jackson, and F. Jaeger (*J. Combin. Theory Ser. B* **35**, 1983, 299–308) and N. Alon and M. Tarsi (*SIAM J. Algebraic Discrete Methods* **6**, 1985, 345–350).

1. INTRODUCTION

The graphs we consider are finite, and may contain loops and multiple edges. A *simple* graph is one without loops or multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *size* of a graph G is the number $|E(G)|$. A subset S of $E(G)$ is an *edge-cut* of G if its removal leaves a graph with more components; S is called a *k-edge-cut* if $|S| = k$. A 1-edge-cut is also called a *bridge*. An *even graph* is one in which every vertex is of even degree. Sometimes, we treat a subgraph as a subset of edges. Hence the *symmetric difference* of two even subgraphs Z_1 and Z_2, denoted by $Z_1 \oplus Z_2$, is the even subgraph $(Z_1 \cup Z_2) \backslash (Z_1 \cap Z_2)$. Let G be a graph. A *cover* of G is a collection **H** of subgraphs of G which together cover each edge of G at least once; **H** is called an *m-cover* of G if each edge of G is covered exactly m times by the subgraphs in **H**, and an *(l, m)-cover* if each edge is covered either exactly l or exactly m times. In this paper, we consider the case where each

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member of **H** is an even subgraph. We generalize, to weighted graphs, a result (Corollary 2.1) of Bermond, Jackson, and Jaeger [2] and Alon and Tarsi [1]. A *weighted graph* is one in which each edge e is assigned a non-negative number $w(e)$, called the *weight* of e. The *weight of a subgraph* is the sum of the weights of its edges.

It is clear that a graph is even if and only if it has a decomposition into edge-disjoint cycles. The following conjecture is known as the "Cycle Double Cover Conjecture":

Conjecture [7]. Every bridgeless graph has a 2-cover by even subgraphs.

In this paper, we shall prove

THEOREM 1. *Every bridgeless weighted graph G has a* $(2, 4)$*-cover by four even subgraphs of total weight at most* $(20/9)\, w(G)$.

If we remove the heaviest of the four even subgraphs in Theorem 1, we obtain a cover of G by three even subgraphs of total weight at most $(5/3)\, w(G)$. Thus,

COROLLARY 1.1. *Every bridgeless weighted graph G has a cover by three even subgraphs of total weight at most* $(5/3)\, w(G)$.

Setting $w(e) = 1$ for every $e \in E(G)$ in the above results, we deduce the following two results on unweighted graphs.

THEOREM 2. *Every bridgeless graph G has a* $(2, 4)$*-cover by four even subgraphs of total size at most* $(20/9)|E(G)|$.

COROLLARY 2.1 [2, 1]. *Every bridgeless graph G has a cover by three even subgraphs of total size at most* $(5/3)|E(G)|$.

Another corollary of Theorem 2 is the following result:

COROLLARY 2.2 [2]. *Every bridgeless graph G has a* 4*-cover by seven even subgraphs.*

Proof. Let G be a bridgeless graph. By Theorem 2, G has a $(2, 4)$-cover by four even subgraphs $\{Z_1, Z_2, Z_3, Z_4\}$. Then $\{Z_1, Z_2, Z_3, Z_4, Z_1 \oplus Z_2, Z_1 \oplus Z_3, Z_1 \oplus Z_4\}$ is a 4-cover of G, as required. ▮

2. PROOF OF THEOREM 1

It suffices to prove the theorem for loopless 2-edge-connected graphs. We apply induction on $|E(G)|$. If $|E(G)| = 2$, $\{G, G, \emptyset, \emptyset\}$ is the required

(2, 4)-cover. Suppose now that the result is true for all loopless 2-edge-connected graphs with fewer edges than G.

If G has a vertex v of degree more than three, then by a result of Fleischner [4] there are two edges e_1, e_2 incident with v such that deleting e_1, e_2 and joining their other ends by a new edge e yields a 2-edge-connected graph G'. Set $w(e) = w(e_1) + w(e_2)$, so that $w(G') = w(G)$. By the induction hypothesis, G' has a (2, 4)-cover with the required properties. This readily yields an appropriate (2, 4)-cover of G.

If G has a 2-edge-cut $\{e_1, e_2\}$, let G' be the weighted graph obtained by contracting e_1 and reassigning to e_2 a new weight $w(e_1) + w(e_2)$, so that $w(G') = w(G)$. If Z' is a even subgraph in G' with $e_2 \in Z'$, let $Z = Z' \cup \{e_1\}$. Then Z is a even subgraph of G since $\{e_1, e_2\}$ is a 2-edge-cut of G. By the induction hypothesis, G' has a (2, 4)-cover with the required property. For each even subgraph Z' containing e_2 in this cover, replace Z' by $Z = Z' \cup \{e_1\}$. This yields an appropriate (2, 4)-cover of G.

Therefore, we may assume that G is a simple, cubic, 3-edge-connected graph. By a result of Edmonds [3], there is a collection $\mathbf{M}$ of $3k$ perfect matchings of G, for some integer $k \geqslant 1$, such that each edge of G belongs to exactly k members of $\mathbf{M}$. Let A be a 3-edge-cut of G. For any $M \in \mathbf{M}$, $|A \cap (E(G) \backslash M)|$ is even because $E(G) \backslash M$ is a 2-factor of G, and so $|A \cap M|$ is odd. It follows that

$$|A \cap M| \geqslant 1 \qquad \text{for every} \quad M \in \mathbf{M}.$$

Since each edge of A belongs to exactly k members of $\mathbf{M}$,

$$\sum_{M \in \mathbf{M}} |A \cap M| = 3k = |\mathbf{M}|.$$

Consequently,

$$|A \cap M| = 1 \qquad \text{for any 3-edge-cut } A \text{ of } G \text{ and every } M \in \mathbf{M}. \tag{1}$$

Since each edge of G belongs to exactly k members of $\mathbf{M}$,

$$\sum_{M \in \mathbf{M}} w(M) = kw(G).$$

Hence there is some $M^* \in \mathbf{M}$ such that

$$w(M^*) \leqslant \frac{kw(G)}{|\mathbf{M}|} = \frac{1}{3} w(G). \tag{2}$$

Let $F^* = E(G) \backslash M^*$ and let G^* be the graph obtained from G by contracting each component of the 2-factor F^* to a single vertex. Then

$$w(G^*) = w(M^*). \tag{3}$$

Since G is 3-edge-connected, so is G^*. Moreover, if A is a 3-edge-cut of G^*, then A is a 3-edge cut of G with $A \subseteq M^*$. This is impossible by (1). Therefore, G^* is 4-edge-connected. It follows from a result of Nash-Williams [6] and Tutte [8] that G^* has two edge-disjoint spanning trees, which implies, as observed by Jaeger [5], that G^* has a 2-cover by three even subgraphs $\{Z_1^*, Z_2^*, Z_3^*\}$. Since these cover every edge of G^* exactly twice, $\sum_{i=1}^{3} w(Z_i^*) = 2w(G^*)$. Without loss of generality, suppose that Z_3^* is the heaviest of these even subgraphs. Then

$$w(Z_1^* \cap Z_2^*) + w(Z_1^* \cup Z_2^*) = w(Z_1^*) + w(Z_2^*) \leqslant \tfrac{4}{3} w(G^*).$$

Because $\{Z_1^*, Z_2^*\}$ is a cover of G^*, $w(Z_1^* \cup Z_2^*) = w(G^*)$. It follows that

$$w(Z_1^* \cap Z_2^*) \leqslant \tfrac{1}{3} w(G^*). \tag{4}$$

Observe that each vertex of G^* corresponds to a cycle of G. It is easy to see that we can add paths in F^* to Z_1^* and Z_2^* and obtain two even subgraphs Z_1 and Z_2 in G, respectively. We claim that $\mathbf{Z} = \{Z_1, Z_2, Z_1 \oplus F^*, Z_2 \oplus F^*\}$ is a (2, 4)-cover of G with the required property. It can be easily checked that $\mathbf{Z}$ covers each edge of G either twice or four times, and that the edges of F^* are covered twice. Thus, if Q denotes the set of edges covered four times,

$$Q \subseteq Z_1^* \cap Z_2^*. \tag{5}$$

By (5), (4), (3), and (2),

$$w(Q) \leqslant w(Z_1^* \cap Z_2^*) \leqslant \tfrac{1}{3} w(G^*) = \tfrac{1}{3} w(M^*) \leqslant \tfrac{1}{9} w(G),$$

and

$$w(\mathbf{Z}) = 2(w(G) + w(Q)) \leqslant \tfrac{20}{9} w(G),$$

as claimed. This completes the proof of Theorem 1. ∎

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